Digital Commons @ LMU and LLS

Digital Commons@
Loyola Marymount University
and Loyola Law School

Mathematics Faculty Works

Mathematics

1-1-2004

A New Proof of a Theorem of Phan

Curtis Bennett
Loyola Marymount University, cbennett@lmu.edu

Sergey Shpectorov
Bowling Green State University - Main Campus

Repository Citation

Bennett, Curtis and Shpectorov, Sergey, "A New Proof of a Theorem of Phan" (2004). *Mathematics Faculty Works.* Paper 12.
http://digitalcommons.lmu.edu/math_fac/12

Recommended Citation

Bennett, C. D., Shpectorov, S. *A New Proof of a Theorem of Phan,* Journal of Group Theory. vol. 7 (2004) pp. 287-310.

This Article is brought to you for free and open access by the Mathematics at Digital Commons@
Loyola Marymount University
and Loyola Law School. It has been accepted for inclusion in Mathematics Faculty Works by an authorized administrator of Digital Commons@
Loyola Marymount University
and Loyola Law School. For more information, please contact digitalcommons@lmu.edu.

J. Group Theory **7** (2004), 287–310

Journal of Group Theory
© de Gruyter 2004

A new proof of a theorem of Phan

Curtis D. Bennett and Sergey Shpectorov*

(Communicated by A. V. Borovik)

Abstract. We apply diagram geometry and amalgam techniques to give a new proof of a theorem of K.-W. Phan, characterizing the special unitary group as a group generated by certain systems of subgroups $\mathrm{SU}(2, q^2)$.

1 Introduction

Suppose that $n > 2$ and that $q > 2$ is a prime power. Consider $G = \mathrm{SU}(n+1, q^2)$ and let $U_i \cong \mathrm{SU}(2, q^2)$, $i = 1, 2, \ldots, n$, be the subgroups of G corresponding to the 2×2 blocks along the main diagonal. Let D_i be the diagonal subgroup in U_i (D_i is a maximal torus of size $q+1$). Then G is generated by the subgroups U_i, and the following hold:

(P1) if $|i-j| > 1$ then $[x, y] = 1$ for all $x \in U_i$ and $y \in U_j$;

(P2) if $|i-j| = 1$ then $U_{ij} = \langle U_i, U_j \rangle$ is isomorphic to $\mathrm{SU}(3, q^2)$; and

(P3) $[x, y] = 1$ for all $x \in D_i$ and $y \in D_j$.

Suppose now that G is an arbitrary group generated by subgroups $U_i \cong \mathrm{SU}(2, q^2)$, $i = 1, 2, \ldots, n$, and that a maximal torus D_i of size $q+1$ is chosen in each U_i. If the conditions (P1)–(P3) above hold for G then we will say that G contains a *Phan system of rank n*, after Kok-Wee Phan, who in 1975 published the following result [8]:

Theorem 1.1. *If G contains a Phan system of rank n with $q > 4$ then G is isomorphic to a factor group of* $\mathrm{SU}(n+1, q^2)$.

Those familiar with the Curtis–Tits theorem will recognize a similarity between that theorem and Phan's theorem. Just as the Curtis–Tits theorem was used in the classification of the finite simple groups, being the principal means for identification of Chevalley groups, Phan's theorem was used for the identification of simple groups

* Both authors were partially supported by the NSA grant #MDA904-01-1-0023.

having a standard component of unitary type. Thus Phan's theorem is important for the revision of the classification led by R. Lyons and R. Solomon.

Over the years it has become apparent that the published proof of Phan's theorem is not entirely satisfactory. Some lemmas rely on heavy computations in the unitary group (and, not surprisingly, the computations are omitted in the published text). Other lemmas are given proofs that are too sketchy, so that even for a specialist it is hard to fill in the gaps. Taking into account the importance of Phan's theorem, it is desirable to give it a new and complete proof, preferably one that is short and transparent.

Ideas for the new proof can be found in the area of flag-transitive diagram geometries and the area of amalgams of groups. In fact, Phan's theorem can be regarded as a characterization of the geometry $\mathcal{N} = \mathcal{N}(n, q)$ of all proper non-singular subspaces in the unitary space for $\mathrm{SU}(n+1, q^2)$. The relation to geometry was first observed by M. Aschbacher [1]. He and K. M. Das did some work toward a new proof of Phan's theorem. In particular, Das [2] proved that $\mathcal{N}$ is simply connected whenever q is odd and $q \neq 3$.

In this paper we present a complete proof of Phan's theorem. In Section 2 we define the geometry $\mathcal{N}$, and in Section 3 we prove that $\mathcal{N}$ is simply connected if $n > 3$, or $n = 3$ and $q > 3$. The case $(n, q) = (5, 2)$ was exceptional for our proof and was covered by a computation in GAP performed by J. Dunlap, whom we thank for this contribution. This extends the result of Das to the case of characteristic 2, and even, to some extent, to the cases $q = 2$ and 3. In Sections 4–12 we carry out a careful analysis of amalgams related to Phan systems and achieve the complete classification. This part was essentially missing from the original paper by Phan, although his theorem implicitly claims that the amalgam is unique.

In fact, we obtain somewhat more than Phan's Theorem. For $q > 4$ our assumptions are slightly weaker than Phan's. Furthermore, we fully cover the case $q = 4$ and obtain partial results for $q = 2$ and $q = 3$. We now give the exact statements of our theorems.

We will say that subgroups U_1 and U_2 of $\mathrm{SU}(3, q^2)$ form a *standard pair* whenever each U_i is the stabilizer in $\mathrm{SU}(3, q^2)$ of a non-singular vector v_i (v_i is then unique up to a scalar factor) and, furthermore, v_1 and v_2 are perpendicular. By Witt's theorem, standard pairs are exactly the conjugates of the pair formed by the two subgroups $\mathrm{SU}(2, q^2)$ arising from the 2×2 blocks on the main diagonal. Standard pairs in $\mathrm{PSU}(3, q^2)$ will be defined as the images under the natural homomorphism of the standard pairs from $\mathrm{SU}(3, q^2)$.

We say that a group G possesses a *weak Phan system* if G contains subgroups $U_i \cong \mathrm{SU}(2, q^2)$, $i = 1, 2, \ldots, n$ and $U_{i,j}$, $1 \leqslant i < j \leqslant n$ so that the following hold:

(wP1) if $|i - j| > 1$ then $U_{i,j}$ is a central product of U_i and U_j;

(wP2) for $i = 1, 2, \ldots, n-1$, U_i and U_{i+1} are contained in $U_{i,i+1}$, which is isomorphic to $\mathrm{SU}(3, q^2)$ or $\mathrm{PSU}(3, q^2)$; moreover, U_i and U_{i+1} form a standard pair in $U_{i,i+1}$; and

(wP3) the subgroups $U_{i,j}$, $1 \leqslant i < j \leqslant n$, generate G.

We have added (wP3) instead of just saying that the subgroups U_i generate G because of the case $q = 2$. When $q = 2$, $\mathrm{SU}(3,4)$ is not generated by a standard pair of subgroups $\mathrm{SU}(2,4)$. This fact influenced the wording of the entire definition: we have not introduced $U_{i,j}$ as $\langle U_i, U_j \rangle$ precisely in order to allow for the case $q = 2$. It is easy to see that conditions (P2) and (P3) imply that U_i and U_{i+1} form a standard pair in $U_{i,i+1} = \langle U_i, U_{i+1} \rangle$. Hence every Phan system leads to a weak Phan system. Consequently, Phan's theorem is implied by the following:

Theorem 1.2. *If G contains a weak Phan system of rank n with $q > 3$ then G is isomorphic to a factor group of* $\mathrm{SU}(n+1, q^2)$.

As we have already mentioned, the Phan system set-up forbids $q = 2$ completely. The formulation of (wP1)–(wP3) allows the case $q = 2$ for weak Phan systems. Thus Theorem 1.2 leaves us with two exceptional cases $q = 2$ and 3 instead of one. For these cases we prove the following:

Theorem 1.3. *Suppose that G contains a weak Phan system of rank $n > 3$ with $q = 2$ or 3. Suppose further that the following conditions are satisfied*:

(1) *for $i \in \{1, \ldots, n-2\}$, the subgroup $\langle U_{i,i+1}, U_{i+1,i+2} \rangle$ is isomorphic to a factor group of* $\mathrm{SU}(4, q^2)$;

(2) *if $q = 2$ then*
   (i) *for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n-1\}$, if $i \notin \{j-1, j, j+1, j+2\}$ then U_i and $U_{j,j+1}$ commute elementwise; and*
   (ii) *for $i \in \{1, \ldots, n-1\}$ and $j \in \{1, \ldots, n-1\}$, if $i \notin \{j-2, j-1, j, j+1, j+2\}$ then $U_{i,i+1}$ and $U_{j,j+1}$ commute elementwise.*

Then G is isomorphic to a factor group of $\mathrm{SU}(n+1, q^2)$.

When $q = 2$, there exist infinite groups G that contain weak Phan systems, and so it seems impossible to achieve a meaningful classification of all such groups G. Thus Theorem 1.3 appears to be best possible when $q = 2$. A complete classification of groups G for $q = 3$ may be feasible. However, it is expected that, when $q = 3$, new examples (that is, other than the factor groups of $\mathrm{SU}(n+1, 3)$) of groups with a weak Phan system exist for all ranks $n > 2$, and these new examples must be constructed before a complete classification can be attempted.

2 The geometry $\mathcal{N}$

Let V be an $(n+1)$-dimensional vector space over the field $\mathrm{GF}(q^2)$, equipped with a non-degenerate unitary form. We define a pregeometry (also called an incidence system) $\mathcal{N} = \mathcal{N}(n, q)$ of rank n with elements of type k for $k \in \{1, \ldots, n\}$, being the non-singular subspaces of V of dimension k. Incidence in $\mathcal{N}$ is defined by containment. We will use the usual geometric terminology (see [9] or [6]). In particular, the elements of $\mathcal{N}$ of type 1 and 2 will be called *points* and *lines* respectively. Recall that

a pregeometry is called a geometry whenever every maximal flag contains one element of each type.

In this section we study the basic properties of $\mathcal{N}$.

Lemma 2.1. *The pregeometry $\mathcal{N}$ is a geometry.*

Proof. Let p be a point. Then every non-degenerate subspace U that properly contains p bijectively corresponds to a non-degenerate subspace in $p^{\perp}$. Namely the map $U \mapsto U \cap p^{\perp}$ is such a bijection, establishing an isomorphism between the residue of p and a similar pregeometry $\mathcal{N}' \cong \mathcal{N}(n-1, q)$. Since every maximal flag clearly contains a point, the induction shows that every maximal flag contains elements of all types.

Recall that the collinearity graph Γ associated with $\mathcal{N}$ is the graph on points of $\mathcal{N}$ in which two points are adjacent whenever they are incident to a common line. Further note that a line of $\mathcal{N}$ contains $q^2 - q$ points.

Lemma 2.2. *If L is a line and a is a point not on L, then a is collinear with at least $q^2 - 2q - 1$ points on L.*

Proof. Let U be the 3-space $\langle a, L\rangle$ and let $W = U \cap a^{\perp}$. Observe that a is not collinear to a point b on L if and only if $X = \langle a, b\rangle$ is singular. On the other hand, if X is a singular 2-space with $y \subset X \subset U$ then $X = \langle y, s\rangle$, where s is a singular 1-space from W (s is the radical of X). So the lemma will follow once we show that the number of these 1-spaces s is at most $q + 1$.

