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Pseudoprime L-Ideals in a Class of F-Rings

Suzanne Larson
Loyola Marymount University, slarson@lmu.edu

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PSEUDOPRIME l-IDEALS IN A CLASS OF f-RINGS

SUZANNE LARSON

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ABSTRACT. In a commutative f-ring, an l-ideal I is called pseudoprime if $ab = 0$ implies $a \in I$ or $b \in I$, and is called square dominated if for every $a \in I$, $|a| \leq x^2$ for some $x \in A$ such that $x^2 \in I$. Several characterizations of pseudoprime l-ideals are given in the class of commutative semiprime f-rings in which minimal prime l-ideals are square dominated. It is shown that the hypothesis imposed on the f-rings, that minimal prime l-ideals are square dominated, cannot be omitted or generalized.

Introduction. Let X be a topological space and $C(X)$ be the f-ring of all continuous real-valued functions on X with coordinatewise operations. The following characterizations of pseudoprime l-ideals of $C(X)$ are known.

(L. Gillman and C. Kohls [**4**, 4.1]) *For an l-ideal I of $C(X)$, the following are equivalent*:

(1) *I is pseudoprime.*

(2) *The prime ideals containing I form a chain.*

(3) *$\sqrt{I}$ is prime.*

In [**11**], Subramanian asks whether this characterization of pseudoprime l-ideals generalizes to semiprime f-rings. The answer in general is no, as can be seen by Example 2.7. In this work, we investigate pseudoprime l-ideals in the class of commutative semiprime f-rings in which minimal prime l-ideals are square dominated. In this class of f-rings, we give some alternate characterizations of pseudoprime l-ideals, and we show that in normal f-rings conditions (2) and (3) characterize pseudoprime l-ideals. We also show that if all prime l-ideals are square dominated, a generalization of condition (3) characterizes pseudoprime l-ideals in archimedean f-algebras. Finally, we show that the hypothesis imposed on our f-rings, that minimal prime l-ideals be square dominated, cannot be omitted or generalized in any way by showing that if any of the characterizations hold in a semiprime f-ring A, then all minimal prime l-ideals of A are square dominated. We assume throughout that all rings are commutative and semiprime.

1. Preliminaries. An f-ring is a lattice ordered ring which is a subdirect product of totally ordered rings. For general information on f-rings see [**2**]. Given an f-ring A and $x \in A$, we let $A^+ = \{a \in A\colon a \geq 0\}$, $x^+ = x \vee 0$, $x^- = (-x) \vee 0$, and $|x| = x \vee (-x)$.

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A ring ideal I of an f-ring A is an l-ideal if $|x| \leq |y|$, $y \in I$ implies $x \in I$. Given any subset $S \subset A$, there is a smallest l-ideal containing S, and we denote this by $\langle S \rangle$.

Suppose A is an f-ring and I, J are l-ideals of A. We let $I \colon J = \{a \in A \colon aJ \subseteq I\}$. The l-ideal I is semiprime (prime) if $a^2 \in I$ ($ab \in I$) implies $a \in I$ ($a \in I$ or $b \in I$). The f-ring A is semiprime (prime) if $\{0\}$ is a semiprime (prime) l-ideal. We let $\sqrt{I}$ denote $\{a \in A \colon a^n \in I \text{ for some } n\}$, the smallest semiprime l-ideal containing I. In [**5**, 3.5], it is shown that:

(1.1) If I is an l-ideal of a semiprime f-ring A, then $\bigcap_{n=1}^{\infty} \langle I^n \rangle$ is semiprime.

It is well known that all l-ideals containing a given prime l-ideal form a chain. The following result is also well known.

(1.2) A prime l-ideal P of a commutative semiprime f-ring is minimal if and only if $a \in P$ implies there is a $b \notin P$ such that $ab = 0$.

