Pseudoprime $L$-Ideals in a Class of $F$-Rings

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PSEUDOPRIME \( l \)-IDEALS IN A CLASS OF \( f \)-RINGS

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ABSTRACT. In a commutative \( f \)-ring, an \( l \)-ideal \( I \) is called pseudoprime if \( ab = 0 \) implies \( a \in I \) or \( b \in I \), and is called square dominated if for every \( a \in I \), \( |a| \leq x^2 \) for some \( x \in A \) such that \( x^2 \in I \). Several characterizations of pseudoprime \( l \)-ideals are given in the class of commutative semiprime \( f \)-rings in which minimal prime \( l \)-ideals are square dominated. It is shown that the hypothesis imposed on the \( f \)-rings, that minimal prime \( l \)-ideals are square dominated, cannot be omitted or generalized.

Introduction. Let \( X \) be a topological space and \( C(X) \) be the \( f \)-ring of all continuous real-valued functions on \( X \) with coordinatewise operations. The following characterizations of pseudoprime \( l \)-ideals of \( C(X) \) are known.

(L. Gillman and C. Kohls [4, 4.1]) For an \( l \)-ideal \( I \) of \( C(X) \), the following are equivalent:

1. \( I \) is pseudoprime.
2. The prime ideals containing \( I \) form a chain.
3. \( \sqrt{I} \) is prime.

In [11], Subramanian asks whether this characterization of pseudoprime \( l \)-ideals generalizes to semiprime \( f \)-rings. The answer in general is no, as can be seen by Example 2.7. In this work, we investigate pseudoprime \( l \)-ideals in the class of commutative semiprime \( f \)-rings in which minimal prime \( l \)-ideals are square dominated. In this class of \( f \)-rings, we give some alternate characterizations of pseudoprime \( l \)-ideals, and we show that in normal \( f \)-rings conditions (2) and (3) characterize pseudoprime \( l \)-ideals. We also show that if all prime \( l \)-ideals are square dominated, a generalization of condition (3) characterizes pseudoprime \( l \)-ideals in archimedean \( f \)-algebras. Finally, we show that the hypothesis imposed on our \( f \)-rings, that minimal prime \( l \)-ideals be square dominated, cannot be omitted or generalized in any way by showing that if any of the characterizations hold in a semiprime \( f \)-ring \( A \), then all minimal prime \( l \)-ideals of \( A \) are square dominated. We assume throughout that all rings are commutative and semiprime.

1. Preliminaries. An \( f \)-ring is a lattice ordered ring which is a subdirect product of totally ordered rings. For general information on \( f \)-rings see [2]. Given an \( f \)-ring \( A \) and \( x \in A \), we let \( A^+ = \{ a \in A : a \geq 0 \} \), \( x^+ = x \lor 0 \), \( x^- = (\neg x) \lor 0 \), and \( |x| = x \lor (\neg x) \).
A ring ideal \( I \) of an \( f \)-ring \( A \) is an \( l \)-ideal if \( |x| \leq |y|, y \in I \) implies \( x \in I \). Given any subset \( S \subset A \), there is a smallest \( l \)-ideal containing \( S \), and we denote this by \((S)\).

Suppose \( A \) is an \( f \)-ring and \( I, J \) are \( l \)-ideals of \( A \). We let \( I: J = \{a \in A: aJ \subseteq I\} \).

The \( l \)-ideal \( I \) is semiprime (prime) if \( a^2 \in I \) \((ab \in I)\) implies \( a \in I \) \((a \in I \text{ or } b \in I)\).

The \( f \)-ring \( A \) is semiprime (prime) if \( \{0\} \) is a semiprime (prime) \( l \)-ideal. We let \( \sqrt{I} \) denote \( \{a \in A: a^n \in I \text{ for some } n\} \), the smallest semiprime \( l \)-ideal containing \( I \).

It is well known that all \( l \)-ideals containing a given prime \( l \)-ideal form a chain. The following result is also well known.

(1.1) If \( I \) is an \( l \)-ideal of a semiprime \( f \)-ring \( A \), then \( \cap_{n=1}^{\infty} (I^n) \) is semiprime.