Since L is a line (a non-singular 2-space), the radical of U has dimension at most 1. Hence the radical of W also has dimension at most 1. By inspection, the number of singular 1-spaces in W is $q + 1$ if W is non-singular and 1 if W is singular.

Lemma 2.3. *Suppose that $n \geqslant 2$. Then the diameter of Γ is 2 provided that $(n, q) \neq (2, 2)$.*

Proof. Suppose first that $n \geqslant 3$ and let a and b be non-collinear points. Then $\langle a, b\rangle$ is a singular but not totally singular subspace of V of codimension at least 2. Hence also $\langle a, b\rangle^{\perp}$ is singular but not totally singular. This means that there is a point c perpendicular to (and hence collinear with) both a and b. Indeed $\langle a, c\rangle$ and $\langle b, c\rangle$ are lines of $\mathcal{N}$.

On the other hand, if $n = 2$ and $q > 2$ then the claim immediately follows from Lemma 2.2.

The case $(n, q) = (2, 2)$ is indeed an exception as Γ is then a disconnected union of four 3-cliques.

Corollary 2.4. *The geometry $\mathcal{N}$ is connected unless $(n, q) = (2, 2)$. Moreover, it is residually connected if $q \neq 2$.*

Proof. The first claim follows from Lemma 2.3, and the second claim follows from that lemma and induction on the rank.

Observe that the group $\Gamma\mathrm{U}(n+1,q^2)$ (i.e., $\mathrm{GU}(n+1,q^2)$ extended by the field automorphisms) acts on $\mathcal{N}$. This action is not faithful as the scalar matrices act trivially.

Lemma 2.5. *The group* $\mathrm{SU}(n+1,q^2)$ *acts flag-transitively on* $\mathcal{N}$.

Proof. Let $G = \mathrm{SU}(n+1,q^2)$. It follows from Witt's theorem that G is transitive on points. Pick a point p. Then G_p contains $\mathrm{SU}(n,q^2)$ acting on the residue of p, which is isomorphic to $\mathcal{N}(n-1,q)$. By induction G_p is flag-transitive on the residue of p and hence G is flag-transitive on $\mathcal{N}$.

To summarize our discussion, we state the following result:

Proposition 2.6. *The pregeometry* $\mathcal{N}$ *is a geometry. It is connected unless* $(n,q) = (2,2)$, *and it is residually connected if* $q \neq 2$. *The group* $\mathrm{SU}(n+1,q^2)$ *acts on* $\mathcal{N}$ *flag-transitively.*

Let $\circ\!\overset{U}{\text{———}}\!\circ$ denote the class of rank 2 geometries $\mathcal{N}(3,q)$. Then the diagram of $\mathcal{N}$ looks as follows:

$$\underset{q^2-q-1}{\circ}\overset{U}{\text{———}}\underset{q^2-q-1}{\circ}\overset{U}{\text{———}}\underset{q^2-q-1}{\circ}\cdots\underset{q^2-q-1}{\circ}\overset{U}{\text{———}}\underset{q^2-q-1}{\circ}.$$

3 Simple connectivity

The purpose of this section is to show that the geometry $\mathcal{N}$ is almost always simply connected.

Proposition 3.1. *Suppose that* $n \geqslant 3$. *Then the geometry* $\mathcal{N}$ *is simply connected unless* $(n,q) = (3,2)$ *or* $(3,3)$.

We will prove this in a series of lemmas. Throughout the rest of this section we assume that $q > 3$ or $n > 3$. One of our lemmas (Lemma 3.6) fails when $(n,q) = (5,2)$. We thank J. Dunlap for verifying the simple connectedness of $\mathcal{N}(5,2)$ on a computer, using GAP ([3]). Due to his computation we shall assume in what follows that $(n,q) \neq (5,2)$.

In order to prove the proposition, we need to show that $\pi_1(\mathcal{N}) = 1$; that is, we need to show that every cycle in the incidence graph of $\mathcal{N}$ is homotopic to the trivial cycle. Recall that two cycles are called *elementary homotopic* if they are obtained from one another by inserting or deleting a 2-cycle (return) or a 3-cycle. The *homotopy relation* is the transitive closure of the elementary homotopy relation.

We will say that a cycle is *geometric* if it is fully contained in $\{a\} \cup \mathrm{res}(a)$ for some $a \in \mathcal{N}$. In other words, all vertices on a geometric cycle are incident to some fixed element $a \in \mathcal{N}$.

Lemma 3.2. *Every geometric cycle is homotopic to the trivial cycle.*

Proof. Suppose that $\gamma = x_1 x_2 \ldots x_k x_1$ is a cycle without returns. If $k \leqslant 3$ then γ is homotopic to the trivial cycle by definition. So we assume that $k > 3$. If $x_1 = a$ or $x_3 = a$ then x_1 is incident to x_3 and so γ is homotopic to a shorter geometric cycle, namely $x_1 x_3 \ldots x_1$. Similarly, if $x_2 = a$ or $x_4 = a$ then γ is homotopic to $x_1 x_2 x_4 \ldots x_1$. Finally, if $a \neq x_i$, $i \leqslant 4$, then γ is homotopic to $x_1 a x_4 \ldots x_1$. Thus, in all cases, γ is homotopic to a shorter geometric cycle, and the claim follows by induction.

Corollary 3.3. *If two cycles are obtained from one another by inserting or deleting a geometric cycle then they are homotopic.*

Fix a point x and let it be our base-point. That is, the cycles forming $\pi_1(\mathcal{N})$ begin and end at x. Let Σ be the subgraph in the incidence graph of $\mathcal{N}$ induced by all points and lines. For an element $a \in \mathcal{N}$ that is neither a point nor a line, define $\Sigma_a = \Sigma \cap \mathrm{res}(a)$. Thus Σ_a consists of all points and lines incident with a.

Lemma 3.4. *Every cycle starting at x is homotopic to a cycle that is fully contained in Σ.*

Proof. For a cycle γ let $s(\gamma)$ be the number of vertices on γ that are neither points nor lines. We will prove the lemma by induction on $s(\gamma)$. If $s(\gamma) = 0$, then γ is fully contained in Σ and there is nothing to prove. Assume that the claim of the lemma holds for all γ with $s(\gamma) < k$, and let $\gamma = x x_1 x_2 \ldots x_m x$ be a cycle with $s(\gamma) = k$. Let x_i be the first vertex on γ that is not contained in Σ. Let a be a point or a line that is incident with both x_i and x_{i+1} (where we take $a = x$ if $i = m$). Let b also be an element of type n that is incident with x_i and x_{i+1}. Since the types of x_{i-1} and a are smaller than the type of x_i, we have that b is incident with x_{i-1} and a. Observe that $\mathrm{res}(b) \cong \mathcal{N}(n-1, q)$. Since $(n, q) \neq (3, 2)$, Lemma 2.4 implies that there is a path α from x_{i-1} to a that is fully contained in Σ_b. By Corollary 3.3, γ is homotopic to $\gamma' = x \ldots x_{i-2} \alpha x_{i+1} \ldots x$. Since $s(\gamma') = k - 1$, the claim of the lemma follows.

In view of Corollary 3.3 and Lemma 3.4, it suffices to show that every cycle in Σ can be decomposed as a product of geometric cycles. Observe that $\mathcal{N}$ is a partial linear space (that is, any two collinear points lie on a unique line). This implies that every cycle in Σ can be recovered from the sequence of points on it. In fact, the cycles in Σ that start from a point and that have no returns correspond bijectively to the cycles in the collinearity graph of $\mathcal{N}$ (call it Γ), having the property that no three consecutive points lie on the same line. This allows us to work with Γ rather than with Σ. For an element $a \in \mathcal{N}$, where a is not a point, let Γ_a be the subgraph in Γ induced by all points incident with a. We will call a cycle in Γ *geometric* if it is fully contained in some Γ_a. As follows from the discussion above, we need to show that

every cycle in Γ can be decomposed as a product of geometric cycles. We will achieve this goal in two steps. We will first show that every triangle (3-cycle) in Γ can be decomposed. After that, it suffices to prove that every cycle in Γ is a product of triangles (and returns).

Recall that our points are non-singular 1-spaces in V. Two points a and b are collinear if and only if the 2-space $\langle a, b\rangle$ is non-singular. In particular, the latter is true whenever a and b are perpendicular; i.e., perpendicular points are collinear. We are now prepared to realize our two-step plan.

Lemma 3.5. *Every triangle in* Γ *is decomposable.*

Proof. Let $\gamma = abca$ be a triangle (3-cycle) in Γ. If the subspace $U = \langle a, b, c\rangle$ is non-singular then γ is geometric. Indeed, $\dim V \geqslant 4$ and hence U is a proper subspace. So suppose that U is singular. Since $\langle a, b\rangle$ is a line, the radical of U is 1-dimensional. Therefore U is contained in a non-singular 4-space. If $n > 3$ then that 4-space is proper and hence γ is geometric. Thus we are down to the case $n = 3$. By assumption, in this case we have $q > 3$.

We deal first with the case where two points on γ (say, a and b) are perpendicular. In this case we say that γ is of *perp type*. Let $W = \langle a, c\rangle^{\perp}$. Since $\langle a, c\rangle$ is a line, W is non-singular, and hence it is a line, too. Clearly all points on W are collinear with a and c. By Lemma 2.2, at least $q^2 - 2q - 1$ of them are also collinear with b. If d is a point on W that is collinear with b then we say that d is *good* if the triangle $dbcd$ is geometric, and that it is *bad* otherwise. We claim that the number of bad points is at most $q + 1$. Indeed, if d is bad then $\langle b, c, d\rangle$ is non-singular with a 1-dimensional radical s. Clearly s is a singular 1-space contained in $\langle b, c\rangle^{\perp}$, which is a line. Hence the number of choices for s is at most $q + 1$. Since $\langle b, c, d\rangle = \langle b, c, s\rangle$, the claim follows.

Thus the number of good points is at least $(q^2 - 2q - 1) - (q + 1) = q^2 - 3q - 2$. Since $q > 3$, good points exist, and we may let d be one of them. Since a is perpendicular to b and d and since $\langle b, d\rangle$ is a line, the 3-space $\langle a, b, d\rangle$ is non-singular. Hence $abda$ is a geometric triangle. Similarly, $adca$ is geometric, since d is perpendicular to a and c. Also $dbcd$ is geometric, since d is good. Thus $abda$, $dbcd$ and $adca$ are all geometric and hence $\gamma = abca$ is decomposable.

Finally, let $\gamma = abca$ be arbitrary. Let $W = \langle a, c\rangle^{\perp}$. By Lemma 2.2, at least $q^2 - 2q - 1$ points on W are collinear with b. Let d be one of these points. Then all three triangles $abda$, $dbcd$ and $adca$ are of perp type. Hence all triangles γ are decomposable.

In view of this lemma, it remains to decompose an arbitrary cycle in Γ as a product of triangles. By Lemma 2.3, the diameter of Γ is 2. By the standard argument, if the diameter of a graph is k then every cycle in it can be decomposed as a product of cycles of length at most $2k + 1$. Hence Proposition 3.1 will follow once we show that 4-cycles and 5-cycles in Γ can be decomposed as products of triangles.

Lemma 3.6. *Every* 4*-cycle in* Γ *is decomposable.*

Proof. Let $\gamma = abcda$ be a 4-cycle. If a is collinear with c, or b is collinear with d, then clearly γ is decomposable. So, without loss of generality, we may assume that a and c are not neighbors, and similarly we may assume b and d are not neighbors.