A subset M of an f-ring A is called an *m-system* if whenever $a, b \in M$ there exists an $x \in A$ such that $axb \in M$. If in A, there is an l-ideal I and an m-system M such that $I \cap M = \varnothing$, then there is a prime l-ideal P such that $I \subset P$, and $P \cap M = \varnothing$.

We call an ideal I *pseudoprime* if $ab = 0$ implies $a \in I$ or $b \in I$. In a semiprime f-ring A, a pseudoprime l-ideal contains a prime l-ideal as shown in [**11**, 2.1]. Also, in a commutative f-ring a pseudoprime and semiprime l-ideal is necessarily a prime l-ideal as shown in [**8**, 4.2] and [**11**, 2.5].

An ideal I of a commutative f-ring A with identity element is a *z-ideal* if whenever $a, b \in A$, are contained in the same set of maximal ideals and $a \in I$ then $b \in I$. In [**10**, 2.7], G. Mason shows

(1.3) In a commutative f-ring A, a z-ideal I is prime if and only if for all $a \in A$, either $a^+ \in I$ or $a^- \in I$.

Henriksen, in [**5**] calls an l-ideal I of an f-ring A square dominated if $I = \{a \in A \colon |a| \leq x^2 \text{ for some } x \in A \text{ such that } x^2 \in I\}$. Every prime square dominated l-ideal P satisfies $P = \langle P^2 \rangle$.

An f-ring (and more generally a Riesz space) A is called normal if $A = \{a^+\}^d + \{a^-\}^d$ for all $a \in A$, or equivalently if $a \wedge b = 0$ implies $A = \{a\}^d + \{b\}^d$. Several conditions equivalent to normality for an f-ring can be found in [**8**, 6.3], and for a Riesz space in [**7**, Theorem 9].

Given an f-ring A and an element $x > 0$ in A, the sequence $\{f_n\}_{n=1}^{\infty}$ is said to *converge x-uniformly* to the element $f \in A$ if for every $\varepsilon > 0$, there exists a positive integer N_ε such that $|f - f_n| < \varepsilon x$ for all $n \geq N_\varepsilon$. An x-uniform Cauchy sequence is defined similarly. If for every $x \geq 0$, every x-uniform Cauchy sequence has a unique limit, then A is said to be *uniformly complete*. We say A is *archimedean* if $a, b \in A^+$ with $na \leq b$ for all n implies $a = 0$. If A is an archimedean f-algebra with identity element, it is well known that A is commutative and semiprime. In [**1**, 4.1(d)], it is shown that:

(1.4) For any archimedean f-algebra A with identity element, there is an embedding e of A into a uniformly complete f-algebra A^* (the uniform completion of A).

2. In this section we give several characterizations of pseudoprime l-ideals in the class of commutative semiprime f-rings with minimal prime l-ideals square dominated. This class contains, of course, all commutative semiprime square root closed f-rings. First we give a lemma that will be used later and that also gives a characterization of commutative semiprime f-rings in which minimal prime l-ideals are square dominated.

LEMMA 2.1. *Let A be a commutative semiprime f-ring.*

(1) *A semiprime l-ideal I is square dominated if every prime l-ideal minimal with respect to containing I is square dominated.*

(2) *Every minimal prime l-ideal of A is square dominated if and only if for every $a \in A^+$, the l-ideal $\{a\}^d = \{b \in A \colon ab = 0\}$ is square dominated.*