(1.2) A prime \( l \)-ideal \( P \) of a commutative semiprime \( f \)-ring is minimal if and only if \( a \in P \) implies there is a \( b \notin P \) such that \( ab = 0 \).

A subset \( M \) of an \( f \)-ring \( A \) is called an \( m \)-system if whenever \( a, b \in M \) there exists an \( x \in A \) such that \( axb \in M \). If \( M \) is a \( l \)-system, then there is a prime \( l \)-ideal \( P \) such that \( I \subseteq P \), and \( P \cap M = \emptyset \).

We call an ideal \( I \) pseudoprime if \( ab = 0 \) implies \( a \in I \) or \( b \in I \). In a semiprime \( f \)-ring \( A \), a pseudoprime \( l \)-ideal contains a prime \( l \)-ideal as shown in [11, 2.1]. Also, in a commutative \( f \)-ring a pseudoprime and semiprime \( l \)-ideal is necessarily a prime \( l \)-ideal as shown in [9, 1.3] and [11, 2.5].

(1.3) In a commutative \( f \)-ring \( A \), a \( z \)-ideal \( I \) is prime if and only if for all \( a \in A \), either \( a^+ \in I \) or \( a^- \in I \).

Henriksen, in [5] calls an \( l \)-ideal \( I \) of an \( f \)-ring \( A \) square dominated if \( I = \{a \in A: |a| \leq \alpha^2 \text{ for some } x \in A \text{ such that } x^2 \in I\} \). Every prime square dominated \( l \)-ideal \( P \) satisfies \( P = (P^2) \).

An \( f \)-ring (and more generally a Riesz space) \( A \) is called normal if \( A = \{a^+, a^+\} \) for all \( a \in A \), or equivalently if \( a \wedge b = 0 \) implies \( A = \{a^+, b^-\} \). Several conditions equivalent to normality for an \( f \)-ring can be found in [8, 6.3], and for a Riesz space in [7, Theorem 9].

Given an \( f \)-ring \( A \) and an element \( x > 0 \) in \( A \), the sequence \( \{f_n\}_{n=1}^{\infty} \) is said to converge \( x \)-uniformly to the element \( f \in A \) if for every \( \varepsilon > 0 \), there exists a positive integer \( N_\varepsilon \) such that \( |f - f_n| < \varepsilon x \) for all \( n \geq N_\varepsilon \). An \( x \)-uniform Cauchy sequence is defined similarly. If for every \( x \geq 0 \), every \( x \)-uniform Cauchy sequence has a unique limit, then \( A \) is said to be uniformly complete. We say \( A \) is archimedean if \( a, b \in A^+ \) with \( na \leq b \) for all \( n \) implies \( a = 0 \). If \( A \) is an archimedean \( f \)-algebra with identity element, it is well known that \( A \) is commutative and semiprime. In [1, 4.1(d)], it is shown that:

(1.4) For any archimedean \( f \)-algebra \( A \) with identity element, there is an embedding \( e \) of \( A \) into a uniformly complete \( f \)-algebra \( A^* \) (the uniform completion of \( A \)).

2. In this section we give several characterizations of pseudoprime \( l \)-ideals in the class of commutative semiprime \( f \)-rings with minimal prime \( l \)-ideals square dominated. This class contains, of course, all commutative semiprime square root
closed f-rings. First we give a lemma that will be used later and that also gives a characterization of commutative semiprime f-rings in which minimal prime l-ideals are square dominated.

**LEMMA 2.1.** Let A be a commutative semiprime f-ring.

1. A semiprime l-ideal I is square dominated if every prime l-ideal minimal with respect to containing I is square dominated.

2. Every minimal prime l-ideal of A is square dominated if and only if for every a ∈ A+, the l-ideal \( \{a\}^d = \{b ∈ A : ab = 0\} \) is square dominated.

