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Suzanne Larson Loyola Marymount University, slarson@lmu.edu

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PSEUDOPRIME *l*-IDEALS IN A CLASS OF *f*-RINGS

SUZANNE LARSON

(Communicated by Louis J. Ratliff, Jr.)

ABSTRACT. In a commutative f-ring, an l-ideal I is called pseudoprime if ab = 0 implies $a \in I$ or $b \in I$, and is called square dominated if for every $a \in I$, $|a| \leq x^2$ for some $x \in A$ such that $x^2 \in I$. Several characterizations of pseudoprime l-ideals are given in the class of commutative semiprime f-rings in which minimal prime l-ideals are square dominated. It is shown that the hypothesis imposed on the f-rings, that minimal prime l-ideals are square dominated, cannot be omitted or generalized.

Introduction. Let X be a topological space and C(X) be the *f*-ring of all continuous real-valued functions on X with coordinatewise operations. The following characterizations of pseudoprime *l*-ideals of C(X) are known.

(L. Gillman and C. Kohls [4, 4.1]) For an *l*-ideal I of C(X), the following are equivalent:

- (1) I is pseudoprime.
- (2) The prime ideals containing I form a chain.
- (3) \sqrt{I} is prime.

In [11], Subramanian asks whether this characterization of pseudoprime *l*-ideals generalizes to semiprime f-rings. The answer in general is no, as can be seen by Example 2.7. In this work, we investigate pseudoprime *l*-ideals in the class of commutative semiprime f-rings in which minimal prime *l*-ideals are square dominated. In this class of f-rings, we give some alternate characterizations of pseudoprime *l*-ideals, and we show that in normal f-rings conditions (2) and (3) characterize pseudoprime *l*-ideals. We also show that if all prime *l*-ideals are square dominated, a generalization of condition (3) characterizes pseudoprime *l*-ideals in archimedean f-algebras. Finally, we show that the hypothesis imposed on our f-rings, that minimal prime *l*-ideals be square dominated, cannot be omitted or generalized in any way by showing that if any of the characterizations hold in a semiprime f-ring A, then all minimal prime *l*-ideals of A are square dominated. We assume throughout that all rings are commutative and semiprime.

1. **Preliminaries.** An *f*-ring is a lattice ordered ring which is a subdirect product of totally ordered rings. For general information on *f*-rings see [2]. Given an *f*-ring *A* and $x \in A$, we let $A^+ = \{a \in A : a \ge 0\}, x^+ = x \lor 0, x^- = (-x) \lor 0$, and $|x| = x \lor (-x)$.

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SUZANNE LARSON

A ring ideal I of an f-ring A is an l-ideal if $|x| \leq |y|$, $y \in I$ implies $x \in I$. Given any subset $S \subset A$, there is a smallest l-ideal containing S, and we denote this by $\langle S \rangle$.

Suppose A is an f-ring and I, J are l-ideals of A. We let $I: J = \{a \in A: aJ \subseteq I\}$. The l-ideal I is semiprime (prime) if $a^2 \in I$ ($ab \in I$) implies $a \in I$ ($a \in I$ or $b \in I$). The f-ring A is semiprime (prime) if $\{0\}$ is a semiprime (prime) l-ideal. We let \sqrt{I} denote $\{a \in A: a^n \in I \text{ for some } n\}$, the smallest semiprime l-ideal containing I. In [5, 3.5], it is shown that:

(1.1) If I is an *l*-ideal of a semiprime f-ring A, then $\bigcap_{n=1}^{\infty} \langle I^n \rangle$ is semiprime.

It is well known that all l-ideals containing a given prime l-ideal form a chain. The following result is also well known.

(1.2) A prime *l*-ideal P of a commutative semiprime f-ring is minimal if and only if $a \in P$ implies there is a $b \notin P$ such that ab = 0.