PROOF. (1) Let $b \in I^+$. Let $M = \{c_1^2 \cdots c_n^2 \colon n \in \mathbf{N};\ b \leq c_i^2\}$. Then M is an m-system. Suppose that $M \cap I = \varnothing$. Then there is a prime l-ideal P such that $I \subseteq P$ and $M \cap P = \varnothing$. Let $P_1 \subseteq P$ be a prime l-ideal minimal with respect to containing I. By hypothesis, P_1 is square dominated, so there is a $p \in A$ such that $b \leq p^2$ and $p^2 \in P_1$. But then $p^2 \in M \cap P$, contrary to assumption. So $M \cap I \neq \varnothing$. Let $c_1^2 \cdots c_n^2 \in M \cap I$, where $b \leq c_i^2$ for each i. Then $b \leq c_1^2 \wedge \cdots \wedge c_n^2 = (|c_1| \wedge \cdots \wedge |c_n|)^2$. Also, $0 \leq (|c_1| \wedge \cdots \wedge |c_n|)^{2n} \leq c_1^2 \cdots c_n^2$. This implies $(|c_1| \wedge \cdots \wedge |c_n|)^{2n} \in I$, and because I is semiprime, $(|c_1| \wedge \cdots \wedge |c_n|)^2 \in I$.

(2) $\Rightarrow$ Suppose that every minimal prime l-ideal is square dominated. Let $a \in A^+$. Suppose P is a prime l-ideal minimal with respect to containing $\{a\}^d$. Then $M = \{b \colon b \in A \backslash P\} \cup \{a^n \colon n \in \mathbf{N}\} \cup \{ba^n \colon b \in A\backslash P,\ n \in \mathbf{N}\}$ is an m-system such that $M \cap \{a\}^d = \varnothing$. So there is a prime l-ideal P_1 satisfying $\{a\}^d \subseteq P_1 \subseteq P$. But our choice of P implies $P_1 = P$ and $a \notin P$.

Now if P_2 is a minimal prime l-ideal contained in P, $a \notin P_2$ implies $\{a\}^d \subseteq P_2$. Hence $P_2 = P$, and P is in fact a minimal prime l-ideal which is square dominated. So every prime l-ideal minimal with respect to containing $\{a\}^d$ is square dominated, and part (1) implies $\{a\}^d$ is square dominated.

$\Leftarrow$ Let P be a minimal prime l-ideal, and $f \in P$. By 1.2, there is a $g \notin P$ such that $fg = 0$. By hypothesis, $\{g\}^d$ is square dominated. So there is $f_1 \in \{g\}^d$ such that $f \leq f_1^2$ and $f_1^2 \in P$. $\square$

Our first characterization of pseudoprime l-ideals follows.

THEOREM 2.2. *Let A be a commutative, semiprime f-ring with identity element and in which every minimal prime l-ideal is square dominated. The following are equivalent for an l-ideal I:*

(1) *I is pseudoprime.*

(2) *$\bigcap_{n=1}^{\infty} \langle I^n \rangle$ is prime.*

(3) *$\langle I\sqrt{I} \rangle$ is pseudoprime.*

(4) *$I : \sqrt{I}$ is pseudoprime and $I : \sqrt{I} \subseteq \sqrt{I}$, or, $\sqrt{I} \subseteq I : \sqrt{I}$ and $\sqrt{I}$ is prime.*

PROOF. (1) $\Rightarrow$ (2). Let P be a minimal prime l-ideal contained in I. Since P is square dominated, $P = \bigcap_{n=1}^{\infty} \langle P^n \rangle \subseteq \bigcap_{n=1}^{\infty} \langle I^n \rangle$. So $\bigcap_{n=1}^{\infty} \langle I^n \rangle$ is pseudoprime. By 1.1, it is also semiprime and therefore prime.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (4). By (3), all l-ideals containing $\langle I\sqrt{I} \rangle$ form a chain. So $I : \sqrt{I} \subseteq \sqrt{I}$ or $\sqrt{I} \subseteq I : \sqrt{I}$.

(4) $\Rightarrow$ (1). Either hypothesis implies that there is a minimal prime l-ideal P contained in $\sqrt{I}$ and also in $I : \sqrt{I}$. Then since P is square dominated, $P = \langle P^2 \rangle \subseteq (I : \sqrt{I})\sqrt{I} \subseteq I$. $\square$

Recall that in a commutative ring, an ideal I is *primary* if $ab \in I$, $a \notin I$ implies $b^n \in I$ for some n. It is well known that $\sqrt{I}$ is prime for every primary l-ideal I. Thus in $C(X)$, every primary l-ideal is pseudoprime. This is not true in general (as can be seen by Example 2.7) and the following corollary gives a condition under which primary l-ideals are pseudoprime.