**PROOF.** (1) Let \( b ∈ I^+ \). Let \( M = \{c_1^n : c_1 \neq 0\} \). Then \( M \) is an m-system. Suppose \( M \cap I = \emptyset \). Then there is a prime l-ideal P such that \( I ⊆ P \) and \( M ∩ P = \emptyset \). Let \( P_1 ⊆ P \) be a prime l-ideal minimal with respect to containing I. By hypothesis, \( P_1 \) is square dominated, so there is a \( p \in A \) such that \( b ≤ p^2 \) and \( p^2 P_1 \). But then \( p^2 \notin M \cap P \), contrary to assumption. So \( M ∩ P = \emptyset \). Let \( c_1^n : c_1^n \in M ∩ I, \) \( b ≤ c_1^n \) for each i. Then \( b ≤ c_1^n ∧ ∨ c_2^n = (|c_1^n| ∧ ∨ |c_2^n|)^2 \). Also, \( 0 ≤ (|c_1^n| ∧ ∨ |c_2^n|)^2 ≤ c_1^n ∧ c_2^n \). This implies \( (|c_1^n| ∧ ∨ |c_2^n|)^2 ∈ I \), and because I is semiprime, \( (|c_1^n| ∧ ∨ |c_2^n|)^2 ∈ I \).

(2) Suppose that every minimal prime l-ideal is square dominated. Let \( a ∈ A^+ \). Suppose \( P \) is a prime l-ideal minimal with respect to containing \( \{a\}^d \). Then \( M = \{b : b ∈ A \setminus P\} \cup \{a^n : n ∈ N\} \cup \{ab^n : b ∈ A \setminus P, n ∈ N\} \) is an m-system such that \( M ∩ \{a\}^d = \emptyset \). So there is a prime l-ideal \( P_1 \) satisfying \( \{a\}^d P_1 P_1 \). By our choice of \( P_1 \) implies \( P_1 = P \) and \( a P_1 \).

Now if \( P_2 \) is a minimal prime l-ideal contained in \( P, a P_2 \) implies \( \{a\}^d P_2 \). Hence \( P_2 = P \), and \( P \) is in fact a minimal prime l-ideal which is square dominated. So every prime l-ideal minimal with respect to containing \( \{a\}^d \) is square dominated, and part (1) implies \( \{a\}^d \) is square dominated.

Our first characterization of pseudoprime l-ideals follows.

**THEOREM 2.2.** Let A be a commutative, semiprime f-ring with identity element and in which every minimal prime l-ideal is square dominated. The following are equivalent for an l-ideal I:

1. I is pseudoprime.
2. \( \bigcap_{n=1}^{∞} (I^n) \) is prime.
3. \( (I: \sqrt{I}) \) is pseudoprime.
4. \( I: \sqrt{I} \) is pseudoprime and \( I: \sqrt{I} \) is prime.

**PROOF.** (1) ⇒ (2). Let P be a minimal prime l-ideal contained in I. Since P is square dominated, \( P = \bigcap_{n=1}^{∞} (P^n) \subseteq \bigcap_{n=1}^{∞} (I^n) \). So \( \bigcap_{n=1}^{∞} (I^n) \) is pseudoprime. By 1.1, it is also semiprime and therefore prime.

(2) ⇒ (3). Trivial.

(3) ⇒ (4). By (3), all l-ideals containing \( (I: \sqrt{I}) \) form a chain. So \( I : \sqrt{I} ⊆ \sqrt{I} \) or \( \sqrt{I} ⊆ I : \sqrt{I} \).

(4) ⇒ (1). Either hypothesis implies that there is a minimal prime l-ideal \( P \) contained in \( \sqrt{I} \) and also in \( I : \sqrt{I} \). Then since \( P \) is square dominated, \( P = \langle P^2 \rangle \subseteq \langle I : \sqrt{I} \rangle \). □
Recall that in a commutative ring, an ideal $I$ is primary if $ab \in I$, $a \notin I$ implies $b^n \in I$ for some $n$. It is well known that $\sqrt{I}$ is prime for every primary $l$-ideal $I$. Thus in $C(X)$, every primary $l$-ideal is pseudoprime. This is not true in general (as can be seen by Example 2.7) and the following corollary gives a condition under which primary $l$-ideals are pseudoprime.

**Corollary 2.2**. Let $A$ be a commutative semiprime $f$-ring with identity element in which every minimal prime $l$-ideal is square dominated. A primary $l$-ideal $I$ is pseudoprime if and only if $I : \sqrt{I}$ is pseudoprime.