A subset M of an f-ring A is called an m-system if whenever $a, b \in M$ there exists an $x \in A$ such that $axb \in M$. If in A, there is an l-ideal I and an m-system M such that $I \cap M = \emptyset$, then there is a prime l-ideal P such that $I \subset P$, and $P \cap M = \emptyset$.

We call an ideal I pseudoprime if ab = 0 implies $a \in I$ or $b \in I$. In a semiprime f-ring A, a pseudoprime l-ideal contains a prime l-ideal as shown in [11, 2.1]. Also, in a commutative f-ring a pseudoprime and semiprime l-ideal is necessarily a prime l-ideal as shown in [8, 4.2] and [11, 2.5].

An ideal I of a commutative f-ring A with identity element is a z-ideal if whenever $a, b \in A$, are contained in the same set of maximal ideals and $a \in I$ then $b \in I$. In [10, 2.7], G. Mason shows

(1.3) In a commutative f-ring A, a z-ideal I is prime if and only if for all $a \in A$, either $a^+ \in I$ or $a^- \in I$.

Henriksen, in [5] calls an *l*-ideal *I* of an *f*-ring *A* square dominated if $I = \{a \in A : |a| \leq x^2 \text{ for some } x \in A \text{ such that } x^2 \in I\}$. Every prime square dominated *l*-ideal *P* satisfies $P = \langle P^2 \rangle$.

An f-ring (and more generally a Riesz space) A is called normal if $A = \{a^+\}^d + \{a^-\}^d$ for all $a \in A$, or equivalently if $a \wedge b = 0$ implies $A = \{a\}^d + \{b\}^d$. Several conditions equivalent to normality for an f-ring can be found in [8, 6.3], and for a Riesz space in [7, Theorem 9].

Given an f-ring A and an element x > 0 in A, the sequence $\{f_n\}_{n=1}^{\infty}$ is said to converge x-uniformly to the element $f \in A$ if for every $\varepsilon > 0$, there exists a positive integer N_{ε} such that $|f - f_n| < \varepsilon x$ for all $n \ge N_{\varepsilon}$. An x-uniform Cauchy sequence is defined similarly. If for every $x \ge 0$, every x-uniform Cauchy sequence has a unique limit, then A is said to be uniformly complete. We say A is archimedean if $a, b \in A^+$ with $na \le b$ for all n implies a = 0. If A is an archimedean f-algebra with identity element, it is well known that A is commutative and semiprime. In [1, 4.1(d)], it is shown that:

(1.4) For any archimedean f-algebra A with identity element, there is an embedding e of A into a uniformly complete f-algebra A^* (the uniform completion of A).

2. In this section we give several characterizations of pseudoprime l-ideals in the class of commutative semiprime f-rings with minimal prime l-ideals square dominated. This class contains, of course, all commutative semiprime square root

closed f-rings. First we give a lemma that will be used later and that also gives a characterization of commutative semiprime f-rings in which minimal prime l-ideals are square dominated.

LEMMA 2.1. Let A be a commutative semiprime f-ring.

(1) A semiprime l-ideal I is square dominated if every prime l-ideal minimal with respect to containing I is square dominated.

(2) Every minimal prime l-ideal of A is square dominated if and only if for every $a \in A^+$, the l-ideal $\{a\}^d = \{b \in A : ab = 0\}$ is square dominated.

PROOF. (1) Let $b \in I^+$. Let $M = \{c_1^2 \cdots c_n^2 : n \in \mathbb{N}; b \leq c_i^2\}$. Then M is an m-system. Suppose that $M \cap I = \emptyset$. Then there is a prime l-ideal P such that $I \subseteq P$ and $M \cap P = \emptyset$. Let $P_1 \subseteq P$ be a prime l-ideal minimal with respect to containing I. By hypothesis, P_1 is square dominated, so there is a $p \in A$ such that $b \leq p^2$ and $p^2 \in P_1$. But then $p^2 \in M \cap P$, contrary to assumption. So $M \cap I \neq \emptyset$. Let $c_1^2 \cdots c_n^2 \in M \cap I$, where $b \leq c_i^2$ for each i. Then $b \leq c_1^2 \wedge \cdots \wedge c_n^2 = (|c_1| \wedge \cdots \wedge |c_n|)^2$. Also, $0 \leq (|c_1| \wedge \cdots \wedge |c_n|)^{2n} \leq c_1^2 \cdots c_n^2$. This implies $(|c_1| \wedge \cdots \wedge |c_n|)^{2n} \in I$, and because I is semiprime, $(|c_1| \wedge \cdots \wedge |c_n|)^2 \in I$.

