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Sums of Semiprime, Z, and D L-Ideals in a Class of F-Rings

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SUMS OF SEMIPRIME, z, AND d l-IDEALS IN A CLASS OF f-RINGS

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ABSTRACT. In this paper it is shown that there is a large class of f-rings in which the sum of any two semiprime l-ideals is semiprime. This result is used to give a class of commutative f-rings with identity element in which the sum of any two z-ideals which are l-ideals is a z-ideal and the sum of any two d-ideals is a d-ideal.

INTRODUCTION

An l-ideal I of an f-ring A is called semiprime if $a^2 \in I$ implies $a \in I$. An ideal I of a commutative ring A with identity element is called a z-ideal if whenever $a, b \in A$ are in the same set of maximal ideals and $a \in I$, then $b \in I$. Given an element a of an f-ring A, let $\{a\}^d = \{x \in A: xa = 0\}$ and $\{a\}^{dd} = \{x \in A: xy = 0 \text{ for all } y \in \{a\}^d\}$. An ideal I of a commutative f-ring is called a d-ideal if $a \in I$ implies $\{a\}^{dd} \subseteq I$.

Several authors have studied the sums of semiprime l-ideals, z-ideals, and d-ideals in various classes of f-rings. In [2, 14.8] it is shown that the sum of two z-ideals in $C(X)$, the f-ring of all real-valued continuous functions defined on the topological space X, is a z-ideal; and in [11, 4.1 and 5.1], Rudd shows that in absolutely convex subrings of $C(X)$, the sum of two semiprime l-ideals is semiprime and the sum of two z-ideals is a z-ideal. Mason studies sums of z-ideals in absolutely convex subrings of the ring of all continuous functions on a topological space and in more general rings in [10]. An example is given in [5, §7] of an f-ring in which there are two (semiprime) z-ideals whose sum is not a z-ideal or semiprime. Huijsmans and de Pagter show in [7, 4.4] that in a normal Riesz space, the sum of two d-ideals is a d-ideal.

In [4, 3.9], Henriksen gives a condition on two semiprime l-ideals of an f-ring which is necessary and sufficient for their sum to be semiprime. Henriksen also notes in [4] that this condition can be difficult to apply globally, and so it seems difficult to use this result to determine in what classes of f-rings are the sum of any two semiprime l-ideals semiprime.

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In this note we show that there is a large class of f-rings, specifically those f-rings in which minimal prime l-ideals are square dominated, in which the sum of any two semiprime l-ideals is semiprime. We use this result to show that in a commutative f-ring with identity element in which minimal prime l-ideals are square dominated, if the sum of any two minimal prime l-ideals is a z-ideal (resp. d-ideal), then the sum of any two z-ideals which are l-ideals is a z-ideal (resp. the sum of any two d-ideals is a d-ideal). As a corollary we show that in a commutative semiprime normal f-ring with identity element, the sum of any two z-ideals which are l-ideals (resp. d-ideals) is a z-ideal (resp. d-ideal).

1. Preliminaries

An f-ring is a lattice-ordered ring which is a subdirect product of totally ordered rings. For general information on f-rings see [1]. Given an f-ring A and $x \in A$, we let $A^+ = \{a \in A\colon a \geq 0\}$, $x^+ = x \vee 0$, $x^- = (-x) \vee 0$, and $|x| = x \vee (-x)$.

A ring ideal I of an f-ring A is an l-ideal if $|x| \leq |y|$, $y \in I$ implies $x \in I$. Given any element $a \in A$ there is a smallest l-ideal containing a, and we denote this by $\langle a \rangle$.

Suppose A is an f-ring and I is an l-ideal of A. The l-ideal I is *semiprime* (*prime*) if $a^2 \in I$ $(ab \in I)$ implies $a \in I$ $(a \in I$ or $b \in I)$. It is well known that in an f-ring, an l-ideal is semiprime if and only if it is an intersection of prime l-ideals, and that all l-ideals containing a given prime l-ideal form a chain. In an f-ring a semiprime l-ideal that contains a prime l-ideal is a prime l-ideal as shown in [12, 2.5], [7, 4.2].

An ideal I of a commutative ring A with identity element is a z-ideal if, whenever $a, b \in A$ are contained in the same set of maximal ideals and $a \in I$, then $b \in I$.

In a commutative f-ring, let $\{a\}^d = \{x \in A\colon ax = 0\}$ and $\{a\}^{dd} = \{x \in A\colon xy = 0$ for all $y \in \{a\}^d\}$. An ideal I of a commutative f-ring is called a d-ideal if $a \in I$ implies $\{a\}^{dd} \subseteq I$.