**Proof.** Assume that $I : \sqrt{I}$ is pseudoprime. If $I = \sqrt{I}$, then $I$ is prime, so we may assume that $I \neq \sqrt{I}$. Let $a \in \sqrt{I} \setminus I$. We will show $I : \sqrt{I} \subseteq \sqrt{I}$. Suppose $b \in I : \sqrt{I}$. Then $ab \in I$, and since $a \notin I$, we must have $b \in \sqrt{I}$. So $I : \sqrt{I} \subseteq \sqrt{I}$. It follows from the previous theorem that $I$ is pseudoprime. □

It is well known that in $C(X)$, an $l$-ideal $I$ is pseudoprime if and only if $\sqrt{I}$ is prime [4, 4.1]. In [11], Subramanian asks whether this characterization of pseudoprime $l$-ideals holds in semiprime $f$-rings. The answer in general is no (even in archimedean $f$-rings), as witnessed by Example 2.7. However, our next goal is to show that the characterization of pseudoprime $l$-ideals as those $l$-ideals $I$ for which $\sqrt{I}$ is prime also holds in a class of normal $f$-rings.

First, we will give two characterizations of normal $f$-rings, one of which is the $f$-ring analogue to a characterization given by Huizingmans in [7]. It should also be noted that the $f$-ring $C(X)$ is normal if and only if the topological space $X$ is an $F$-space and that the characterizations of normal $f$-rings given next are similar to two characterizations of $F$-spaces given in [3, 14.25]. If $P$ is any prime ideal, the $P$ component of 0 is $O_P = \{a \in A : \exists b \notin P \text{ such that } ab = 0\}$.

**Theorem 2.4**. Let $A$ be a commutative semiprime $f$-ring with identity element. The following are equivalent.

1. $A$ is normal.
2. Every ideal $O_P$, where $P$ is a (proper) prime $l$-ideal, is prime.
3. Every ideal $O_M$, where $M$ is a maximal $l$-ideal, is prime.

**Proof.** (1) $\Rightarrow$ (2). Since $O_P$ is a $z$-ideal, 1.3 implies that it will suffice to show that for all $a \in A$, either $a^+ \in O_P$ or $a^- \in O_P$. Suppose $a \in A$. If $a^+ \notin O_P$ and $a^- \notin O_P$, then $\{a^+\}^d \subseteq P$ and $\{a^-\}^d \subseteq P$. But this would imply $A = \{a^+\}^d + \{a^-\}^d \subseteq P$, contrary to hypothesis.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). Suppose $A$ is not normal. Then there is an element $a \in A$ such that $\{a^+\}^d + \{a^-\}^d \neq A$. Let $M$ be a maximal $l$-ideal containing the $l$-ideal $\{a^+\}^d + \{a^-\}^d$. Then $O_M$ is a $z$-ideal such that $a^+ \notin O_M$ and $a^- \notin O_M$. By 1.3, $O_M$ is not prime, contrary to our hypothesis. □

This characterization allows us to make the following observation.

**Lemma 2.5.** If $A$ is a commutative, semiprime normal $f$-ring with identity element then every minimal prime $l$-ideal of $A$ is square dominated.

**Proof.** Let $P$ be a minimal prime $l$-ideal of $A$. Since the $l$-ideals containing $P$ form a chain, there is a unique maximal $l$-ideal $M$ containing $P$. Now $O_M \subseteq P$ and, by the previous theorem $O_M$ is prime. Therefore, $O_M = P$. We will show $O_M$
is square dominated. Let \( a \in O^+_M \). Then there exists a \( b > 0 \) such that \( b \notin M \) and \( ab = 0 \). Then \( a \land b = 0 \) and \( \{a\}^d + \{b\}^d = A \). Thus, there exists \( x \in \{a\}^d, y \in \{b\}^d \) such that \( x + y = a \lor 1 \). It follows from \( y \in \{b\}^d \) and \( b \notin M \), that \( y \in O_M \). We have \( y \in O_M \), and \( a < \frac{y}{2} \).