(2) \Rightarrow Suppose that every minimal prime *l*-ideal is square dominated. Let $a \in A^+$. Suppose *P* is a prime *l*-ideal minimal with respect to containing $\{a\}^d$. Then $M = \{b: b \in A \setminus P\} \cup \{a^n: n \in \mathbb{N}\} \cup \{ba^n: b \in A \setminus P, n \in \mathbb{N}\}$ is an *m*-system such that $M \cap \{a\}^d = \emptyset$. So there is a prime *l*-ideal P_1 satisfying $\{a\}^d \subseteq P_1 \subseteq P$. But our choice of *P* implies $P_1 = P$ and $a \notin P$.

Now if P_2 is a minimal prime *l*-ideal contained in P, $a \notin P_2$ implies $\{a\}^d \subseteq P_2$. Hence $P_2 = P$, and P is in fact a minimal prime *l*-ideal which is square dominated. So every prime *l*-ideal minimal with respect to containing $\{a\}^d$ is square dominated, and part (1) implies $\{a\}^d$ is square dominated.

 \Leftarrow Let P be a minimal prime *l*-ideal, and $f \in P$. By 1.2, there is a $g \notin P$ such that fg = 0. By hypothesis, $\{g\}^d$ is square dominated. So there is $f_1 \in \{g\}^d$ such that $f \leq f_1^2$ and $f_1^2 \in P$. \Box

Our first characterization of pseudoprime *l*-ideals follows.

THEOREM 2.2. Let A be a commutative, semiprime f-ring with identity element and in which every minimal prime l-ideal is square dominated. The following are equivalent for an l-ideal I:

(1) I is pseudoprime.

- (2) $\bigcap_{n=1}^{\infty} \langle I^n \rangle$ is prime.
- (3) $\langle I\sqrt{I}\rangle$ is pseudoprime.

(4) $I: \sqrt{I}$ is pseudoprime and $I: \sqrt{I} \subseteq \sqrt{I}$, or, $\sqrt{I} \subseteq I: \sqrt{I}$ and \sqrt{I} is prime.

PROOF. (1) \Rightarrow (2). Let *P* be a minimal prime *l*-ideal contained in *I*. Since *P* is square dominated, $P = \bigcap_{n=1}^{\infty} \langle P^n \rangle \subseteq \bigcap_{n=1}^{\infty} \langle I^n \rangle$. So $\bigcap_{n=1}^{\infty} \langle I^n \rangle$ is pseudoprime. By 1.1, it is also semiprime and therefore prime.

 $(2) \Rightarrow (3)$. Trivial.

(3) \Rightarrow (4). By (3), all *l*-ideals containing $\langle I\sqrt{I}\rangle$ form a chain. So $I: \sqrt{I} \subseteq \sqrt{I}$ or $\sqrt{I} \subseteq I: \sqrt{I}$.

(4) \Rightarrow (1). Either hypothesis implies that there is a minimal prime *l*-ideal *P* contained in \sqrt{I} and also in $I: \sqrt{I}$. Then since *P* is square dominated, $P = \langle P^2 \rangle \subseteq (I:\sqrt{I})\sqrt{I} \subseteq I$. \Box

SUZANNE LARSON

Recall that in a commutative ring, an ideal I is primary if $ab \in I$, $a \notin I$ implies $b^n \in I$ for some n. It is well known that \sqrt{I} is prime for every primary *l*-ideal I. Thus in C(X), every primary *l*-ideal is pseudoprime. This is not true in general (as can be seen by Example 2.7) and the following corollary gives a condition under which primary *l*-ideals are pseudoprime.