COROLLARY 2.3. *Let A be a commutative semiprime f-ring with identity element in which every minimal prime l-ideal is square dominated. A primary l-ideal I is pseudoprime if and only if $I : \sqrt{I}$ is pseudoprime.*

PROOF. Assume that $I : \sqrt{I}$ is pseudoprime. If $I = \sqrt{I}$, then I is prime, so we may assume that $I \neq \sqrt{I}$. Let $a \in \sqrt{I}\backslash I$. We will show $I : \sqrt{I} \subseteq \sqrt{I}$. Suppose $b \in I : \sqrt{I}$. Then $ab \in I$, and since $a \notin I$, we must have $b \in \sqrt{I}$. So $I : \sqrt{I} \subseteq \sqrt{I}$. It follows from the previous theorem that I is pseudoprime. □

It is well known that in $C(X)$, an l-ideal I is pseudoprime if and only if $\sqrt{I}$ is prime [**4**, 4.1]. In [**11**], Subramanian asks whether this characterization of pseudoprime l-ideals holds in semiprime f-rings. The answer in general is no (even in archimedean f-rings), as witnessed by Example 2.7. However, our next goal is to show that the characterization of pseudoprime l-ideals as those l-ideals I for which $\sqrt{I}$ is prime also holds in a class of normal f-rings.

First, we will give two characterizations of normal f-rings, one of which is the f-ring analogue to a characterization given by Huijsmans in [**7**]. It should also be noted that the f-ring $C(X)$ is normal if and only if the topological space X is an F-space and that the characterizations of normal f-rings given next are similar to two characterizations of F-spaces given in [**3**, 14.25]. If P is any prime ideal, the P component of 0 is $O_P = \{a \in A\colon \exists b \notin P \text{ such that } ab = 0\}$.

THEOREM 2.4. *Let A be a commutative semiprime f-ring with identity element. The following are equivalent.*

(1) *A is normal.*

(2) *Every ideal O_P, where P is a (proper) prime l-ideal, is prime.*

(3) *Every ideal O_M, where M is a maximal l-ideal, is prime.*

PROOF. (1) $\Rightarrow$ (2). Since O_P is a z-ideal, 1.3 implies that it will suffice to show that for all $a \in A$, either $a^+ \in O_P$ or $a^- \in O_P$. Suppose $a \in A$. If $a^+ \notin O_P$ and $a^- \notin O_P$, then $\{a^+\}^d \subseteq P$ and $\{a^-\}^d \subseteq P$. But this would imply $A = \{a^+\}^d + \{a^-\}^d \subseteq P$, contrary to hypothesis.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). Suppose A is not normal. Then there is an element $a \in A$ such that $\{a^+\}^d + \{a^-\}^d \neq A$. Let M be a maximal l-ideal containing the l-ideal $\{a^+\}^d + \{a^-\}^d$. Then O_M is a z-ideal such that $a^+ \notin O_M$ and $a^- \notin O_M$. By 1.3, O_M is not prime, contrary to our hypothesis. □

This characterization allows us to make the following observation.

LEMMA 2.5. *If A is a commutative, semiprime normal f-ring with identity element then every minimal prime l-ideal of A is square dominated.*

PROOF. Let P be a minimal prime l-ideal of A. Since the l-ideals containing P form a chain, there is a unique maximal l-ideal M containing P. Now $O_M \subseteq P$ and, by the previous theorem O_M is prime. Therefore, $O_M = P$. We will show O_M

is square dominated. Let $a \in O_M^+$. Then there exists a $b > 0$ such that $b \notin M$ and $ab = 0$. Then $a \wedge b = 0$ and $\{a\}^d + \{b\}^d = A$. Thus, there exists $x \in \{a\}^d$, $y \in \{b\}^d$ such that $x + y = a \vee 1$. It follows from $y \in \{b\}^d$ and $b \notin M$, that $y \in O_M$. We have $y \in O_M$, and $a \leq y^2$. □

In [**9**, 3.6] it is shown that if I is an l-ideal of a commutative semiprime f-ring with identity element such that $I = I : \sqrt{I}$, then I is an intersection of primary l-ideals. We will use this fact in the proof of the next theorem.