Henriksen, in [4] calls an l-ideal I of an f-ring A *square dominated* if $I = \{a \in A\colon |a| \leq x^2$ for some $x \in A$ such that $x^2 \in I\}$. Two characterizations now follow describing those commutative semiprime f-rings in which all minimal prime l-ideals are square dominated. Parts (1) and (2) of the following lemma are shown to be equivalent in [8, 2.1]. That part (3) of the following lemma is equivalent to part (1) follows easily from the equivalence of parts (1) and (2).

Lemma 1.1. *Let A be a commutative semiprime f-ring. The following are equivalent*:

(1) *Every minimal prime l-ideal of A is square dominated*
(2) *For every $a \in A^+$ the l-ideal $\{a\}^d$ is square dominated*

(3) *The l-ideal* $O_P = \{a \in A\colon$ *there exists a* $b \notin P$ *such that* $ab = 0\}$ *is square dominated for all prime l-ideals* P *of* A.

2.

We begin with two results that will be needed when showing that in an f-ring in which minimal prime l-ideals are square dominated, the sum of two semiprime l-ideals is semiprime.

Theorem 2.1. *Let* A *be an* f*-ring. In* A*, the sum of a semiprime l-ideal and a square-dominated semiprime l-ideal is semiprime.*

Proof. Suppose that I, J are semiprime l-ideals and that J is square dominated. Let $a^2 \in I + J$ with $a \geq 0$. Then $a^2 \leq i + j$ for some $i \in I^+$, $j \in J^+$. Since J is square dominated, $j \leq j_1^2$ for some $j_1 \in A^+$ with $j_1^2 \in J$. So $a^2 \leq i + j_1^2$. Let $x = a - (a \wedge j_1)$ and $y = a \wedge j_1$. Since J is a semiprime l-ideal, $j_1 \in J$ and $y \in J$. Now for any positive elements a, j_1 of any totally ordered ring, $a \wedge j_1 = a$ or $a \wedge j_1 = j_1$. In the first case $(a - (a \wedge j_1))^2 = 0$, and in the second case $(a - (a \wedge j_1))^2 = (a - j_1)^2 = a^2 - aj_1 - j_1 a + j_1^2 \leq a^2 - 2j_1^2 + j_1^2 = a^2 - j_1^2$. Therefore in any totally ordered ring, $(a - (a \wedge j_1))^2 \leq 0 \vee (a^2 - j_1^2)$. This implies that in the f-ring A, $x^2 = (a - (a \wedge j_1))^2 \leq 0 \vee (a^2 - j_1^2) \leq i$. Thus, $x^2 \in I$ and hence $x \in I$. We have $a = x + y$ with $x \in I$ and $y \in J$. Therefore $a \in I + J$. □

Lemma 2.2. *Let* A *be an* f*-ring in which minimal prime l-ideals are square dominated. In* A*, the sum of any two prime l-ideals is prime.*

Proof. Let I and J be prime l-ideals of A. Let I_1, J_1 be minimal prime l-ideals contained in I, J respectively. We will show $I + J$ is an intersection of prime l-ideals. To do so, we let $z \in A$ such that $z \notin I + J$ and we will show there is a prime l-ideal containing $I + J$ but not z. The l-ideal $I_1 + J_1$ is prime, and the prime l-ideals containing it form a chain. By the maximal principle, there is a prime l-ideal Q containing $I_1 + J_1$ which is maximal with respect to not containing z. By the previous theorem, $I + J_1$ is semiprime. It also contains a prime l-ideal and is therefore prime. Similarly, $I_1 + J$ is prime. Thus $I \subseteq I + J_1 \subseteq Q$ and $J \subseteq I_1 + J \subseteq Q$. This implies that $I + J \subseteq Q$ and $z \notin Q$. Therefore $I + J$ is an intersection of prime l-ideals. So it is semiprime. It also contains a prime l-ideal and is therefore prime. □

In [3, 4.7], Gillman and Kohls show that in $C(X)$, the f-ring of all real-valued continuous functions defined on the topological space X, an l-ideal is an interesection of l-ideals, each of which contains a prime l-ideal. Their proof easily generalizes to prove that in an f-ring, an l-ideal which contains all nilpotent elements of the f-ring is an intersection of l-ideals, each of which contains a prime l-ideal. We will make use of this result in the proof of the following theorem.