COROLLARY 2.3. Let A be a commutative semiprime f-ring with identity element in which every minimal prime l-ideal is square dominated. A primary l-ideal I is pseudoprime if and only if $I : \sqrt{I}$ is pseudoprime.

PROOF. Assume that $I : \sqrt{I}$ is pseudoprime. If $I = \sqrt{I}$, then I is prime, so we may assume that $I \neq \sqrt{I}$. Let $a \in \sqrt{I} \setminus I$. We will show $I : \sqrt{I} \subseteq \sqrt{I}$. Suppose $b \in I : \sqrt{I}$. Then $ab \in I$, and since $a \notin I$, we must have $b \in \sqrt{I}$. So $I : \sqrt{I} \subseteq \sqrt{I}$. It follows from the previous theorem that I is pseudoprime. \Box

It is well known that in C(X), an *l*-ideal *I* is pseudoprime if and only if \sqrt{I} is prime [4, 4.1]. In [11], Subramanian asks whether this characterization of pseudoprime *l*-ideals holds in semiprime *f*-rings. The answer in general is no (even in archimedean *f*-rings), as witnessed by Example 2.7. However, our next goal is to show that the characterization of pseudoprime *l*-ideals as those *l*-ideals *I* for which \sqrt{I} is prime also holds in a class of normal *f*-rings.

First, we will give two characterizations of normal f-rings, one of which is the f-ring analogue to a characterization given by Huijsmans in [7]. It should also be noted that the f-ring C(X) is normal if and only if the topological space X is an F-space and that the characterizations of normal f-rings given next are similar to two characterizations of F-spaces given in [3, 14.25]. If P is any prime ideal, the P component of 0 is $O_P = \{a \in A : \exists b \notin P \text{ such that } ab = 0\}.$

THEOREM 2.4. Let A be a commutative semiprime f-ring with identity element. The following are equivalent.

(1) A is normal.

(2) Every ideal O_P , where P is a (proper) prime l-ideal, is prime.

(3) Every ideal O_M , where M is a maximal l-ideal, is prime.

PROOF. (1) \Rightarrow (2). Since O_P is a z-ideal, 1.3 implies that it will suffice to show that for all $a \in A$, either $a^+ \in O_P$ or $a^- \in O_P$. Suppose $a \in A$. If $a^+ \notin O_P$ and $a^- \notin O_P$, then $\{a^+\}^d \subseteq P$ and $\{a^-\}^d \subseteq P$. But this would imply $A = \{a^+\}^d + \{a^-\}^d \subseteq P$, contrary to hypothesis.

 $(2) \Rightarrow (3)$. Obvious.

 $(3) \Rightarrow (1)$. Suppose A is not normal. Then there is an element $a \in A$ such that $\{a^+\}^d + \{a^-\}^d \neq A$. Let M be a maximal *l*-ideal containing the *l*-ideal $\{a^+\}^d + \{a^-\}^d$. Then O_M is a z-ideal such that $a^+ \notin O_M$ and $a^- \notin O_M$. By 1.3, O_M is not prime, contrary to our hypothesis. \Box

This characterization allows us to make the following observation.

LEMMA 2.5. If A is a commutative, semiprime normal f-ring with identity element then every minimal prime l-ideal of A is square dominated.