THEOREM 2.6. *Let A be a commutative semiprime normal f-ring with identity element. The following are equivalent for an l-ideal I.*

(1) *I is pseudoprime.*

(2) *The prime l-ideals containing I form a chain.*

(3) *$\sqrt{I}$ is prime.*

PROOF. We need only show (3) $\Rightarrow$ (1). Suppose that $\sqrt{I}$ is prime. Let P be a minimal prime l-ideal contained in $\sqrt{I}$ and $J = P \cap I$. We will show that J is pseudoprime. Knowing that $\sqrt{J}$ is square dominated (Lemma 2.5), it is not hard to show that $J : \sqrt{J} = (J : \sqrt{J}) : \sqrt{J : \sqrt{J}}$. Let M be the maximal l-ideal containing P. For any $x \in M \backslash J : \sqrt{J}$, there is a primary l-ideal Q_x containing $J : \sqrt{J}$, but not x by the result mentioned above [**9**, 3.6]. The l-ideals containing P form a chain, so $P = \sqrt{J} \subseteq \sqrt{J : \sqrt{J}} \subseteq \sqrt{Q_x} \subseteq M$. Knowing $Q_x \subseteq M$, it is easy to show that $O_M \subseteq Q_x$ for all x. Now $J : \sqrt{J} = M \cap (\bigcap Q_x)$, so $O_M \subseteq J : \sqrt{J}$. By Theorem 2.4, O_M is a prime l-ideal. Thus $J : \sqrt{J}$ is a pseudoprime l-ideal. Also, since the l-ideals containing O_M form a chain, we have $J : \sqrt{J} \subseteq \sqrt{J}$ or $\sqrt{J} \subseteq J : \sqrt{J}$. In either case, Theorem 2.1 now implies that J is pseudoprime. □

In particular, this result implies that in an f-ring satisfying the hypotheses of the theorem, every primary l-ideal is pseudoprime.

The next example shows that the hypothesis of normality cannot be left out of this theorem. It also shows that the characterization of pseudoprime l-ideals as being those I for which $\sqrt{I}$ is prime does not hold in archimedean f-algebras. In fact, primary l-ideals in archimedean f-algebras are not necessarily pseudoprime.

EXAMPLE 2.7. Let $B = \{f \in C[0,1]\colon \exists x_f \in (0,1)$ such that $f(x) = \sum_{i=1}^n a_i x^{r_i}$, where $a_i \in \mathbf{R}$, $r_i \in \mathbf{Q}$ for all $x \in [0, x_f]\}$. Let $P = \{f \in B\colon f(0) = 0\}$. Then P is a prime l-ideal of B. Let $A = \{(f,g) \in B \times B\colon f - g \in P\}$. Then as shown in [**6**], A is a semiprime archimedean f-algebra with identity element. It is not hard to show that every prime l-ideal of A is square dominated.

Let $I = \{(f,g) \in A\colon f(x), g(x) \leq nx^2$ for some n, $\forall x \in [0, x_f \wedge x_g]\}$. Then I is an l-ideal of A. We will show that I is primary. Suppose $(f,g)(h,k) \in I$, and $(f,g) \notin I$. For every n, either $f(x) \not\leq nx^2$ or $g(x) \not\leq nx^2$ on $[0,a)$ for any $a \in (0,1)$. Suppose $f(x) \not\leq nx^2$. Then $h(0) = 0$ and $h \in P$. But $h - k \in P$, so $k \in P$. This implies there must exist some N such that $(h,k)^N \in I$. Hence I is primary, and $\sqrt{I}$ is prime. Yet I is not pseudoprime since $(x,0)(0,x) = (0,0)$ while $(x,0) \notin I$ and $(0,x) \notin I$.