In [9, 3.6] it is shown that if \( I \) is an \( l \)-ideal of a commutative semiprime \( f \)-ring with identity element such that \( I = I : \sqrt{I} \), then \( I \) is an intersection of primary \( l \)-ideals. We will use this fact in the proof of the next theorem.

**Theorem 2.6.** Let \( A \) be a commutative semiprime normal \( f \)-ring with identity element. The following are equivalent for an \( l \)-ideal \( I \).

1. \( I \) is pseudoprime.
2. The prime \( l \)-ideals containing \( I \) form a chain.
3. \( \sqrt{I} \) is prime.

**Proof.** We need only show (3) \( \Rightarrow \) (1). Suppose that \( \sqrt{I} \) is prime. Let \( P \) be a minimal prime \( l \)-ideal contained in \( \sqrt{I} \) and \( J = P \cap I \). We will show that \( J \) is pseudoprime. Knowing that \( \sqrt{J} \) is square dominated (Lemma 2.5), it is not hard to show that \( J : \sqrt{J} = (J : \sqrt{J}) : \sqrt{J} \). Let \( M \) be the maximal \( l \)-ideal containing \( P \). For any \( x \in M \setminus J : \sqrt{J} \), there is a primary \( l \)-ideal \( Q \) containing \( J : \sqrt{J} \), but not \( x \) by the result mentioned above [9, 3.6]. The \( l \)-ideals containing \( P \) form a chain, so \( P = \sqrt{J} \subseteq J : \sqrt{J} \subseteq Q_x \subseteq M \). Knowing \( Q_x \subseteq M \), it is easy to show that \( O_M \subseteq Q_x \) for all \( x \). Now \( J : \sqrt{J} = M \cap (\bigcup Q_x) \), so \( O_M \subseteq J : \sqrt{J} \). By Theorem 2.4, \( O_M \) is a prime \( l \)-ideal. Thus \( J : \sqrt{J} \) is a pseudoprime \( l \)-ideal. Also, since the \( l \)-ideals containing \( O_M \) form a chain, we have \( J : \sqrt{J} \subseteq \sqrt{J} \) or \( \sqrt{J} \subseteq J : \sqrt{J} \). In either case, Theorem 2.1 now implies that \( J \) is pseudoprime.

In particular, this result implies that in an \( f \)-ring satisfying the hypotheses of the theorem, every primary \( l \)-ideal is pseudoprime.

The next example shows that the hypothesis of normality cannot be left out of this theorem. It also shows that the characterization of pseudoprime \( l \)-ideals as being those \( I \) for which \( \sqrt{I} \) is prime does not hold in archimedean \( f \)-algebras. In fact, primary \( l \)-ideals in archimedean \( f \)-algebras are not necessarily pseudoprime.

**Example 2.7.** Let \( B = \{ f \in C[0,1] : \exists x_f \in (0,1) \text{ such that } f(x) = \sum_{i=1}^{n} a_i x_i^{r_i}, \text{ where } a_i \in \mathbb{R}, r_i \in \mathbb{Q} \text{ for all } x \in [0,x_f] \} \). Let \( P = \{ f \in B : f(0) = 0 \} \). Then \( P \) is a prime \( l \)-ideal of \( B \). Let \( A = \{ (f,g) \in B \times B : f - g \in P \} \). Then as shown in [6], \( A \) is a semifinite archimedean \( f \)-algebra with identity element. It is not hard to show that every prime \( l \)-ideal of \( A \) is square dominated.

Let \( I = \{(f,g) \in A : f(x), g(x) \leq nx^2 \text{ for some } n, \forall x \in [0, x_f \land x_g] \} \). Then \( I \) is an \( l \)-ideal of \( A \). We will show that \( I \) is primary. Suppose \( (f,g)(h,k) \in I \), and \( (f,g) \notin I \). For every \( n \), either \( f(x) \notin nx^2 \) or \( g(x) \notin nx^2 \) on \( [0, a] \) for any \( a \in (0,1) \). Suppose \( f(x) \notin nx^2 \). Then \( h(0) = 0 \) and \( h \in P \). But \( h - k \in P \), so \( k \in P \). This implies there must exist some \( N \) such that \( (h,k)^N \in I \). Hence \( I \) is primary, and \( \sqrt{I} \) is prime. Yet \( I \) is not pseudoprime since \( (x,0)(0,x) = (0,0) \) while \( (x,0) \notin I \) and \( (0,x) \notin I \).