PROOF. Let P be a minimal prime l-ideal of A. Since the l-ideals containing P form a chain, there is a unique maximal l-ideal M containing P. Now $O_M \subseteq P$ and, by the previous theorem O_M is prime. Therefore, $O_M = P$. We will show O_M

688

is square dominated. Let $a \in O_M^+$. Then there exists a b > 0 such that $b \notin M$ and ab = 0. Then $a \wedge b = 0$ and $\{a\}^d + \{b\}^d = A$. Thus, there exists $x \in \{a\}^d$, $y \in \{b\}^d$ such that $x + y = a \vee 1$. It follows from $y \in \{b\}^d$ and $b \notin M$, that $y \in O_M$. We have $y \in O_M$, and $a \leq y^2$. \Box

In [9, 3.6] it is shown that if I is an *l*-ideal of a commutative semiprime *f*-ring with identity element such that $I = I : \sqrt{I}$, then I is an intersection of primary *l*-ideals. We will use this fact in the proof of the next theorem.

THEOREM 2.6. Let A be a commutative semiprime normal f-ring with identity element. The following are equivalent for an l-ideal I.

- (1) I is pseudoprime.
- (2) The prime l-ideals containing I form a chain.
- (3) \sqrt{I} is prime.

PROOF. We need only show $(3) \Rightarrow (1)$. Suppose that \sqrt{I} is prime. Let P be a minimal prime *l*-ideal contained in \sqrt{I} and $J = P \cap I$. We will show that J is pseudoprime. Knowing that \sqrt{J} is square dominated (Lemma 2.5), it is not hard to show that $J : \sqrt{J} = (J : \sqrt{J}) : \sqrt{J} : \sqrt{J}$. Let M be the maximal *l*-ideal containing P. For any $x \in M \setminus J : \sqrt{J}$, there is a primary *l*-ideal Q_x containing $J : \sqrt{J}$, but not x by the result mentioned above [9, 3.6]. The *l*-ideals containing P form a chain, so $P = \sqrt{J} \subseteq \sqrt{J} : \sqrt{J} \subseteq \sqrt{Q_x} \subseteq M$. Knowing $Q_x \subseteq M$, it is easy to show that $O_M \subseteq Q_x$ for all x. Now $J : \sqrt{J} = M \cap (\bigcap Q_x)$, so $O_M \subseteq J : \sqrt{J}$. By Theorem 2.4, O_M is a prime *l*-ideal. Thus $J : \sqrt{J}$ is a pseudoprime *l*-ideal. Also, since the *l*-ideals containing O_M form a chain, we have $J : \sqrt{J} \subseteq \sqrt{J}$ or $\sqrt{J} \subseteq J : \sqrt{J}$. In either case, Theorem 2.1 now implies that J is pseudoprime. \Box

In particular, this result implies that in an f-ring satisfying the hypotheses of the theorem, every primary l-ideal is pseudoprime.

The next example shows that the hypothesis of normality cannot be left out of this theorem. It also shows that the characterization of pseudoprime *l*-ideals as being those I for which \sqrt{I} is prime does not hold in archimedean f-algebras. In fact, primary *l*-ideals in archimedean f-algebras are not necessarily pseudoprime.

EXAMPLE 2.7. Let $B = \{f \in C[0,1] : \exists x_f \in (0,1) \text{ such that } f(x) = \sum_{i=1}^n a_i x^{r_i}, where <math>a_i \in \mathbf{R}, r_i \in \mathbf{Q} \text{ for all } x \in [0, x_f]\}$. Let $P = \{f \in B : f(0) = 0\}$. Then P is a prime *l*-ideal of *B*. Let $A = \{(f,g) \in B \times B : f - g \in P\}$. Then as shown in [6], *A* is a semiprime archimedean *f*-algebra with identity element. It is not hard to show that every prime *l*-ideal of *A* is square dominated.

Let $I = \{(f,g) \in A: f(x), g(x) \leq nx^2 \text{ for some } n, \forall x \in [0, x_f \land x_g]\}$. Then I is an *l*-ideal of A. We will show that I is primary. Suppose $(f,g)(h,k) \in I$, and $(f,g) \notin I$. For every n, either $f(x) \nleq nx^2$ or $g(x) \nleq nx^2$ on [0,a) for any $a \in (0,1)$. Suppose $f(x) \nleq nx^2$. Then h(0) = 0 and $h \in P$. But $h - k \in P$, so $k \in P$. This implies there must exist some N such that $(h,k)^N \in I$. Hence I is primary, and \sqrt{I} is prime. Yet I is not pseudoprime since (x,0)(0,x) = (0,0) while $(x,0) \notin I$ and $(0,x) \notin I$.