To see directly that A is not normal, consider the element $(x,-x)$: $\{(x,-x)^+\}^d + \{(x,-x)^-\}^d = \{(x,0)\}^d + \{(0,x)\}^d = \{(f,g)\colon f,g \in P\} \neq A$. □

We have found some generalizations of the condition that $\sqrt{I}$ be prime that characterize pseudoprime l-ideals in archimedean f-algebras in which minimal prime l-ideals are square dominated. However, each of these conditions are difficult to verify for most l-ideals. If we strengthen our hypotheses to insist that all prime l-ideals of the f-algebra are square dominated, we can give the following generalization to the condition which is a characterization of pseudoprime l-ideals.

THEOREM 2.8. *Let A be an archimedean f-algebra with identity element in which prime l-ideals are square dominated. If I is an l-ideal then I is pseudoprime if and only if when $\{f_{1j}\}_{j=1}^{\infty}, \dots, \{f_{nj}\}_{j=1}^{\infty}$ are increasing positive Cauchy sequences such that $f_{1j}f_{2j}\cdots f_{nj} = 0$ for all j, there is a sequence $\{f_{lj}\}_{j=1}^{\infty}$, for which there exists a positive integer N such that $f_{lj}^N \in I$ for all j.*

PROOF. Let A^* denote the uniform completion of A, and $e\colon A \to A^*$ be the embedding.

$\Leftarrow$ Suppose that whenever $\{f_{1j}\}_{j=1}^{\infty}, \dots, \{f_{nj}\}_{j=1}^{\infty}$ are increasing positive Cauchy sequences such that $f_{1j}f_{2j}\cdots f_{nj} = 0$ for all j, there is a sequence $\{f_{lj}\}_{j=1}^{\infty}$, for which there exists a positive integer N such that $f_{lj}^N \in I$ for all j. In A^*, let $M = \{x_1 \cdots x_m\colon$ for each x_i, $\forall j \exists a_{ij} \in A$ such that $0 \le e(a_{ij}) \le x_i^j$ and $a_{ij}^j \notin I\}$. Then M is an m-system. First we will show that $M \cap \{0\} = \varnothing$. Suppose that for $i = 1, 2, \dots, m$, $x_i \in A^*$, such that for every $j \in \mathbf{N}$, there is an $a_{ij} \in A$ such that $0 \le e(a_{ij}) \le x_i^j$ and $a_{ij}^j \notin I$. Define

$$f_{ij} = \sum_{k=1}^{j} \frac{1}{2^k}(a_{ik} \wedge 1) \quad \text{for } i = 1, 2, \dots, m,\ j = 1, 2, \dots.$$

Then $0 \le (1/2^j)e(a_{ij} \wedge 1) \le e(f_{ij}) \le x_i$, for $i = 1, 2, \dots, m$, $j = 1, 2, \dots$, and $\{f_{ij}\}_{j=1}^{\infty}$ is an increasing positive Cauchy sequence for $i = 1, 2, \dots, m$. Now for each sequence $\{f_{ij}\}_{j=1}^{\infty}$, there cannot be a natural number N such that $f_{ij}^N \in I$ for all j, because the existence of such an N would imply that $(a_{ij} \wedge 1)^N$, and hence a_{ij}^N, is in I for all j. So by hypothesis, $f_{1j}f_{2j}\cdots f_{mj} \neq 0$ for some j. This, and the fact that $0 \le e(f_{1j})e(f_{2j})\cdots e(f_{mj}) \le x_1x_2\cdots x_m$ implies $x_1x_2\cdots x_m \neq 0$. Therefore, $M \cap \{0\} = \varnothing$.