To see directly that \( A \) is not normal, consider the element \( (x,-x) : \{(x,-x)^+\}^d + \{(x,-x)^-\}^d = \{(x,0)\}^d + \{(0,x)\}^d = \{(f,g) : f, g \in P \} \neq A \).
We have found some generalizations of the condition that \( \sqrt{I} \) be prime that characterize pseudoprime \( l \)-ideals in archimedean \( f \)-algebras in which minimal prime \( l \)-ideals are square dominated. However, each of these conditions are difficult to verify for most \( l \)-ideals. If we strengthen our hypotheses to insist that all prime \( l \)-ideals of the \( f \)-algebra are square dominated, we can give the following generalization to the condition which is a characterization of pseudoprime \( l \)-ideals.

**Theorem 2.8.** Let \( A \) be an archimedean \( f \)-algebra with identity element in which prime \( l \)-ideals are square dominated. If \( I \) is an \( l \)-ideal then \( I \) is pseudoprime if and only if when \( \{f_{ij}\}_{j=1}^{\infty}, \{f_{nj}\}_{j=1}^{\infty} \) are increasing positive Cauchy sequences such that \( f_{1j}f_{2j} \cdots f_{nj} = 0 \) for all \( j \), there is a sequence \( \{f_{ij}\}_{j=1}^{\infty} \), for which there exists a positive integer \( N \) such that \( f_{ij}^N \in I \) for all \( j \).

**Proof.** Let \( A^* \) denote the uniform completion of \( A \), and \( e: A \rightarrow A^* \) be the embedding.

\( \Leftarrow \) Suppose that whenever \( \{f_{ij}\}_{j=1}^{\infty}, \{f_{nj}\}_{j=1}^{\infty} \) are increasing positive Cauchy sequences such that \( f_{1j}f_{2j} \cdots f_{nj} = 0 \) for all \( j \), there is a sequence \( \{f_{ij}\}_{j=1}^{\infty} \), for which there exists a positive integer \( N \) such that \( f_{ij}^N \in I \) for all \( j \). In \( A^* \), let \( M = \{x_1 \cdots x_m: \text{for each } x_i, \forall j \exists a_{ij} \in A \text{ such that } 0 < e(a_{ij}) < x_i \text{ and } a_{ij} \notin I \} \).

Then \( M \) is an \( m \)-system. First we will show that \( M \cap \{0\} = \emptyset \). Suppose that for \( i = 1, 2, \ldots, m \), \( x_i \in A^* \), such that for every \( j \in \mathbb{N} \), there is an \( a_{ij} \in A \) such that \( 0 < e(a_{ij}) < x_i \) and \( a_{ij} \notin I \). Define

\[
 f_{ij} = \sum_{k=1}^{j} \frac{1}{2^k} (a_{ik} \land 1) \text{ for } i = 1, 2, \ldots, m, \ j = 1, 2, \ldots
\]

Then \( 0 \leq (1/2)^j e(a_{ij} \land 1) \leq e(f_{ij}) \leq x_i, \text{ for } i = 1, 2, \ldots, m, \ j = 1, 2, \ldots \), and \( \{f_{ij}\}_{j=1}^{\infty} \) is an increasing positive Cauchy sequence for \( i = 1, 2, \ldots, m \). Now for each sequence \( \{f_{ij}\}_{j=1}^{\infty} \), there cannot be a natural number \( N \) such that \( f_{ij}^N \in I \) for all \( j \), because the existence of such an \( N \) would imply that \( (a_{ij} \land 1)^N \), and hence \( a_{ij}^N \), is in \( I \) for all \( j \). So by hypothesis, \( f_{1j}f_{2j} \cdots f_{mj} \neq 0 \) for some \( j \). This, and the fact that \( 0 \leq e(f_{1j})e(f_{2j}) \cdots e(f_{mj}) \leq x_1x_2 \cdots x_m \) implies \( x_1x_2 \cdots x_m \neq 0 \). Therefore, \( M \cap \{0\} = \emptyset \).