Let L be a line contained in the non-singular $(n-1)$-space $\langle a, b\rangle^{\perp}$. If c is on L, then it is collinear with all of the points on L. If c is not on L, then Lemma 2.2 implies that c is collinear with at least $q^2 - 2q - 1$ points on L. In each case c is collinear with all but at most $q+1$ points on L. Clearly the same holds for d. Therefore there are at least $(q^2 - q) - 2(q+1) = q^2 - 3q - 2$ points on L that are collinear with both c and d. If $q > 3$ then $q^2 - 3q - 2 > 0$ and hence we can choose a point e on L that is collinear with c and d. Since L is contained in $\langle a, b\rangle^{\perp}$, e is also collinear with a and b, and hence γ is decomposable. It remains to deal with the cases $q = 2$ and $q = 3$. Recall that in these cases we have $n > 3$.

Notice that $U = \langle a, b, c\rangle$ is a 3-space whose radical has dimension at most 1. Therefore $U^{\perp}$ is not totally singular as $\dim V \geqslant 5$. This means that there is a point e perpendicular to a, b and c. Similarly, there is a point f perpendicular to a, d and c. Observe that γ decomposes as a product of the 4-cycle $\gamma' = aecfa$ and four triangles $abea$, $ebce$, $fcdf$ and $afda$. Thus it remains to decompose γ'. If e and f are not collinear this is not the desired decomposition. So suppose that $\langle e, f\rangle$ is singular. If $W = \langle a, e, c, f\rangle$ is 3-dimensional then $W^{\perp}$ is not totally singular, and hence there is a point perpendicular to all four points a, e, c and f. Thus we may assume that $\dim W = 4$. However this means that W is the orthogonal direct sum of two singular 2-spaces, $\langle a, c\rangle$ and $\langle e, f\rangle$. Hence the radical of W is of dimension 2. This shows that $\dim V \geqslant 6$; that is, $n \geqslant 5$. If $\dim V \geqslant 7$ then $W^{\perp}$ is not totally singular, and hence again there is a point perpendicular to all four points on γ'. Thus we may assume that $\dim V = 6$. Since $(n, q) \neq (5, 2)$ by assumption, we then have $q = 3$.

Observe that $\langle a, e, c\rangle^{\perp}$ is 3-dimensional and its radical is 1-dimensional. Hence it contains a line L. According to Lemma 2.2, f is collinear with a point on L, and that point is then collinear with all of a, e, c and f.

We now deal with the 5-cycles.

Lemma 3.7. *Every 5-cycle in Γ is decomposable.*

Proof. Let $\gamma = abcdea$ be a 5-cycle. We claim that there is a point f that is collinear with a, c and d. Indeed, if $n > 3$ then $\langle a, c, d\rangle^{\perp}$ is not totally singular, and hence f can be chosen in it. If $n = 3$ then $q > 3$. In this case $L = \langle c, d\rangle^{\perp}$ is a line and we can take as f any point in L that is collinear with a. Such a point exists by Lemma 2.2, and the claim follows.

Now it follows from Lemmas 3.5 and 3.6 that γ is decomposable, since it is the product of $abcfa$, $fcdf$ and $afdea$.

This completes the proof of Proposition 3.1.

Before moving on to the portion of the paper using amalgams, we state and prove a technical lemma for use in Section 12. We borrow this lemma from [4], which is logically dependent on the results of the present paper. To avoid any questions of circular logic, we borrow this lemma together with its proof.

Recall that the *direct sum* $\Gamma_1 \oplus \Gamma_2$ of geometries Γ_1 and Γ_2 is defined as follows. The type set (respectively, element set) of $\Gamma_1 \oplus \Gamma_2$ is the disjoint union of the type sets (respectively, element sets) of Γ_1 and Γ_2. The incidence relation on $\Gamma_1 \oplus \Gamma_2$ is given by the incidence relations on Γ_1 and Γ_2 together with the extra requirement that every element of Γ_1 is incident with every element of Γ_2.

Lemma 3.8. *Assume that* $\Sigma = \Sigma_1 \oplus \Sigma_2$ *with* Σ_1 *connected of rank at least* 2. *Then* Σ *is simply connected.*

Proof. Certainly Σ is connected. Choose a base-point $x \in \Sigma_1$. We first notice that every cycle $xx_1 \dots x_{n-1}x$ fully contained in Σ_1 is null-homotopic. Indeed, if $y \in \Sigma_2$ then y is incident to x and every x_i.

Thus it suffices to show that every cycle $xx_1 \dots x_{n-1}x$ is homotopic to a cycle contained in Σ_1. We proceed by induction on the number of elements on the cycle that are not in Σ_1. Suppose that s is minimal such that $x_s \notin \Sigma_1$. Let $y \in \Sigma_1$ be such that $y \neq x_{s+1}$ and y is incident with x_{s+1}. (Recall that Σ_1 has rank at least 2.) Notice that y is incident with x_s. Since the residue of x_s contains Σ_1 (and Σ_1 is connected), there exists a path $x_{s-1}y_1 \dots y_{k-1}y$ fully contained in Σ_1. Furthermore, this path is homotopic to the path $x_{s-1}x_s y$, since all elements on it are incident with x_s. Thus our original path is homotopic to the path $xx_1 \dots x_{s-1}y_1 \dots y_{k-1}yx_{s+1} \dots x_{n-1}x$. This path has fewer elements outside Σ_1, and our lemma follows.

4 Amalgams: preliminaries

A general definition of an amalgam of groups can be found, say, in [10]. In this paper we need the simplest kind of amalgams, defined as follows. An *amalgam* $\mathscr{A}$ is a set with a partial operation of multiplication and a collection of subsets $\{G_i\}_{i \in I}$, such that the following hold:

(1) $\mathscr{A} = \bigcup_{i \in I} G_i$;

(2) the product ab is defined if and only if $a, b \in G_i$ for some $i \in I$;

(3) the restriction of the multiplication to each G_i turns G_i into a group; and

(4) $G_i \cap G_j$ is a subgroup both in G_i and G_j for all $i, j \in I$.

We call the groups G_i the *members* of the amalgam $\mathscr{A}$. Let $\mathscr{A} = \bigcup_{i \in I} G_i$ and $\mathscr{B} = \bigcup_{i \in I} H_i$ be two amalgams over the same index set I. A mapping $\phi : \mathscr{A} \to \mathscr{B}$ is an *amalgam homomorphism* if for every $i \in I$ the restriction of ϕ to G_i is a homomorphism from G_i to H_i. If ϕ is bijective and it establishes an isomorphism between each G_i and the corresponding H_i, then ϕ is an *amalgam isomorphism*. An *automorphism* of $\mathscr{A}$ is an isomorphism of $\mathscr{A}$ onto itself. As usual, the automorphisms of $\mathscr{A}$ form the automorphism group, $\mathrm{Aut}(\mathscr{A})$.

An amalgam $\mathscr{B} = \bigcup_{i \in I} H_i$ is a *quotient* of the amalgam $\mathscr{A} = \bigcup_{i \in I} G_i$ if there is a homomorphism ϕ from $\mathscr{A}$ to $\mathscr{B}$ such that the restriction of ϕ to every G_i maps G_i onto H_i.

A group G is called a *completion* of $\mathscr{A}$ if there exists a mapping $\pi : \mathscr{A} \to G$ such that

(1) for all $i \in I$ the restriction of π to G_i is a homomorphism of G_i to G; and

(2) $\pi(\mathscr{A})$ generates G.

Among all completions of $\mathscr{A}$ there is a 'largest' one, having the following presentation:

$$U(\mathscr{A}) = \langle u_g, g \in \mathscr{A} \mid u_x u_y = u_z, \text{ whenever } xy = z \text{ in } \mathscr{A} \rangle.$$

Since we can map $\mathscr{A}$ to $U(\mathscr{A})$ via $\psi : g \mapsto u_g$, $U(\mathscr{A})$ is indeed a completion of $\mathscr{A}$. If G is an arbitrary completion of $\mathscr{A}$ and π is the corresponding mapping from $\mathscr{A}$ to G then the mapping $u_g \mapsto \pi(g)$ leads to a surjective homomorphism $\hat{\pi}$ from $U(\mathscr{A})$ to G. Thus every completion of $\mathscr{A}$ is an image of $U(\mathscr{A})$. Because of this, $U(\mathscr{A})$ is called the *universal completion* of $\mathscr{A}$. We say that $\mathscr{A}$ *collapses* if $U(\mathscr{A}) = 1$. (Some adjustments to this notion for the case $q = 2$ will be necessary in Section 5.) Notice that if $\mathscr{B}$ is a quotient of $\mathscr{A}$ then $U(\mathscr{B})$ is (isomorphic to) a factor group of $U(\mathscr{A})$. In particular, if $\mathscr{B}$ does not collapse then neither does $\mathscr{A}$.

Suppose that Γ is a geometry and $G \leqslant \operatorname{Aut}\Gamma$ is a flag-transitive group. Corresponding to Γ and G, there is an amalgam $\mathscr{A} = \mathscr{A}(\Gamma, G)$, defined as follows. Let F be a maximal flag in Γ. Define $\mathscr{A}$ to be the union $\bigcup_{x \in F} G_x$, where G_x denotes the stabilizer of x in G. Since G is flag-transitive, it follows that $\mathscr{A}$ is independent (up to conjugation) of the choice of F. In general, for $\varnothing \neq F_0 \subseteq F$, we call G_{F_0} a *parabolic subgroup*, or just a parabolic. Parabolics are ordered by inclusion, which corresponds to the reverse inclusion of the associated flags. The maximal parabolics are the stabilizers of one-element subflags and thus we call $\mathscr{A}$ the *amalgam of maximal parabolics*. Notice that every parabolic G_{F_0} is an intersection of maximal parabolics, $G_{F_0} = \bigcap_{x \in F_0} G_x$, and hence we can also view $\mathscr{A}$ as the union of all parabolics G_{F_0}, where $\varnothing \neq F_0 \subseteq F$. The *rank* of the parabolic G_{F_0} is defined to be the rank of the residue of F_0 in Γ. If the rank of Γ is n then the maximal parabolics have rank $n - 1$, the smallest parabolic G_F has rank 0 and is called the *Borel subgroup*, and parabolics of rank 1 are called the *minimal parabolics*.