Thus there is a prime l-ideal in A^* containing $\{0\}$ and disjoint from M. Let P be a minimal prime l-ideal with this property. We will show $P \cap e(A) \subseteq e(I)$. Let $f \in A^+$ such that $e(f) \in P \cap e(A)$.

By 1.2, there is a $g \notin P$ such that $e(f)g = 0$. In A, $\{g\}_A^d = \{x \in A\colon xg = 0\}$ is a semiprime l-ideal and Lemma 2.1(1) implies it is square dominated. So $f \le f_1^2$ for some $f_1 \in A^+$ with $f_1^2 \in \{g\}_A^d$. Since $\{g\}_A^d$ is semiprime, $f_1 \in \{g\}_A^d$. As before, $f_1 \le f_2^2$ for some $f_2 \in A^+$ with $f_2^2 \in \{g\}_A^d$. Again $f_2 \in \{g\}_A^d$. So $f \le f_1^2 \le f_2^4$. Continuing this for $i = 3, 4, \dots$, we find $f \le f_1^2 \le f_2^4 \le \cdots \le f_i^{2^i} \le \cdots$. Define

$$h_j = \sum_{i=1}^{j} \frac{1}{2^i}(f_i \wedge 1) \quad \text{for } j = 1, 2, \dots.$$

Then $\{h_j\}_{m=1}^{\infty}$ is an increasing Cauchy sequence in A and $\{e(h_j)\}_{j=1}^{\infty}$ converges to an element $h \in A^*$. Now $hg = 0$ and $g \notin P$, implying $h \in P$.

We assert that there exists some positive integer M such that

$$(1/2^M)(f_M \wedge 1)^{2^M} \in I.$$

For if not, $0 \leq (1/2^i)(f_i \wedge 1)^{2^i} \leq (f_i \wedge 1)^{i^2}$ would imply $(f_i \wedge 1)^{i^2} = ((f_i \wedge 1)^i)^i \notin I$ for all $i \neq 3$. We would then have

$$0 \leq (1/2^{i^2})(f_i \wedge 1)^i \leq h^i \quad \text{and} \quad ((1/2^{i^2})(f_i \wedge 1)^i)^i \notin I \quad \text{for all } i.$$

But this would imply $h \in M$, contrary to the fact that $M \cap P = \varnothing$. So there exists some positive integer M such that $(1/2^M)(f_M \wedge 1)^{2^M} \in I$.

We now know $(1/2^M)(f \wedge 1) \leq (1/2^M)(f_M \wedge 1)^{2^M}$ implies $(1/2^M)(f \wedge 1) \in I$. Therefore $(f \wedge 1) \in I$ and $f = (f \wedge 1)(f \vee 1) \in I$. We have $P \cap e(A) \subseteq e(I)$ and $P \cap e(A)$ is prime in $e(A)$. Therefore $e(I)$ is pseudoprime in A^*. This implies I is pseudoprime in A.

$\Rightarrow$ Suppose that I is pseudoprime and that $\{f_{1j}\}_{j=1}^{\infty}, \ldots, \{f_{nj}\}_{j=1}^{\infty}$ are increasing positive Cauchy sequences in A with $f_{1j}f_{2j}\cdots f_{nj} = 0$ for all j. Each of the sequences $\{e(f_{ij})\}_{j=1}^{\infty}$ converges to some element f_i in A^*. Let P be a prime l-ideal contained in I. Then $M = \{e(a) \colon a \in A,\ a \notin P\}$ is an m-system in A^*. So there is a prime l-ideal P^* of A^* such that $P \subseteq P^*$ and $P^* \cap M = \varnothing$. Now $e(f_1)e(f_2)\cdots e(f_n) = 0$ and so $e(f_m) \in P^*$ for some m. Thus $e(f_{mj}) \leq e(f_m)$ implies that $e(f_{mj}) \in P^* \cap e(A) = e(P)$ for all j. □

It is not difficult to show directly that in a uniformly complete f-ring, the property characterizing a pseudoprime l-ideal I given in this theorem is equivalent to the property that $\sqrt{I}$ is prime.