To see directly that A is not normal, consider the element (x, -x): $\{(x, -x)^+\}^d + \{(x, -x)^-\}^d = \{(x, 0)\}^d + \{(0, x)\}^d = \{(f, g): f, g \in P\} \neq A$. \Box

SUZANNE LARSON

We have found some generalizations of the condition that \sqrt{I} be prime that characterize pseudoprime *l*-ideals in archimedean *f*-algebras in which minimal prime *l*ideals are square dominated. However, each of these conditions are difficult to verify for most *l*-ideals. If we strengthen our hypotheses to insist that all prime *l*-ideals of the *f*-algebra are square dominated, we can give the following generalization to the condition which is a characterization of pseudoprime *l*-ideals.

THEOREM 2.8. Let A be an archimedean f-algebra with identity element in which prime l-ideals are square dominated. If I is an l-ideal then I is pseudoprime if and only if when $\{f_{1j}\}_{j=1}^{\infty}, \ldots, \{f_{nj}\}_{j=1}^{\infty}$ are increasing positive Cauchy sequences such that $f_{1j}f_{2j}\cdots f_{nj}=0$ for all j, there is a sequence $\{f_{lj}\}_{j=1}^{\infty}$, for which there exists a positive integer N such that $f_{lj}^N \in I$ for all j.

PROOF. Let A^* denote the uniform completion of A, and $e: A \to A^*$ be the embedding.

 \in Suppose that whenever $\{f_{1j}\}_{j=1}^{\infty}, \ldots, \{f_{nj}\}_{j=1}^{\infty}$ are increasing positive Cauchy sequences such that $f_{1j}f_{2j}\cdots f_{nj}=0$ for all j, there is a sequence $\{f_{lj}\}_{j=1}^{\infty}$, for which there exists a positive integer N such that $f_{lj}^N \in I$ for all j. In A^* , let $M = \{x_1 \cdots x_m: \text{ for each } x_i, \forall j \exists a_{ij} \in A \text{ such that } 0 \leq e(a_{ij}) \leq x_i^j \text{ and } a_{ij}^j \notin I\}$. Then M is an m-system. First we will show that $M \cap \{0\} = \emptyset$. Suppose that for $i = 1, 2, \ldots, m, x_i \in A^*$, such that for every $j \in \mathbb{N}$, there is an $a_{ij} \in A$ such that $0 \leq e(a_{ij}) \leq x_i^j$ and $a_{ij}^j \notin I$. Define

$$f_{ij} = \sum_{k=1}^{j} \frac{1}{2^k} (a_{ik} \wedge 1)$$
 for $i = 1, 2, ..., m, \ j = 1, 2, ..., m$

Then $0 \leq (1/2^j)e(a_{ij} \wedge 1) \leq e(f_{ij}) \leq x_i$, for i = 1, 2, ..., m, j = 1, 2, ..., m, j = 1, 2, ..., m. Now for $\{f_{ij}\}_{j=1}^{\infty}$ is an increasing positive Cauchy sequence for i = 1, 2, ..., m. Now for each sequence $\{f_{ij}\}_{j=1}^{\infty}$, there cannot be a natural number N such that $f_{ij}^N \in I$ for all j, because the existence of such an N would imply that $(a_{ij} \wedge 1)^N$, and hence a_{ij}^N , is in I for all j. So by hypothesis, $f_{1j}f_{2j}\cdots f_{mj} \neq 0$ for some j. This, and the fact that $0 \leq e(f_{1j})e(f_{2j})\cdots e(f_{mj}) \leq x_1x_2\cdots x_m$ implies $x_1x_2\cdots x_m \neq 0$. Therefore, $M \cap \{0\} = \emptyset$.