Let us conclude this section by describing the parabolics in the case of $G = \mathrm{SU}(n+1, q^2)$ acting on our geometry $\mathscr{N}$. Recall that according to Lemma 2.5, G acts flag-transitively on $\mathscr{N}$. Let $\mathscr{E} = \{e_1, \ldots, e_{n+1}\}$ be an orthonormal basis in the natural unitary space V for G. Without loss of generality, we may assume that this is the standard basis, so that every element of G is naturally an $(n+1) \times (n+1)$ unitary matrix. In order to define the parabolics of G, we must first choose a maximal flag F. Let $F = \{E_1, E_2, \ldots, E_n\}$, where

$$E_i = \langle e_1, \ldots, e_i \rangle.$$

Notice that every subspace E_i is non-degenerate; hence F is indeed a maximal flag of $\mathscr{N}$. Choose a subflag F_0 of F, say $F_0 = \{E_{i_1}, E_{i_2}, \ldots, E_{i_k}\}$. Without loss we assume that $i_1 < i_2 < \cdots < i_k$. This subflag corresponds to a decomposition

$$V = E_{i_1} \oplus (E_{i_2} \cap E_{i_1}^{\perp}) \oplus (E_{i_3} \cap E_{i_2}^{\perp}) \oplus \cdots \oplus E_{i_k}^{\perp}.$$

Let us denote the members of this decomposition by $V_0, \ldots, V_k$, so that

$$V_j = \langle e_{i_j+1}, \ldots, e_{i_{j+1}} \rangle \quad \text{for } j = 0, \ldots, k,$$

where $i_0 = 0$ and $i_{k+1} = n+1$. The corresponding parabolic subgroup G_{F_0} is the full stabilizer of this decomposition; namely, it is a block-diagonal subgroup with blocks of size $m_0 = i_1$, $m_1 = i_2 - i_1, \ldots, m_{k-1} = i_k - i_{k-1}$, $m_k = n+1-i_k$. Because our group is $\mathrm{SU}(n+1, q^2)$, the determinant of the entire matrix must be 1, and thus

$$G_{F_0} \cong (\mathrm{GU}(m_0, q^2) \times \mathrm{GU}(m_1, q^2) \times \cdots \times \mathrm{GU}(m_{k-1}, q^2) \times \mathrm{GU}(m_k, q^2))^+.$$

The Borel subgroup in this case is simply the group D of diagonal matrices, and its order is $(q+1)^n$. The minimal parabolics arise when all but one of the blocks have size 1, and the remaining block has size 2. Thus, a minimal parabolic is a product of $\mathrm{SU}(2, q^2)$ and D. The parabolics of rank 2, on the other hand, come in two sorts. Either the generic matrix in the parabolic has two blocks of size 2 giving rise to a subgroup of the form $\mathrm{SU}(2, q^2) \times \mathrm{SU}(2, q^2)$ extended by D, or the generic matrix has one block of size 3, giving rise to a subgroup of the form $\mathrm{SU}(3, q^2)$, again extended by D. This shows that our geometry $\mathscr{N}$ leads to a configuration like that of Phan. The subgroups U_i and $U_{i,j}$ from the Phan system are normal subgroups in the parabolics of rank 1 and rank 2, and the corresponding parabolic is always the product of D with U_i or $U_{i,j}$.

For a more detailed discussion of amalgams and related methods see [7, Part II].

5 Phan amalgams

Our goal in this section is to translate the above into the language of amalgams and derive a more general class of amalgams, which we call Phan amalgams and which we shall classify.

As before, let $G = \mathrm{SU}(n+1, q^2)$, and let V be the natural unitary space for G with an orthonormal basis $\mathscr{E} = \{e_1, \ldots, e_{n+1}\}$. A decomposition $V = \bigoplus V_i$ is called *compatible* (with $\mathscr{E}$) if each V_i is spanned by a subset of $\mathscr{E}$ of the form $\{e_j, e_{j+1}, \ldots, e_k\}$ for some j with $1 \leqslant j \leqslant k \leqslant n+1$, exactly the decompositions we saw in the previous section. The compatible decompositions are indexed by subsets of $I = \{1, \ldots, n\}$. Indeed, a subset $J = \{i_1 < i_2 < \cdots < i_k\}$ of I defines the decomposition with the following $k+1$ summands:

$$V_0 = \langle e_1, \ldots, e_{i_1} \rangle, \quad V_1 = \langle e_{i_1+1}, \ldots, e_{i_2} \rangle, \ldots, \quad V_k = \langle e_{i_k+1}, \ldots, e_{n+1} \rangle.$$

We denote the decomposition corresponding to the subset J by $\mathscr{D}_J$. Write, as above, $F = \{U_1, U_2, \ldots, U_n\}$, where $U_i = \langle e_1, \ldots, e_i \rangle$. If $F_0 = \{U_{i_1}, U_{i_2}, \ldots, U_{i_k}\}$ is the subflag of F then the type of F_0 is exactly the set J. The relation between F_0 and the decomposition $\mathscr{D}_J$ can be described as follows. To find F_0 in terms of

$\mathscr{D}_J$ we set $U_{i_j} = \bigoplus_{s=0}^{j-1} V_s$. Conversely, $\mathscr{D}_J$ is obtained from F by taking $V_0 = U_{i_1}$, $V_1 = U_{i_2} \cap U_{i_1}^{\perp}, \ldots, V_{k-1} = U_{i_k} \cap U_{i_{k-1}}^{\perp}$ and $V_k = U_{i_k}^{\perp}$.

For a non-degenerate subspace U of V, let $\mathrm{SU}(U)$ denote the subgroup of G consisting of all elements stabilizing U and acting trivially on $U^{\perp}$. Clearly $\mathrm{SU}(U) \cong \mathrm{SU}(m, q^2)$, where $m = \dim U$. For $J \subseteq I$, let $L_J = \prod \mathrm{SU}(V_i)$ where $V = \bigoplus V_i$ is the decomposition $\mathscr{D}_J$. Notice that if F_0 is the subflag of F of type J, then the parabolic G_{F_0} is equal to $L_J D$, where $D = G_F$ is the diagonal group defined above. The *level* of a subgroup L_J is by definition $n - |J|$, which is the size of the complement $\bar{J}$ of J in I. Clearly the level of L_J coincides with the rank of the parabolic G_{F_0}.

Let S be a subset of the power set of $I = \{1, \ldots, n\}$ closed under supersets (that is, if $A \in S$ then every B with $A \subset B \subseteq I$ is also in S). The *standard Phan amalgam of shape* S is the amalgam $\hat{\mathscr{A}}_S = \bigcup_{J \in S} L_J$. In the particular case where $S = S_k$ consists of all subsets $J \subset I$ with $|\bar{J}| \leqslant k$, we call $\hat{\mathscr{A}}_S$ the *standard Phan amalgam of level k* (and rank n) and we denote it by $\hat{\mathscr{A}}_k = \hat{\mathscr{A}}(n, k, q)$. This is the amalgam formed by all subgroups L_J of level at most k. The shape S_k will be called the *straight level k shape*. If $S \supseteq S_k$ then we say that S is of level (at least) k.

By an arbitrary *Phan amalgam* of shape S we will understand an amalgam $\mathscr{A} = \bigcup_{J \in S} U_J$ where U_J is a group isomorphic to a quotient of L_J over a subgroup of the center of L_J. Furthermore, if $J \subset J'$, then we require that $U_{J'}$ be contained in U_J, namely, that $U_{J'}$ be the image of $L_{J'}$ under the natural homomorphism from L_J onto U_J.

To make our notation compatible with Phan's, we let $U_i = U_{\overline{\{i\}}}$ for $i \in I$, and similarly we let $U_{i,j} = U_{\overline{\{i,j\}}}$ for $i, j \in I$, $i < j$.

For example, when $S = S_2$, we have the following configuration: $\mathscr{A}$ contains subgroups U_i for $1 \leqslant i \leqslant n$ and $U_{i,j}$ for $1 \leqslant i < j \leqslant n$, so that

(1) for $i \in I$, $U_i \cong \mathrm{SU}(2, q^2)$;

(2) for $i, j \in I$, with $i < j$, we have
   (a) if $j - i > 1$ then $U_{i,j}$ is a central product of U_i and U_j;
   (b) if $j - i = 1$ then $U_{i,j} \cong \mathrm{SU}(3, q^2)$ or $\mathrm{PSU}(3, q^2)$; moreover U_i and U_j form a standard pair in $U_{i,j}$.

(Notice that in (1) the group U_i cannot be isomorphic to $\mathrm{PSU}(2, q^2)$ as can be seen in $U_{j,j+1}$ where $j = i$ or $i - 1$.)

If a group G contains a weak Phan system $U_1, \ldots, U_n$ then $\mathscr{A} = \bigcup_{(i,j) \in I} U_{i,j}$ is a Phan amalgam of level 2, where the groups $U_{i,i+1}$ are as in (wP2) and $U_{i,j} = U_i U_j$ if $j - i > 1$. This amalgam $\mathscr{A}$ does not collapse, because G is a quotient of its universal completion.

The converse is also true: a Phan amalgam that does not collapse leads to a group with a weak Phan system.

Lemma 5.1. *Suppose that $\mathscr{A}$ is a Phan amalgam with $q \neq 2$ and of shape S with $S_2 \subseteq S$. Suppose further that G is a non-trivial completion of $\mathscr{A}$ via a mapping π. Then $\pi(U_1), \ldots, \pi(U_n)$ form a weak Phan system in G.*

Proof. It suffices to see that π is injective on every U_i. Indeed, suppose that $u \in U_i^{\#}$ and $\pi(u) = 1$. Let $j = i - 1$ or $i + 1$. Since $q \neq 2$ we have $\langle U_i, U_j \rangle \cong \mathrm{SU}(3, q^2)$, which is quasisimple (indeed $\langle U_i, U_j \rangle$ is $U_{i,j}$ or $U_{j,i}$). Since $1 \neq u \in U_i$, we have $u \notin Z(\langle U_i, U_j \rangle)$. This implies that $\pi(\langle U_i, U_j \rangle) = 1$, that is, $\pi(U_i) = 1$ and $\pi(U_j) = 1$. Clearly this leads to a contradiction. Thus π is injective on every U_i.

The conclusion of this lemma is false when $q = 2$, since a standard pair in $\mathrm{SU}(3, 2^2)$ generates in this group a normal subgroup of index 4. (See Lemma 2.3, Corollary 2.4 and the comment between them.) Since we need the conclusion of this lemma in what follows, we modify our notion of a non-collapsing Phan amalgam. Namely, when $q = 2$, a Phan amalgam is called *non-collapsing* if there exists a completion into which the groups U_i map injectively under the corresponding mapping π.

6 Characteristic completions

Suppose that $\mathscr{A}$ is an amalgam. A completion G of $\mathscr{A}$ is called *characteristic* if and only if every automorphism of $\mathscr{A}$ extends to an automorphism of G. Notice that since G is generated by (the image of) $\mathscr{A}$, the extension is unique. Clearly the universal completion is always characteristic, but there may be other characteristic completions as well. In particular, in this section we prove the following result.

Proposition 6.1. *The group $G = \mathrm{SU}(n + 1, q^2)$, $n \geqslant 2$, is a characteristic completion of the standard Phan amalgam $\hat{\mathscr{A}}_S$ for any shape $S \supseteq S_2$.*

Later on we will show that, in most (but not all) cases, $\mathrm{SU}(n + 1, q^2)$ is the universal completion of $\hat{\mathscr{A}}_k$. To prove the above proposition we will need a lemma.

Lemma 6.2. *Let $G = \mathrm{SU}(3, q^2)$, and let U_1, U_2 be a standard pair in G. Then*

(1) *the joint stabilizer T in $\mathrm{Aut}\, G$ of U_1 and U_2 is an extension of a group of order $(q + 1)^2$, which consists of diagonal automorphisms, by the field automorphisms;*

(2) *the centralizer in T of U_1 is of order $q + 1$.*

Proof. Every automorphism of G comes from a semi-linear transformation of the natural module. Since $\langle e_3 \rangle$ is the only 1-dimensional subspace left invariant by U_1 and $\langle e_1 \rangle$ is the only 1-dimensional subspace left invariant by U_2, both $\langle e_1 \rangle$ and $\langle e_3 \rangle$ (and hence also $\langle e_2 \rangle$) are stabilized by T. Since any automorphism $t \in T$ is then the product of a diagonal matrix (over the basis $\{e_1, e_2, e_3\}$) with a field automorphism ϕ acting on the coordinates (under the same basis), taking a diagonal matrix $M = \mathrm{diag}(a, a^{-1}, 1)$ in U_1, we see that for M to commute with t requires $a^{\phi} = a$. Since a is arbitrary, ϕ is trivial, and (1) holds.

It is now clear that the centralizer in T of U_1 consists of all the diagonal matrices $\mathrm{diag}(b, b, b^{-2})$ and (2) holds.