Finally we show that the hypothesis that minimal prime l-ideals be square dominated cannot be dropped or generalized in any way in any of our theorems characterizing pseudoprime l-ideals.

LEMMA 2.9. *Let A be a semiprime f-ring. If in A, a pseudoprime l-ideal I is characterized by being an l-ideal that satisfies any one of the following conditions, then every minimal prime l-ideal of A is square dominated.*

(1) *$\bigcap_{n=1}^{\infty}\langle I^n\rangle$ is prime.*

(2) *$\langle I\sqrt{I}\rangle$ is pseudoprime.*

(3) *$I : \sqrt{I}$ is pseudoprime and $I : \sqrt{I} \subseteq \sqrt{I}$, or, $\sqrt{I} \subseteq I : \sqrt{I}$ and $\sqrt{I}$ is prime.*

(4) *The prime l-ideals containing I form a chain.*

(5) *$\sqrt{I}$ is prime.*

Also, if A is an archimedean f-ring, and if in A, a pseudoprime l-ideal I is characterized by being an l-ideal that satisfies the following condition, then every minimal prime l-ideal of A is square dominated.

(6) *Whenever $\{f_{1j}\}_{j=1}^{\infty}, \ldots, \{f_{nj}\}_{j=1}^{\infty}$ are increasing positive Cauchy sequences such that $f_{1j}f_{2j}\cdots f_{nj} = 0$ for all j, there is a sequence $\{f_{lj}\}_{j=1}^{\infty}$, for which there exists a positive integer N such that $f_{lj}^N \in I$ for all j.*

PROOF. Let P be a minimal prime l-ideal. Note that in each case it will suffice to show that $\langle P^2\rangle$ is pseudoprime since $\langle P^2\rangle \subseteq P$ and P being a minimal prime l-ideal will then imply $\langle P^2\rangle = P$.

If characterization (1) holds, then $\bigcap_{n=1}^{\infty}\langle P^n\rangle$ is a prime l-ideal contained in $\langle P^2\rangle$. So $\langle P^2\rangle$ is pseudoprime.

Characterization (2) implies $\langle P\sqrt{P}\rangle = \langle P^2\rangle$ is pseudoprime.

Suppose characterization (3) holds. Note that $\langle P^2\rangle : \sqrt{\langle P^2\rangle} = \langle P^2\rangle : P \supseteq \sqrt{\langle P^2\rangle} = P$. So $\langle P^2\rangle : \sqrt{\langle P^2\rangle}$ is pseudoprime. Then characterization (3) implies $\langle P^2\rangle$ is pseudoprime.

Suppose characterization (4) holds. Every prime l-ideal containing $\langle P^2\rangle$ also contains P. So the prime l-ideals containing $\langle P^2\rangle$ form a chain. Characterization (4) implies $\langle P^2\rangle$ is pseudoprime.

Suppose characterization (5) holds. Then $\sqrt{\langle P^2\rangle} = P$ is prime, implying that $\langle P^2\rangle$ is pseudoprime.

Finally, suppose that A is an archimedean f-algebra and that characterization (6) holds. Suppose that $\{f_{1j}\}_{j=1}^{\infty}, \dots, \{f_{nj}\}_{j=1}^{\infty}$ are increasing positive Cauchy sequences such that $f_{1j}f_{2j}\cdots f_{nj} = 0$ for all j. Then for some m, $f_{mj} \in P$ for all j. But then $f_{mj}^2 \in \langle P^2\rangle$ for all j. So characterization (6) implies that $\langle P^2\rangle$ is pseudoprime. □

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DEPARTMENT OF MATHEMATICS, LOYOLA MARYMOUNT UNIVERSITY, LOS ANGELES, CALIFORNIA 90045