Thus there is a prime *l*-ideal in A^* containing $\{0\}$ and disjoint from M. Let P be a minimal prime *l*-ideal with this property. We will show $P \cap e(A) \subseteq e(I)$. Let $f \in A^+$ such that $e(f) \in P \cap e(A)$.

By 1.2, there is a $g \notin P$ such that e(f)g = 0. In A, $\{g\}_A^d = \{x \in A : xg = 0\}$ is a semiprime *l*-ideal and Lemma 2.1(1) implies it is square dominated. So $f \leq f_1^2$ for some $f_1 \in A^+$ with $f_1^2 \in \{g\}_A^d$. Since $\{g\}_A^d$ is semiprime, $f_1 \in \{g\}_A^d$. As before, $f_1 \leq f_2^2$ for some $f_2 \in A^+$ with $f_2^2 \in \{g\}_A^d$. Again $f_2 \in \{g\}_A^d$. So $f \leq f_1^2 \leq f_2^4$. Continuing this for $i = 3, 4, \ldots$, we find $f \leq f_1^2 \leq f_2^4 \leq \cdots \leq f_i^{2^i} \leq \cdots$. Define

$$h_j = \sum_{i=1}^j \frac{1}{2^i} (f_i \wedge 1)$$
 for $j = 1, 2, \dots$

Then $\{h_j\}_{m=1}^{\infty}$ is an increasing Cauchy sequence in A and $\{e(h_j)\}_{j=1}^{\infty}$ converges to an element $h \in A^*$. Now hg = 0 and $g \notin P$, implying $h \in P$.

We assert that there exists some positive integer M such that

$$(1/2^M)(f_M \wedge 1)^{2^M} \in I.$$

For if not, $0 \leq (1/2^i)(f_i \wedge 1)^{2^i} \leq (f_i \wedge 1)^{i^2}$ would imply $(f_i \wedge 1)^{i^2} = ((f_i \wedge 1)^i)^i \notin I$ for all $i \neq 3$. We would then have

$$0 \le (1/2^{i^2})(f_i \wedge 1)^i \le h^i$$
 and $((1/2^{i^2})(f_i \wedge 1)^i)^i \notin I$ for all *i*.

But this would imply $h \in M$, contrary to the fact that $M \cap P = \emptyset$. So there exists some positive integer M such that $(1/2^M)(f_M \wedge 1)^{2^M} \in I$.

We now know $(1/2^M)(f \wedge 1) \leq (1/2^M)(f_M \wedge 1)^{2^M}$ implies $(1/2^M)(f \wedge 1) \in I$. Therefore $(f \wedge 1) \in I$ and $f = (f \wedge 1)(f \vee 1) \in I$. We have $P \cap e(A) \subseteq e(I)$ and $P \cap e(A)$ is prime in e(A). Therefore e(I) is pseudoprime in A^* . This implies I is pseudoprime in A.

 \Rightarrow Suppose that I is pseudoprime and that $\{f_{1j}\}_{j=1}^{\infty}, \ldots, \{f_{nj}\}_{j=1}^{\infty}$ are increasing positive Cauchy sequences in A with $f_{1j}f_{2j}\cdots f_{nj} = 0$ for all j. Each of the sequences $\{e(f_{ij})\}_{j=1}^{\infty}$ converges to some element f_i in A^* . Let P be a prime lideal contained in *I*. Then $M = \{e(a): a \in A, a \notin P\}$ is an *m*-system in A^* . So there is a prime *l*-ideal P^* of A^* such that $P \subseteq P^*$ and $P^* \cap M = \emptyset$. Now $e(f_1)e(f_2)\cdots e(f_n)=0$ and so $e(f_m)\in P^*$ for some m. Thus $e(f_{mj})\leq e(f_m)$ implies that $e(f_{mj}) \in P^* \cap e(A) = e(P)$ for all j. \Box

It is not difficult to show directly that in a uniformly complete f-ring, the property characterizing a pseudoprime l-ideal I given in this theorem is equivalent to the property that \sqrt{I} is prime.