We now prove Proposition 6.1.

Proof of Proposition 6.1. We begin with the case of straight level 2, the case $\hat{\mathscr{A}}_2$. We claim that the group D of automorphisms of $\hat{\mathscr{A}}_2$ is of order $(q+1)^n \cdot f$ where $q^2 = p^f$, p a prime. Clearly D has at least this order. Indeed the diagonal and field automorphisms of G induce a group of this order on $\hat{\mathscr{A}}_2$. Thus we only need to show that the order of D is at most $(q+1)^n \cdot f$. We argue by induction on n. We start the induction with $n = 2$, in which case D is the group of automorphisms of $\mathrm{SU}(3, q^2)$ stabilizing U_1 and U_2. Hence the claim follows from Lemma 6.2 (1) in this case. Suppose that the claim holds for $n = k$, and let $n = k+1$. Let $\hat{\mathscr{B}}$ be the amalgam of all members of $\hat{\mathscr{A}}_2$ which are contained in the upper left $n \times n$ block of G. Then $\hat{\mathscr{B}}$ is a similar standard Phan amalgam of straight level 2 for $n = k$, and in particular, the claim will follow if we prove that $C = C_D(\hat{\mathscr{B}})$ has order at most $q+1$. Let $U = L_{n-1,n} \cong \mathrm{SU}(3, q^2)$ be the member of $\hat{\mathscr{A}}_2$ containing U_{n-1} and U_n. By Lemma 6.2 (2), C induces on U a group of order at most $q+1$ since C acts trivially on U_{n-1}. The other members of $\hat{\mathscr{A}}_2$ that are not in $\hat{\mathscr{B}}$ are of the form $U_i \times U_n$, for $i < n-1$. Clearly every element of C that acts trivially on U_n acts trivially on every such group.

Thus D has order as claimed, and as already mentioned, every element of D is induced by an automorphism of G.

Now we consider an arbitrary shape $S \supset S_2$. Observe that $\hat{\mathscr{A}}_2$ is a subamalgam of $\hat{\mathscr{A}}_S$ (that is, every member of $\hat{\mathscr{A}}_2$ is also a member of $\hat{\mathscr{A}}_S$) and hence every automorphism ϕ of $\hat{\mathscr{A}}_S$ induces an automorphism ψ of $\hat{\mathscr{A}}_2$. By the above, ψ extends uniquely to an automorphism π of G. Let $H = L_J$ be one of the members of $\hat{\mathscr{A}}_S$. We claim that $\pi|_H = \phi|_H$ so that π is an extension of ϕ. Indeed $H = \prod \mathrm{SU}(V_i)$, where $V = \bigoplus V_i$ is the compatible decomposition $\mathscr{D}_J$ defined by J. It suffices to show that π and ϕ agree on every $\mathrm{SU}(V_i)$. If $\dim(V_i) \leqslant 3$ then $\mathrm{SU}(V_i)$ is a member of $\hat{\mathscr{A}}_2$ and there is nothing to show. Suppose $\dim(V_i) > 3$. By the above $K = \mathrm{SU}(V_i)$ is a characteristic completion of the amalgam formed by all members of $\hat{\mathscr{A}}_2$ contained in K. Thus there is a unique extension of $\phi|_K$ to K. This shows that π and ψ must agree on K, as both are extensions of ϕ.

Corollary 6.3. *For $J \subset I = \{1, \ldots, n\}$ with $|\bar{J}| \geqslant 3$, the group L_J is a characteristic completion of the amalgam $\bigcup_{J' \supset J} L_{J'}$.*

Proof. Let $V = \bigoplus V_i$ be the compatible decomposition $\mathscr{D}_J$ defined by J. If only one summand V_i has dimension greater than 1, then $L_J = \mathrm{SU}(V_i)$, and the claim follows immediately from Proposition 6.1. Now suppose that at least two factors of L_J are non-trivial. Then every $H_i = \mathrm{SU}(V_i)$ is a member of the amalgam $\mathscr{B} = \bigcup_{J' \supset J} L_{J'}$. So if ϕ is an automorphism of $\mathscr{B}$, then its action on each H_i is known, and this defines a unique automorphism ψ of $L_J = \prod H_i$. Since every $L_{J'}$ is the direct product of its intersections with the factors H_i, it is clear that ψ extends ϕ.

We will use the notion of a characteristic completion via the following technical lemma.

Lemma 6.4. *For $i = 1, 2$, let $\mathscr{A}_i$ be an amalgam and let G_i be a completion of $\mathscr{A}_i$ via the mapping π_i. Suppose that there exist isomorphisms $\psi : \mathscr{A}_1 \to \mathscr{A}_2$ and $\phi : G_1 \to G_2$ such*

that $\phi\pi_1 = \pi_2\psi$. If G_1 is a characteristic completion of $\mathscr{A}_1$, then for any isomorphism $\psi' : \mathscr{A}_1 \to \mathscr{A}_2$ there exists a unique isomorphism $\phi' : G_1 \to G_2$ such that $\phi'\pi_1 = \pi_2\psi'$.

Proof. Consider $\alpha = (\psi')^{-1}\psi$. This is an automorphism of $\mathscr{A}_1$. Since G_1 is a characteristic completion, α extends to an automorphism of G_1; that is, there is an automorphism β of G_1 such that $\pi_1\alpha = \beta\pi_1$. A simple check shows that $\phi' = \phi\beta^{-1}$ satisfies the requirement of the lemma.

7 Unambiguous Phan amalgams

The definition of a Phan amalgam leaves some ambiguity as to what is the exact structure of each U_J. For example, in the straight level 2 case, when $j - i > 1$, either U_i and U_j have trivial intersection, or they have a common central involution. Similarly, when $j - i = 1$, $U_{i,j}$ may be either $\mathrm{SU}(3, q^2)$ or $\mathrm{PSU}(3, q^2)$. Finally, the intersections of the members of the amalgam might be larger than expected. We call a Phan amalgam *unambiguous* if (1) every U_J is isomorphic to the corresponding L_J (cf. Section 5); and (2) $U_J \cap U_{J'} = U_{J\cup J'}$ for all J and J'.

By a *covering* of a Phan amalgam $\mathscr{A} = \bigcup_{J\in S} U_J$ of shape S we mean a second Phan amalgam $\tilde{\mathscr{A}} = \bigcup_{J\in S} \tilde{U}_J$ of the same shape S, together with an amalgam homomorphism $\pi : \tilde{\mathscr{A}} \to \mathscr{A}$, such that π induces a surjective homomorphism of $\tilde{U}_J$ onto U_J for every $J \in S$. We call two coverings $(\mathscr{A}_1, \pi_1)$ and $(\mathscr{A}_2, \pi_2)$ of $\mathscr{A}$ *equivalent* if there is an isomorphism ϕ of $\mathscr{A}_1$ onto $\mathscr{A}_2$ such that $\pi_1 = \pi_2\phi$.

Proposition 7.1. *Every Phan amalgam $\mathscr{A}$ has a unique* (*up to the above equivalence*) *unambiguous covering $\tilde{\mathscr{A}}$.*

Proof. We proceed by induction on $|S|$, where S is the shape of $\mathscr{A} = \bigcup_{J\in S} U_J$. Our basis is the case $S = \varnothing$, which corresponds to an empty amalgam $\mathscr{A}$. Vacuously, this amalgam is unambiguous. Suppose now that S is a non-empty shape, and that for every shape $S' \subset S$ the claim holds. Let J be a minimal (under inclusion) element of S and set $S' = S\backslash\{J\}$ and $\mathscr{A}' = \bigcup_{J'\in S'} U_{J'}$. Then S' is a shape, and $\mathscr{A}'$ is a Phan subamalgam in $\mathscr{A}$ of shape S'.

By the inductive assumption, there is a (unique) unambiguous covering Phan amalgam $(\tilde{\mathscr{A}}' = \bigcup_{J'\in S'} U'_{J'}, \pi')$ of $\mathscr{A}'$. We will find an unambiguous covering $(\tilde{\mathscr{A}}, \pi)$ of $\mathscr{A}$ by gluing a copy of L_J to $\tilde{\mathscr{A}}'$ and by extending π' to the new member of the amalgam. To glue L_J to the amalgam $\tilde{\mathscr{A}}'$, we need to construct an isomorphism from the subamalgam $\mathscr{B} = \bigcup_{J'\supset J} U'_{J'}$ of $\tilde{\mathscr{A}}'$ onto the corresponding amalgam $\mathscr{C} = \bigcup_{J'\supset J} L_{J'}$ of subgroups of L_J. By the definition of a Phan amalgam, there is a homomorphism ψ from L_J onto U_J mapping $\mathscr{C}$ onto $\mathscr{D} = \bigcup_{J'\supset J} U_{J'}$. Note that $\mathscr{D}$ is a Phan amalgam of shape $\{J' \mid J' \supset J\}$. Note further that $(\mathscr{B}, \pi'|_{\mathscr{B}})$ and $(\mathscr{C}, \psi|_{\mathscr{C}})$ are two unambiguous coverings of $\mathscr{D}$. By induction, the uniqueness of the unambiguous covering holds, so that there is an amalgam isomorphism ϕ from $\mathscr{B}$ onto $\mathscr{C}$ such that $\psi\phi = \pi'|_{\mathscr{B}}$. Clearly ϕ tells us how to glue L_J to $\tilde{\mathscr{A}}'$ to produce $\tilde{\mathscr{A}}$, and furthermore, as π we can take the union of ψ and π'. The condition $\psi\phi = \pi'|_{\mathscr{B}}$ guarantees that ψ and π' agree on the intersection $\mathscr{B} = \mathscr{C}$ (identified via ϕ). Finally, notice that $\tilde{\mathscr{A}}$ is

an unambiguous Phan amalgam of type S, so that $(\tilde{\mathscr{A}}, \pi)$ is an unambiguous covering of $\mathscr{A}$.

This completes the proof of the existence of an unambiguous covering $\tilde{\mathscr{A}}$. Now we prove the uniqueness. Suppose that we have two such coverings $\tilde{\mathscr{B}} = \bigcup_{J \in S} B_J$ and $\tilde{\mathscr{C}} = \bigcup_{J \in S} C_J$ with corresponding amalgam homomorphism π_1 and π_2 onto $\mathscr{A}$. Select J as in the previous paragraph, and define $S' = S \backslash \{J\}$. Let $\mathscr{A}'$, $\tilde{\mathscr{B}}'$ and $\tilde{\mathscr{C}}'$ be the subamalgams of shape S' in $\mathscr{A}$, $\tilde{\mathscr{B}}$ and $\tilde{\mathscr{C}}$, respectively. By induction, there exists an isomorphism ϕ from $\tilde{\mathscr{B}}'$ onto $\tilde{\mathscr{C}}'$ such that $\pi_1|_{\tilde{\mathscr{B}}'} = \pi_2\phi$. It suffices to extend ϕ to B_J.

We have two cases. First assume that the decomposition $\mathscr{D}_J$ has more than one summand of dimension greater than 1. In this case, $B_J \cong L_J$ is isomorphic to a direct product of $L_{J'}$ and $L_{J''}$ for suitable supersets J' and J'' of J. Clearly ϕ is already known on $B_{J'}$ and $B_{J''}$, and so ϕ extends uniquely to B_J. Since every member B_K with $K \supset J$ is a direct product of its intersections with $B_{J'}$ and $B_{J''}$, this extension, which we also denote ϕ, will be a well-defined amalgam isomorphism from $\mathscr{B}$ to $\mathscr{C}$, and furthermore $\pi_2 = \pi_1\phi$.