Thus there is a prime \( l \)-ideal in \( A^* \) containing \( \{0\} \) and disjoint from \( M \). Let \( P \) be a minimal prime \( l \)-ideal with this property. We will show \( P \cap e(A) \subseteq e(I) \). Let \( f \in A^+ \) such that \( e(f) \in P \cap e(A) \).

By 1.2, there is a \( g \notin P \) such that \( e(f)g = 0 \). In \( A \), \( \{g\}_A = \{x \in A: xg = 0\} \) is a semiprime \( l \)-ideal and Lemma 2.1(1) implies it is square dominated. So \( f \leq f_1^2 \) for some \( f_1 \in A^+ \) with \( f_1 \in \{g\}_A \). Since \( \{g\}_A \) is semiprime, \( f_1 \in \{g\}_A \). As before, \( f_1 \leq f_2^2 \) for some \( f_2 \in A^+ \) with \( f_2 \in \{g\}_A \). Again \( f_2 \in \{g\}_A \). So \( f \leq f_1^2 \leq f_2^2 \). Continuing this for \( i = 3, 4, \ldots, \) we find \( f \leq f_1^2 \leq f_2^2 \leq \cdots \leq f_n^2 \leq \cdots \). Define

\[
 h_j = \sum_{i=1}^{j} \frac{1}{2^i} (f_i \land 1) \text{ for } j = 1, 2, \ldots
\]

Then \( \{h_j\}_{j=1}^{\infty} \) is an increasing Cauchy sequence in \( A \) and \( e(h_j) \) converges to an element \( h \in A^* \). Now \( hg = 0 \) and \( g \notin P \), implying \( h \in P \).
We assert that there exists some positive integer $M$ such that
\[(1/2^M)(f_M \land 1)^{2^M} \in I.\]
For if not, $0 \leq (1/2^i)(f_i \land 1)^{2^i} \leq (f_i \land 1)^{i^2} = ((f_i \land 1)^i)^i \not\in I$
for all $i \neq 3$. We would then have
\[0 \leq (1/2^i)(f_i \land 1)^i \leq h^i \quad \text{and} \quad ((1/2^i)(f_i \land 1)^i)^i \not\in I \quad \text{for all } i.\]
But this would imply $h \in M$, contrary to the fact that $M \cap P = \emptyset$. So there exists
some positive integer $M$ such that $(1/2^M)(f_M \land 1)^{2^M} \in I$.

We now know $(1/2^M)(f \land 1) \leq (1/2^M)(f_M \land 1)^{2^M}$ implies $(1/2^M)(f \land 1) \in I$. Therefore $(f \land 1) \in I$ and $f = (f \land 1)(f \lor 1) \in I$. We have $P \cap e(A) \subseteq e(I)$ and $P \cap e(A)$ is prime in $e(A)$. Therefore $e(I)$ is pseudoprime in $A^*$. This implies $I$ is pseudoprime in $A$.

⇒ Suppose that $I$ is pseudoprime and that \( \{f_{ij}\}_{j=1}^{\infty}, \{f_{nj}\}_{j=1}^{\infty} \) are increasing
positive Cauchy sequences in $A$ with $f_{ij}f_{2j} \cdots f_{nj} = 0$ for all $j$. Each of the
sequences $\{e(f_{ij})\}_{j=1}^{\infty}$ converges to some element $f_i$ in $A^*$. Let $P$ be a prime $l$-
ideal contained in $I$. Then $M = \{e(a) : a \in A, a \notin P\}$ is an $m$-system in $A^*$. So there is a prime $l$-ideal $P^*$ of $A^*$ such that $P \subseteq P^*$ and $P^* \cap M = \emptyset$. Now $e(f_{1j})e(f_{2j}) \cdots e(f_{nj}) = 0$ and so $e(f_{mj}) \in P^*$ for some $m$. Thus $e(f_{mj}) \leq e(f_m)$ implies that $e(f_{mj}) \in P^* \cap e(A) = e(P)$ for all $j$. \(\Box\)

It is not difficult to show directly that in a uniformly complete $f$-ring, the prop-
erty characterizing a pseudoprime $l$-ideal $I$ given in this theorem is equivalent to
the property that $\sqrt{I}$ is prime.