Finally we show that the hypothesis that minimal prime *l*-ideals be square dominated cannot be dropped or generalized in any way in any of our theorems characterizing pseudoprime *l*-ideals.

LEMMA 2.9. Let A be a semiprime f-ring. If in A, a pseudoprime l-ideal I is characterized by being an l-ideal that satisfies any one of the following conditions, then every minimal prime l-ideal of A is square dominated.

(1) $\bigcap_{n=1}^{\infty} \langle I^n \rangle$ is prime. (2) $\langle I \sqrt{I} \rangle$ is pseudoprime.

(3) $I: \sqrt{I}$ is pseudoprime and $I: \sqrt{I} \subseteq \sqrt{I}$, or, $\sqrt{I} \subseteq I: \sqrt{I}$ and \sqrt{I} is prime.

(4) The prime l-ideals containing I form a chain.

(5) \sqrt{I} is prime.

Also, if A is an archimedean f-ring, and if in A, a pseudoprime l-ideal I is characterized by being an l-ideal that satisfies the following condition, then every minimal prime l-ideal of A is square dominated.

(6) Whenever $\{f_{1j}\}_{j=1}^{\infty}, \ldots, \{f_{nj}\}_{j=1}^{\infty}$ are increasing positive Cauchy sequences such that $f_{1j}f_{2j}\cdots f_{nj}=0$ for all j, there is a sequence $\{f_{lj}\}_{j=1}^{\infty}$, for which there exists a positive integer N such that $f_{lj}^N \in I$ for all j.

PROOF. Let P be a minimal prime l-ideal. Note that in each case it will suffice to show that $\langle P^2 \rangle$ is pseudoprime since $\langle P^2 \rangle \subseteq P$ and P being a minimal prime *l*-ideal will then imply $\langle P^2 \rangle = P$.

If characterization (1) holds, then $\bigcap_{n=1}^{\infty} \langle P^n \rangle$ is a prime *l*-ideal contained in $\langle P^2 \rangle$. So $\langle P^2 \rangle$ is pseudoprime.

Characterization (2) implies $\langle P\sqrt{P}\rangle = \langle P^2\rangle$ is pseudoprime.

Suppose characterization (3) holds. Note that $\langle P^2 \rangle : \sqrt{\langle P^2 \rangle} = \langle P^2 \rangle : P \supseteq \sqrt{\langle P^2 \rangle} = P$. So $\langle P^2 \rangle : \sqrt{\langle P^2 \rangle}$ is pseudoprime. Then characterization (3) implies $\langle P^2 \rangle$ is pseudoprime.

Suppose characterization (4) holds. Every prime *l*-ideal containing $\langle P^2 \rangle$ also contains P. So the prime *l*-ideals containing $\langle P^2 \rangle$ form a chain. Characterization (4) implies $\langle P^2 \rangle$ is pseudoprime.

Suppose characterization (5) holds. Then $\sqrt{\langle P^2 \rangle} = P$ is prime, implying that $\langle P^2 \rangle$ is pseudoprime.

Finally, suppose that A is an archimedean f-algebra and that characterization (6) holds. Suppose that $\{f_{1j}\}_{j=1}^{\infty}, \ldots, \{f_{nj}\}_{j=1}^{\infty}$ are increasing positive Cauchy sequences such that $f_{1j}f_{2j}\cdots f_{nj}=0$ for all j. Then for some $m, f_{mj} \in P$ for all j. But then $f_{mj}^2 \in \langle P^2 \rangle$ for all j. So characterization (6) implies that $\langle P^2 \rangle$ is pseudoprime. \Box

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DEPARTMENT OF MATHEMATICS, LOYOLA MARYMOUNT UNIVERSITY, LOS ANGELES, CALIFORNIA 90045