In the second case, $\mathscr{D}_J$ has a unique summand of dimension $m \geqslant 2$. In this case $B_J \cong C_J \cong L_J \cong \mathrm{SU}(m, q^2)$. Choose an arbitrary isomorphism $\psi : B_J \to C_J$, and consider the mapping $\alpha : U_J \to U_J$ defined by $\alpha(u) = \pi_2\phi\pi_1^{-1}(u)$. Notice that α is well-defined automorphism of U_J, because the fibers of π_1 are cosets of the kernel of π_1, and ϕ takes them to cosets of the kernel of π_2 (since ϕ takes the kernel of π_1 to the kernel of π_2, these being subgroups of equal order in the cyclic centers of B_J and C_J, respectively). Notice that every automorphism of U_J lifts to a unique automorphism of $C_J \cong \mathrm{SU}(m, q^2)$. (Indeed, with finitely many exceptions C_J is the largest perfect central extension of U_J, implying the claim in those cases; the exceptional cases can be verified with a case-by-case analysis.) Thus there is an automorphism β of C_J such that $\pi_2\beta = \alpha\pi_2|_{C_J}$. Define θ on B_J as $\beta^{-1}\psi$. First, by definition we have $\pi_1|_{B_J} = \pi_2\theta$. Next, for every $J' \supset J$ we have that $\theta^{-1}\phi_{B_{J'}}$ is a lifting to $B_{J'}$ of the identity automorphism of $U_{J'}$, and hence it is the identity. This shows that ϕ and θ agree on every subgroup $B_{J'}$, which allows us to extend ϕ to the entire $\tilde{\mathscr{B}}$ by defining it on B_J as θ.

Since $\mathscr{A}$ is a quotient of $\tilde{\mathscr{A}}$ and $\mathscr{A}$ does not collapse, neither does its unambiguous covering $\tilde{\mathscr{A}}$. We can now state our uniqueness result for the amalgams arising from the weak Phan systems.

Theorem 7.2. *If $\mathscr{A}$ is a non-collapsing Phan amalgam of shape $S \supseteq S_2$ then its unique unambiguous covering $\tilde{\mathscr{A}}$ is isomorphic to the standard Phan amalgam $\hat{\mathscr{A}}_S$.*

In Sections 8–10 we will prove this theorem by establishing the uniqueness of an unambiguous non-collapsing Phan amalgam for each choice of n, q and $S \supseteq S_2$.

8 Goldschmidt's lemma

Let G_1 and G_2 be two groups and S_1 and S_2 be subgroups in G_1 and G_2 respectively so that $S_1 \cong S_2$. If ψ is an isomorphism from S_1 to S_2, then we can construct an amalgam $\mathscr{A} = G_1 \cup G_2$ by identifying $x \in S_1$ with $\psi(x) \in S_2$. The natural question is

this: given G_1, G_2, S_1 and S_2, how many non-isomorphic amalgams can be constructed in this way when we take all possible ψ?

Let us fix one isomorphism ψ. Then any other isomorphism can be obtained by composing ψ with an automorphism of S_1. Let A_1 be the group of those automorphisms of S_1 which are induced by an automorphism of G_1 normalizing S_1. Similarly, let A_2 be the group of automorphisms of S_1 obtained as follows. We take all the automorphisms of S_1 of the form $\psi^{-1}\phi\psi$, where ϕ is an automorphism of S_2 induced by an automorphism of G_2 normalizing S_2. The following is a lemma from [5]:

Lemma 8.1. *The number of non-isomorphic amalgams $\mathscr{A}$ coincides with the number of double cosets of A_1 and A_2 in* $\mathrm{Aut}(S_1)$.

We apply this lemma in the case where $G_1 \cong G_2 \cong \mathrm{SU}(n, q^2)$, $n \geqslant 3$, and $S_1 \cong S_2 \cong \mathrm{SU}(n-1, q^2)$. In this case, S_i is embedded in G_i as the stabilizer of a non-singular vector. Every automorphism of S_1 is a product of inner, diagonal and field automorphisms. All of these are induced by automorphisms of G_1. Thus $A_1 = \mathrm{Aut}(S_1)$ and so from the above lemma the amalgam is unique. Hence Lemma 8.1 has the following corollary:

Corollary 8.2. *For $n \geqslant 3$, up to isomorphism there exists a unique amalgam $\mathscr{A} = G_1 \cup G_2$, where $G_1 \cong G_2 \cong \mathrm{SU}(n, q^2)$, $S = G_1 \cap G_2$ is isomorphic to* $\mathrm{SU}(n-1, q^2)$, *and S is embedded in both groups G_i as the stabilizer of a non-singular vector.*

9 The case $n = 3$

In this section and the next, we prove Theorem 7.2 for the straight level 2 case, $S = S_2$. The general case is then a simple extension using Corollary 6.3. Let $\mathscr{A} = \bigcup_{(i,j)\in I} U_{i,j}$ be an unambiguous Phan amalgam of shape S_2 that does not collapse. We will establish the uniqueness of $\mathscr{A}$ up to isomorphism in a series of lemmas. When $n = 2$, the amalgam is unique by definition. In this section we shall deal with the case $n = 3$.

Let $n = 3$. For $i \in \{1, 2, 3\}$, let $L_i = U_{j,k}$, where $\{i, j, k\} = \{1, 2, 3\}$ and $j < k$. Since $\mathscr{A}$ is unambiguous, each subgroup U_i coincides with $L_j \cap L_k$.

Define $D_1 = N_{U_1}(U_2)$ (this makes sense since $U_1, U_2 \leqslant U_{1,2}$; the same applies to the subsequent definitions) and $D_3 = N_{U_3}(U_2)$. Since U_1 and U_2 form a standard pair in L_3, it follows that D_1 has order $q+1$, and it is a maximal torus in U_1. Symmetrically, D_3 is a maximal torus of order $q+1$ in U_3. We also define $D_2^1 = N_{U_2}(U_1)$ and $D_2^3 = N_{U_2}(U_3)$. Again these are maximal tori of size $q+1$ in U_2. The following lemma gives us an extra condition on $\mathscr{A}$ that holds because $\mathscr{A}$ does not collapse.

Lemma 9.1. $D_2^1 = D_2^3$.

Proof. Let G be a completion of $\mathscr{A}$ and let π be the corresponding mapping from $\mathscr{A}$ to G. Since $\mathscr{A}$ is non-collapsing, we may assume that π is injective on every U_i (see Lemma 5.1 and the subsequent comment). Observe that $D_2^i = C_{U_2}(D_i)$ (viewed as

subgroups of L_{4-i}), for $i = 1, 3$. Thus $\pi(D_2^i) = C_{\pi(U_2)}(\pi(D_i))$. Since D_1 and D_3 commute elementwise in L_2, we have that $\pi(D_1)$ and $\pi(D_3)$ commute elementwise, too. Hence $\pi(D_2^1) = C_{\pi(U_2)}(\pi(D_1))$ must be invariant under $\pi(D_3)$. Since U_2 is invariant under D_3 (in L_1) and since π is injective on U_2, it follows that D_2^1 is invariant under D_3 (again as subgroups of L_1).

Notice that D_2^1 and D_3 are both cyclic of order $q+1$. Since $\mathrm{Aut}(Z_{q+1})$ has order $\phi(q+1) < q+1$ we have that D_3 contains a non-trivial element d acting trivially on D_2^1. Now it is easy to check in $L_1 \cong \mathrm{SU}(3, q^2)$ that the only elements commuting with d in U_2 are those contained in D_2^3. Hence $D_2^1 = D_2^3$.

In view of this lemma we can use the notation $D_2 = D_2^1 = D_2^3$.

Recall that our goal is the uniqueness of $\mathscr{A}$. Now suppose that there is a second amalgam $\mathscr{A}' = L_1' \cup L_2' \cup L_3'$. By Corollary 8.2, the amalgams $\mathscr{B} = L_1 \cup L_3$ and $\mathscr{B}' = L_1' \cup L_3'$ are isomorphic via an amalgam isomorphism ψ. Clearly $\psi(U_2) = \psi(L_1 \cap L_3) = L_1' \cap L_3' = U_2'$. We claim that without loss of generality we may also assume that $\psi(U_1) = U_1'$. Indeed, by definition, U_1' and U_2' form a standard pair in L_3' (as U_1 and U_2 do in L_3). This means that U_2' is the stabilizer of a non-singular vector v in the natural module for $L_3' \cong \mathrm{SU}(3, q^2)$. Moreover U_1' is the stabilizer of a non-singular vector u lying in $v^\perp$. Since U_2' is transitive on non-singular 1-spaces in $v^\perp$, there must be an element $g \in U_2'$ such that g conjugates U_1' to $\psi(U_1)$. Clearly conjugation by g is an automorphism ϕ of the amalgam $\mathscr{B}'$. Thus, by substituting (if necessary) ψ with $\phi \circ \psi$, we can indeed assume that $\psi(U_1) = U_1'$.

Our next goal is to show that, again up to an automorphism of $\mathscr{B}'$, we can assume that $\psi(U_3) = U_3'$. Let V be the natural 3-dimensional unitary space for L_1'. Since $\psi(U_1) = U_1'$ and $\psi(U_2) = U_2'$, we have $\psi(D_2) = D_2' = N_{U_2'}(U_1') = N_{U_2'}(U_3')$. Here we use the fact that $\mathscr{A}$ and $\mathscr{A}'$ do not collapse (see Lemma 9.1). Observe that $D' = D_2'$ is cyclic and let d be a generator of D'. Since $D' \leqslant U_2'$, it fixes a non-singular vector $u \in V$. Also D' normalizes U_3', and hence D' stabilizes the 1-space $\langle v \rangle$, where $v \in V$ is a vector fixed by U_3'. Notice that u and v are perpendicular, since U_2' and U_3' form a standard pair in L_1'. Let $\langle w \rangle = \langle u, v \rangle^\perp$. Clearly $\langle w \rangle$ is also stabilized by D'. Hence d is diagonal with respect to the orthogonal basis $\{u, v, w\}$, with eigenvalues 1, a, a^{-1}. Since d has order $q+1 \neq 2$ we have $a \neq a^{-1}$, and hence $\langle u \rangle$, $\langle v \rangle$ and $\langle w \rangle$ are the only 1-spaces stabilized by D'. Let U_3'' be the stabilizer in L_1' of w.

Lemma 9.2. *$\psi(U_3) = U_3'$ or $\psi(U_3) = U_3''$.*

Proof. Since U_2' and $\psi(U_3)$ form a standard pair in L_1' and since

$$D' = \psi(D_2) = N_{U_2'}(\psi(U_3)),$$

we conclude that D' must stabilize the 1-space fixed by $\psi(U_3)$. Clearly this 1-space must be either $\langle v \rangle$ or $\langle w \rangle$.

Without loss of generality, $\{u, v, w\}$ is an orthonormal basis. Let $g \in U_2'$ have matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

with respect to the basis $\{u, v, w\}$. Conjugation by g induces on U_2' the same action as the contragredient automorphism (or the field automorphism of order 2, since these automorphisms coincide on the unitary group). Define an automorphism π of the amalgam $\mathscr{B}'$ as follows: on L_1', π is the composition of the conjugation by g and the field involution and on L_3, π is the trivial automorphism. This is well defined since the action of π on $L_1' \cap L_3' = U_2'$ is trivial in both cases. Clearly π stabilizes U_1' and U_2' and interchanges U_3' and U_3''. Thus, taking if necessary $\pi \circ \psi$ in place of ψ, we may assume that $\psi(U_3) = U_3'$.