Finally we show that the hypothesis that minimal prime $l$-ideals be square dom-
inated cannot be dropped or generalized in any way in any of our theorems char-
acterizing pseudoprime $l$-ideals.

**Lemma 2.9.** Let $A$ be a semiprime $f$-ring. If in $A$, a pseudoprime $l$-ideal $I$ is
characterized by being an $l$-ideal that satisfies any one of the following conditions,
then every minimal prime $l$-ideal of $A$ is square dominated.

1. $\bigcap_{n=1}^{\infty} (P^n)$ is prime.
2. $(I, \sqrt{I})$ is pseudoprime.
3. $I : \sqrt{I}$ is pseudoprime and $I : \sqrt{I} \subseteq \sqrt{I}$, or, $\sqrt{I} \subseteq I : \sqrt{I}$ and $\sqrt{I}$ is prime.
4. The prime $l$-ideals containing $I$ form a chain.
5. $\sqrt{I}$ is prime.

Also, if $A$ is an archimedean $f$-ring, and if in $A$, a pseudoprime $l$-ideal $I$ is
characterized by being an $l$-ideal that satisfies the following condition, then every
minimal prime $l$-ideal of $A$ is square dominated.

6. Whenever $\{f_{1j}\}_{j=1}^{\infty}, \{f_{nj}\}_{j=1}^{\infty}$ are increasing positive Cauchy sequences
such that $f_{1j}f_{2j} \cdots f_{nj} = 0$ for all $j$, there is a sequence $\{f_{ij}\}_{j=1}^{\infty}$, for which there
exists a positive integer $N$ such that $f_{ij}^N \in I$ for all $j$.

**Proof.** Let $P$ be a minimal prime $l$-ideal. Note that in each case it will suffice
to show that $\langle P^2 \rangle$ is pseudoprime since $\langle P^2 \rangle \subseteq P$ and $P$ being a minimal prime
$l$-ideal will then imply $\langle P^2 \rangle = P$.

If characterization (1) holds, then $\bigcap_{n=1}^{\infty} (P^n)$ is a prime $l$-ideal contained in $\langle P^2 \rangle$. So $\langle P^2 \rangle$ is pseudoprime.
Characterization (2) implies \( \langle P\sqrt{P} \rangle = \langle P^2 \rangle \) is pseudoprime.

Suppose characterization (3) holds. Note that \( \langle P^2 \rangle : \sqrt{\langle P^2 \rangle} = \langle P^2 \rangle : P \geq \sqrt{\langle P^2 \rangle} = P \). So \( \langle P^2 \rangle : \sqrt{\langle P^2 \rangle} \) is pseudoprime. Then characterization (3) implies \( \langle P^2 \rangle \) is pseudoprime.

Suppose characterization (4) holds. Every prime l-ideal containing \( \langle P^2 \rangle \) also contains \( P \). So the prime l-ideals containing \( \langle P^2 \rangle \) form a chain. Characterization (4) implies \( \langle P^2 \rangle \) is pseudoprime.

Suppose characterization (5) holds. Then \( \sqrt{\langle P^2 \rangle} = P \) is prime, implying that \( \langle P^2 \rangle \) is pseudoprime.

Finally, suppose that \( A \) is an archimedean f-algebra and that characterization (6) holds. Suppose that \( \{f_{1j}\}_{j=1}^{\infty}, \ldots, \{f_{nj}\}_{j=1}^{\infty} \) are increasing positive Cauchy sequences such that \( f_{1j}f_{2j} \cdots f_{nj} = 0 \) for all \( j \). Then for some \( m \), \( f_{mj} \in P \) for all \( j \). But then \( f_{mj}^2 \in \langle P^2 \rangle \) for all \( j \). So characterization (6) implies that \( \langle P^2 \rangle \) is pseudoprime. \( \square \)

REFERENCES


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