Proposition 9.3. *If $n = 3$ then the amalgam $\mathscr{A}$ is unique up to isomorphism.*

Proof. We need to show that $\mathscr{A}$ and $\mathscr{A}'$ are isomorphic. By the above we have an isomorphism $\psi : \mathscr{B} \to \mathscr{B}'$ which takes every U_i to U_i'. Since $L_2 = U_1 \times U_3$ and $L_2' = U_1' \times U_3'$, ψ clearly extends to an isomorphism from $\mathscr{A}$ to $\mathscr{A}'$.

10 The case $n > 3$

In this section we complete the proof of the uniqueness of a non-collapsing unambiguous Phan amalgam $\mathscr{A}$ of shape S_2. We prove this by induction, the case $n = 3$ being the basis of induction. Thus we assume that the claim holds for $n = k \geqslant 3$. Assume that $n = k + 1$. Let $\mathscr{A}$ be a non-collapsing unambiguous Phan amalgam of shape S_2. Our first step is to extend the amalgam $\mathscr{A}$ by adding to it two new members, both isomorphic to $\mathrm{SU}(n, q^2)$.

Lemma 10.1. *There exists a unique amalgam $\mathscr{B} = \mathscr{A} \cup H_1 \cup H_2$, where $H_i \cong \mathrm{SU}(n, q^2)$, H_1 contains and the subgroups $U_{i,j}$, $1 \leqslant i < j \leqslant n-1$ and is generated by them, and similarly H_2 contains the subgroups $U_{i,j}$, $2 \leqslant i < j \leqslant n$ and is generated by them.*

Proof. Let $\mathscr{B}_1 = \bigcup_{1 \leqslant i < j \leqslant n-1} U_{i,j}$, $\mathscr{B}_2 = \bigcup_{2 \leqslant i < j \leqslant n} U_{i,j}$, and $\mathscr{C} = \mathscr{B}_1 \cap \mathscr{B}_2$. By the inductive assumption, $\mathscr{B}_i$ is isomorphic to the amalgam found in $\mathrm{SU}(n, q^2)$. Furthermore, by Proposition 6.1, $\mathrm{SU}(n, q^2)$ is a characteristic completion of that amalgam. Thus there exists an injective amalgam homomorphism $\pi_i : \mathscr{B}_i \to H_i$, where $H_i \cong \mathrm{SU}(n, q^2)$. We glue H_i to the amalgam $\mathscr{A}$ via π_i. Notice that π_i sends $\mathscr{C}$ into the subgroup K_i of H_i that is isomorphic to $\mathrm{SU}(n-1, q^2)$. Since the copies of $\mathscr{C}$ in K_1 and K_2 are standard Phan amalgams, we have an isomorphism $\phi : K_1 \to K_2$ that takes $\pi_1(\mathscr{C})$ to $\pi_2(\mathscr{C})$. Let ψ be the restriction of ϕ to $\mathscr{C}$. Consider $\mathscr{A}_1 = \pi_1(\mathscr{C})$ and $\mathscr{A}_2 = \pi_2(\mathscr{C})$ together with their embeddings into K_1 and K_2. Applying Lemma 6.4 with ϕ and ψ as above and $\psi' = \pi_2|_{\mathscr{C}}(\pi_1|_{\mathscr{C}})^{-1}$, there exists an isomorphism

$\phi' : K_1 \to K_2$ such that $\phi'|_{\mathscr{A}_1} = \psi'$. Thus $\phi'\pi_1 = \pi_2$. Identifying K_1 with K_2 via ϕ' we obtain our unique amalgam $\mathscr{B}$.

We now start proving the uniqueness of the amalgam $\mathscr{A}$. Suppose that we have two Phan amalgams $\mathscr{A}$ and $\mathscr{A}'$ (where we will use the prime notation for all groups in $\mathscr{A}'$). Extend $\mathscr{A}$ and $\mathscr{A}'$ to amalgams $\mathscr{B} = \mathscr{A} \cup H_1 \cup H_2$ and $\mathscr{B}' = \mathscr{A}' \cup H_1' \cup H_2'$ as in Lemma 10.1. Corollary 8.2 gives us an isomorphism ϕ from $H_1 \cup H_2$ onto $H_1' \cup H_2'$. By the inductive assumption, the subgroups $U_{i,j}$ with $1 < i < j < n$ form a standard Phan amalgam in $H_1 \cap H_2$. Similarly the subgroups $U_{i,j}'$ with $1 < i < j < n$ form a standard Phan amalgam in $H_1' \cap H_2'$. Thus $\bigcup_{1<i<j<n} U_{i,j}'$ and $\bigcup_{1<i<j<n} \phi(U_{i,j})$ are standard Phan amalgams in $H_1' \cap H_2'$. These amalgams correspond to two choices of an orthonormal basis in the natural unitary space for $H_1' \cap H_2'$. So adjusting ϕ, if necessary, by an inner automorphism of $H_1' \cap H_2'$, we may assume that $\phi(U_{i,j}) = U_{i,j}'$ for $1 < i < j < n$. Notice that U_1 is the centralizer in H_1 of $\langle U_3, \ldots, U_{n-1} \rangle$ (and the same is true for U_1' in H_1'). Therefore $\phi(U_1) = C_{H_1'}(\langle U_3', \ldots, U_{n-1}' \rangle) = U_1'$. Similarly $\phi(U_n) = U_n'$. We claim that ϕ extends to an isomorphism from $\mathscr{A}$ to $\mathscr{A}'$. Indeed, ϕ is already defined on all $U_{i,j}$ with $1 < i < j < n$. Moreover inside H_1' we see that $\phi(U_{1,i})$ is $U_{1,i}'$ for $i < n$ (if $q > 2$ this is immediate since $U_{1,i} = \langle U_1, U_i \rangle$, and if $q = 2$, the claim still holds, because ϕ is induced by a semi-linear transformation between the corresponding unitary spaces). Similarly in H_2' we see that $\phi(U_{j,n}) = U_{j,n}'$. It remains to note that $U_{1,n}$ is the direct product of U_1 and U_n so that ϕ extends to an isomorphism of $\mathscr{A}$ to $\mathscr{A}'$. Thus we have shown

Proposition 10.2. *If $n > 3$ then the amalgam $\mathscr{A}$ is unique up to isomorphism.*

11 Arbitrary shape

Having completed the proof of Theorem 7.2 for the case $S = S_2$, we are now ready to attack the general case.

Proposition 11.1. *If $S \supseteq S_2$, then $\hat{\mathscr{A}}_S$ is the unique* (*up to isomorphism*) *unambiguous non-collapsing Phan amalgam of shape S.*

Proof. We just prove the uniqueness of the amalgam. Suppose that $\mathscr{A} = \bigcup_{J \in S} U_J$ and $\mathscr{A}' = \bigcup_{J \in S} U_J'$ are non-collapsing unambiguous Phan amalgams of shape S. If $S = S_2$, then the claim follows from Propositions 9.3 and 10.2. Otherwise, let $J \in S$ be such that $|\bar{J}| > 3$ and J is minimal under inclusion. Let $T = S \backslash \{J\}$, $\mathscr{B} = \bigcup_{J' \in T} U_{J'}$ and $\mathscr{B}' = \bigcup_{J' \in T} U_{J'}'$ be the subamalgams of shape T of $\mathscr{A}$ and $\mathscr{A}'$ respectively. We can assume by induction that $\mathscr{B}$ and $\mathscr{B}'$ are isomorphic, so that there is an isomorphism $\theta : \mathscr{B} \to \mathscr{B}'$. Thus we need to extend θ to the missing member U_J. Let $\mathscr{C} = \mathscr{B} \cap U_J$ and $\mathscr{C}' = \mathscr{B}' \cap U_J'$. Observe that θ establishes an isomorphism between $\mathscr{C}$ and $\mathscr{C}'$. Notice that U_J and U_J' are isomorphic, because $\mathscr{A}$ and $\mathscr{A}'$ are unambiguous and hence both U_J and U_J' are isomorphic to L_J. Furthermore, from the definition of a Phan amalgam, an isomorphism $\phi : U_J \to U_J'$ can be chosen so that ϕ takes $\mathscr{C}$ to $\mathscr{C}'$. Applying Lemma 6.4 to amalgams $\mathscr{C}$ and $\mathscr{C}'$, completions U_J

and U_J' (for which π_1 and π_2 are just the inclusion mappings), and isomorphisms ϕ and $\psi = \phi|_{\mathscr{C}}$, we obtain that, for $\psi' = \theta|_{\mathscr{C}}$, there exists an isomorphism $\phi' : U_J \to U_J'$ such that $\psi' = \phi'|_{\mathscr{C}}$. Here we also use Corollary 6.3, which gives us that $U_J \cong L_J$ is a characteristic completion of $\mathscr{C}$.

Since $\theta|_{\mathscr{C}} = \psi' = \phi'|_{\mathscr{C}}$, the union of θ and ψ' is an amalgam isomorphism between $\mathscr{A}$ and $\mathscr{A}'$.

This completes the proof of Theorem 7.2

12 The universal completion of $\hat{\mathscr{A}}_S$

In this section we complete the proofs of Theorems 1.2 and 1.3. *Throughout this section* $n \geqslant 3$ *and* $(n,q) \neq (3,2),(3,3)$. Let $S(q)$ be defined as follows. If $q > 3$ let $S(q) = S_2$, the set of all non-empty subsets $J \subseteq \{1,\dots,n\}$ with $|\bar{J}| \leqslant 2$. Let $S(3)$ be obtained from S_2 by adding all subsets $\overline{\{i, i+1, i+2\}}$ with $i \in \{1,\dots,n-2\}$. Finally, let $S(2)$ be obtained by adding to $S(3)$ all subsets $\overline{\{i\} \cup \{j, j+1\}}$, where $i \in \{1,\dots,n\}$, $j \in \{1,\dots,n-1\}$ and $i \notin \{j-1, j, j+1, j+2\}$, and all subsets

$$\overline{\{i, i+1\} \cup \{j, j+1\}}, \quad \text{where } i \in \{1,\dots,n-1\},\ j \in \{1,\dots,n-1\}$$

and $i \notin \{j-2, j-1, j, j+1, j+2\}$. In all cases $S(q)$ is closed under supersets, so that $S(q)$ is a shape as defined in Section 5.

Now the assumptions of Theorems 1.2 and 1.3 amount to the following: G contains a Phan amalgam $\mathscr{A} = \bigcup_{J \in S(q)} U_J$ of shape $S(q)$ and, furthermore, G is generated by $\mathscr{A}$, that is, G is a completion of $\mathscr{A}$. By Proposition 7.1, $\mathscr{A}$ admits an unambiguous covering $\tilde{\mathscr{A}}$. Clearly G is also a completion of $\tilde{\mathscr{A}}$, and thus $\tilde{\mathscr{A}}$ is non-collapsing. By Theorem 7.2, $\tilde{\mathscr{A}}$ is isomorphic to the standard Phan amalgam $\hat{\mathscr{A}}_{S(q)}$. Thus Theorems 1.2 and 1.3 follow from the following proposition:

Proposition 12.1. *For every shape* $S \supseteq S(q)$, *the universal completion of* $\hat{\mathscr{A}}_S$ *is isomorphic to* $\mathrm{SU}(n+1, q^2)$.

In proving Proposition 12.1, our principal tool will be the following observation due to Tits [11].

Proposition 12.2. *Let* G *be a group acting flag-transitively on a geometry* Γ, *and let* F *be a maximal flag of* Γ. *For* $F_0 \subset F$ *let* G_{F_0} *be the elementwise stabilizer of* F_0 *in* G. *Then* G *is the universal completion of the amalgam* $\bigcup_{\varnothing \neq F_0 \subset F} G_{F_0}$ *if and only if* Γ *is simply connected.*

We will apply this with Γ equal to various residues of $\mathscr{N} = \mathscr{N}(n,q)$, the geometry defined in Section 2. We select a maximal flag $F = \{E_1,\dots,E_n\}$ of $\mathscr{N}$ as in Section 4. That is, for a fixed orthonormal basis $\mathscr{E} = \{e_1,\dots,e_{n+1}\}$, we take $E_i = \langle e_1,\dots,e_i\rangle$.

Proposition 12.2 and Proposition 3.1 yield that $G = \mathrm{SU}(n+1, q^2)$ is the universal completion of the amalgam $\bigcup_{\varnothing \neq F_0 \subset F} G_{F_0}$. Recall that G_{F_0} denotes the stabilizer in G of the flag F_0, and we call such subgroups parabolics. Notice that each subflag F_0 corresponds to a compatible (see Section 5) decomposition $\mathscr{D}_J$ for a subset $J \subseteq \{1, \ldots, n\}$. Here J is the type of the flag F_0. We will also use the notation $F(J)$ for F_0 when we wish to stress the type of F_0.

Lemma 12.3. *Suppose that S is a shape such that $S \supseteq S(q)$. Then the universal completion of the amalgam $\hat{\mathscr{B}}_S = \bigcup_{J \in S} G_{F(J)}$ is isomorphic to* $\mathrm{SU}(n+1, q^2)$.

Proof. Let $\hat{G}$ be the universal completion of $\hat{\mathscr{B}}_S$. Since $\mathrm{SU}(n+1, q^2)$ is a completion of $\hat{\mathscr{B}}_S$ it follows that $\hat{\mathscr{B}}_S$ embeds into $\hat{G}$. It suffices to show that $\hat{\mathscr{B}}_S$ can be extended inside $\hat{G}$ to the full amalgam $\bigcup_{\varnothing \neq F_0 \subset F} G_{F_0}$. By induction it suffices to show that we can extend S by one new subset. Let T be a largest subset of $I = \{1, \ldots, n\}$ not contained in S. Then $S' = S \cup \{T\}$ is again a type. Observe that the amalgam $\bigcup_{J \supset T} G_{F(J)}$ is fully contained in $\hat{\mathscr{B}}_S$, and this amalgam is the amalgam of all parabolics of $G_{F(T)}$ acting on the residue $\mathscr{R}$ of $F(T)$ in $\mathscr{N}$ ($\mathscr{R}$ consists of all elements x not in $F(T)$ such that $\{x\} \cup F(T)$ is again a flag). We claim that $\mathscr{R}$ is simply connected. Indeed, as T is not in $S(q)$, the residue has rank $k \geqslant 3$. If its diagram is connected, then $\mathscr{R}$ is isomorphic to the geometry $\mathscr{N}(k, q)$ and the claim follows from Proposition 3.1. If the diagram of $\mathscr{R}$ has three or more connected components, then $\mathscr{R}$ can be written as a direct sum of two geometries, one of which is connected. Thus the claim follows Lemma 3.8. Finally, if the diagram of $\mathscr{R}$ has exactly two connected components, then $\mathscr{R}$ can still be represented as a direct sum of two geometries with one of the geometries connected. This last property is due to the assumption that $S \supset S(q)$ and our choice for $S(q)$. Thus in all cases $\mathscr{R}$ is simply connected. Now Proposition 12.2 implies that the subgroup of $\hat{G}$ generated by the images of all subgroups $G_{F(J)}$, for $J \supset T$, is a quotient of $G_{F(T)}$. Since in the completion $\mathrm{SU}(n+1, q^2)$ a similar subgroup is isomorphic to $G_{F(T)}$, the same assertion must hold in $\hat{G}$. Thus $\hat{\mathscr{B}}_S$ extends inside $\hat{G}$ to $\hat{\mathscr{B}}_{S'}$. This establishes the result.

We now need to pass from the amalgam $\hat{\mathscr{B}}_S$ to the amalgam $\hat{\mathscr{A}}_S$. Notice that every member L_J of the amalgam $\hat{\mathscr{A}}_S$ is normal in the corresponding member $G_{F(J)}$ of the amalgam $\hat{\mathscr{B}}_S$. Furthermore, $G_{F(J)}$ is equal to $L_J D$ where D is the diagonal subgroup in $\mathrm{SU}(n+1, q^2)$.

Lemma 12.4. *Suppose that $S \supseteq S(q)$. Then the universal completion of $\hat{\mathscr{B}}_S$ is also the universal completion of $\hat{\mathscr{A}}_S$.*

Proof. Let $\tilde{G}$ be the universal completion of $\hat{\mathscr{A}}_S$. Since $\mathrm{SU}(n+1, q^2)$ is a completion of $\hat{\mathscr{A}}_S$, the image of $\hat{\mathscr{A}}_S$ in $\tilde{G}$ is isomorphic to $\hat{\mathscr{A}}_S$. Hence it suffices to show that the image of $\hat{\mathscr{A}}_S$ in $\tilde{G}$ can be extended to a copy of $\hat{\mathscr{B}}_S$.

Let D_i be the intersection of $U_i = L_{\bar{i}}$ with the diagonal subgroup D. Notice that D (and hence every D_i) normalizes every L_J. We adopt the tilde convention, so that

for any element x or subgroup H from $\hat{\mathscr{A}}_S$, $\tilde{x}$ and $\tilde{H}$ denote their images in $\tilde{G}$. Notice that D is not a subgroup of the amalgam $\hat{\mathscr{A}}$, and consequently we cannot use this convention to define $\tilde{D}$; therefore we define it indirectly as follows. Let $\tilde{D}$ be equal to the product of all the subgroups $\tilde{D}_i$. (Notice that $D_i \subset \hat{\mathscr{A}}_S$ and hence $\tilde{D}_i$ is defined.) We claim that $\tilde{D}$ is a direct product of the subgroups $\tilde{D}_i$, and in particular that it is an abelian group of order $(q+1)^n$. Indeed D_i and D_j are both contained in $U_{ij} = L_{\overline{ij}}$ and they commute elementwise. Therefore $\tilde{D}_i$ and $\tilde{D}_j$ also commute elementwise. This proves that $\tilde{D}$ is abelian of order at most $(q+1)^n$. Since $\mathrm{SU}(n+1, q^2)$ is a completion of $\hat{\mathscr{A}}_S$, and since in this completion the image of $\tilde{D}$ coincides with D, we see that the size of $\tilde{D}$ is at least $(q+1)^n$. This proves our claim.

In a similar spirit, for $J \in S$ define $\tilde{G}_{F(J)}$ to be the product of the subgroups $\tilde{L}_J$ with $\tilde{D}$. For this definition to make sense, we must show that every $\tilde{D}_i$ normalizes $\tilde{L}_J$. First suppose that $q \neq 2$. Then L_J is generated by the subgroups U_s with $s \notin J$. Inside U_{is} we see that D_i normalizes U_s. Hence $\tilde{D}_i$ normalizes every $\tilde{U}_s$, implying that $\tilde{D}_i$ normalizes $\tilde{L}_J$. If $q = 2$, then a similar proof works, using that $S(q)$ contains all subsets of the form $\overline{\{i, s, s+1\}}$ and that L_J is generated by the subgroups U_s with $s \notin J$ together with all subgroups $U_{s,s+1}$ where s, $s+1 \notin J$. Thus the subgroups $\tilde{G}_{F(J)}$ are well defined.

We claim that with respect to the natural homomorphism from $\tilde{G}$ onto $\mathrm{SU}(n+1, q^2)$, $\tilde{G}_{F(J)}$ isomorphically maps onto the group $G_{F(J)}$. This map is clearly surjective, and so it suffices to show that it is injective, or that the order of $\tilde{G}_{F(J)}$ is at most the order of $G_{F(J)}$. Indeed, $|G_{F(J)}| = |D|\,|L_J|/|D \cap L_J|$. Similarly,

$$|\tilde{G}_{F(J)}| = |\tilde{D}|\,|\tilde{L}_J|/|\tilde{D} \cap \tilde{L}_J|.$$

Notice that $D \cap L_J$ is the product of the subgroups D_i with $i \notin J$ and so clearly every such $\tilde{D}_i$ is contained in $\tilde{D} \cap \tilde{L}_J$. Therefore $|\tilde{D} \cap \tilde{L}_J| \geqslant |D \cap L_J|$, and hence $|G_{F(J)}| \geqslant |\tilde{G}_{F(J)}|$ as desired. So we have proved that every $\tilde{G}_{F(J)}$ is isomorphic to $G_{F(J)}$. Now it is clear that the natural homomorphism from $\tilde{G}$ onto $\mathrm{SU}(n+1, q^2)$ induces an isomorphism from the amalgam $\bigcup_{J \in S} \tilde{G}_{F(J)}$ onto the amalgam $\hat{\mathscr{B}}_S$. (Clearly the subgroups $\tilde{G}_{F(J)}$ and $\tilde{G}_{F(J')}$ cannot have a smaller intersection than the corresponding parabolics $G_{F(J)}$ and $G_{F(J')}$.) Thus we have extended the image of the amalgam $\hat{\mathscr{A}}_S$ to a copy of $\hat{\mathscr{B}}_S$. This proves that $\hat{\mathscr{A}}_S$ and $\hat{\mathscr{B}}_S$ have isomorphic completions.

Lemmas 12.3 and 12.4 imply Proposition 12.1, thus completing the proof of Theorems 1.2 and 1.3.

References

[1] M. Aschbacher. Notes on the Curtis–Tits–Phan theorems. Preprint.

[2] K. M. Das. Homotopy and homology of p-group complexes. Ph.D. thesis, Caltech (1994).

[3] The GAP Group. *GAP—Groups, Algorithms, and Programming*, Version 4.2 (Aachen, St Andrews, 2000) (http://www-gap.dcs.st-and.ac.uk/~gap).

[4] R. Gramlich, C. Hoffmann and S. Shpectorov. A Phan-type theorem for $\mathrm{Sp}(2n, q)$. *J. Algebra*, to appear.
[5] D. Goldschmidt. Automorphisms of trivalent graphs. *Ann. of Math.* (2) **111** (1980), 377–406.
[6] A. A. Ivanov. *Geometry of sporadic groups*. I. *Petersen and tilde geometries* (Cambridge University Press, 1999).
[7] A. A. Ivanov and S. Shpectorov. *Geometry of sporadic groups*. II. *Representations and amalgams* (Cambridge University Press, 2002).
[8] K.-W. Phan. On groups generated by three-dimensional unitary groups I. *J. Austral. Math. Soc. Ser. A* **23** (1977), 67–77.
[9] A. Pasini. *Diagram geometries* (Clarendon Press, 1994).
[10] J.-P. Serre. *Trees* (Springer-Verlag, 1980).
[11] J. Tits. Ensembles ordonnés, immeubles et sommes amalgameés. *Bull. Soc. Math. Belg. Sér. A* **38** (1986), 367–387.

Received 17 May, 2002; revised 14 August, 2003

C. D. Bennett, Department of Mathematics, Suite 2700, University Hall, Loyola Marymount University, One LMU Drive, Los Angeles, CA 90045, U.S.A.
E-mail: cbennett@lmu.edu

S. Shpectorov, Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403, U.S.A.
E-mail: sergey@bgnet.bgsu.edu