Theorem 2.3. *Let A be an f-ring in which minimal prime l-ideals are square dominated. In A, the sum of any two semiprime l-ideals is semiprime.*

Proof. Let I, J be semiprime l-ideals. We will show $I + J$ is an intersection of prime l-ideals. To do so, we let $z \in A$ such that $z \notin I + J$ and we show that there is a prime l-ideal containing $I + J$ but not z. By Gillman and Kohl's result mentioned above, there is an l-ideal Q containing $I + J$ and containing a prime l-ideal but not containing z. Let P be a minimal prime l-ideal contained in Q. By Theorem 2.1, $P + I$ is semiprime. Also, it contains a prime l-ideal and so is prime. Similarly, $P + J$ is prime. Then by the previous lemma, $(P + I) + (P + J)$ is prime. Since $(P + I) + (P + J) \subseteq Q$, $z \notin (P + I) + (P + J)$ and $I + J \subseteq (P + I) + (P + J)$. $\square$

The converse of the previous theorem does not hold, as we show next.

Example 2.4. In $C([0, 1])$, denote by i the function $i(x) = x$, and by e the function $e(x) = 1$. Let $A = \{f \in C([0, 1]) : f = ae + g \text{ where } a \in \mathbf{R},\ g \in \langle i \rangle\}$ with coordinate operations. Then A is a commutative semiprime f-ring.

We will show that the sum of two semiprime l-ideals of A is semiprime. So suppose I, J are semiprime l-ideals. If I or J contains an element $f = ae+g$ such that $a \neq 0$, then it can be shown that I or J is square dominated. Then by Theorem 2.1, $I + J$ is semiprime. So we may now suppose that both $I, J \subseteq \langle i \rangle$. If $f^2 \in I + J$, then there is $i_1 \in I^+$, $j_1 \in J^+$ such that $f^2 = i_1 + j_1$. Also, $f \in \langle i \rangle$ which implies $|f| \leq ni$ and $f^2 \leq n^2 i^2$ for some $n \in \mathbf{N}$. So $i_1 \leq n^2 i^2$ and $j_1 \leq n^2 i^2$. Therefore $\sqrt{i_1} \leq ni$ and $\sqrt{i_1} \in A$. Since I is semiprime, $\sqrt{i_1} \in I$. Similarly, $\sqrt{j_1} \in J$. So $f \leq \sqrt{i_1} + \sqrt{j_1}$ implies $f \in I + J$. Thus $I + J$ is semiprime.

Next we show that not every minimal prime l-ideal of A is square dominated. Let f be a function such that $0 \leq f \leq i$, $f(x) = 0$ for all $x \in [1/4, 1]$, $f(x) = 0$ for all $x \in [1/(4n + 2), 1/4n]$, and $f(1/(4n + 3)) = 1/(4n + 3)$ for all $n \in \mathbf{N}$. Also, let g be a function such that $0 \leq g \leq i$, $g(x) = 0$ for all $x \in [1/4, 1]$, $g(1/(4n + 1)) = 1/(4n + 1)$, and $g(x) = 0$ for all $x \in [1/(4n + 4), 1/(4n + 2)]$ for all $n \in \mathbf{N}$. Then $g \in \{f\}^d$, and there is no element $h \in A$ which satisfies $g \leq h^2$ and $h^2 \in \{f\}^d$. So $\{f\}^d$ is not square dominated, and Lemma 1.1 implies that not every minimal prime l-ideal of A is square dominated.

Next we turn our attention to the sum of two z-ideals which are l-ideals and to the sum of two d-ideals. Note that in a commutative f-ring with identity element, a z-ideal is not always an l-ideal. However it can easily be seen that in a commutative f-ring with identity element, if every maximal ideal is an l-ideal (or equivalently if for all $x \geq 1$, x^{-1} exists), then a z-ideal is always an l-ideal. In a commutative f-ring with identity element, every d-ideal is an l-ideal. G. Mason has established three results concerning z-ideals which we will use in the proof of the next theorem. The first is as follows.

(α) In a commutative ring with identity element, every z-ideal is semiprime [9, 1.0].

The second was proven for a commutative ring with identity element, and the third was proven for a commutative ring with identity element in which the prime ideals containing a given prime form a chain. With only very slight modifications to the proofs, these results can be given in the context of f-rings.

(β) If, in a commutative f-ring with identity element, P is minimal in the class of prime l-ideals containing a z-ideal I which is an l-ideal, then P is also a z-ideal. In particular, minimal prime l-ideals are z-ideals [9, 1.1].

(γ) If, in a commutative f-ring with identity element, the sum of any two minimal prime l-ideals is a prime z-ideal, then the sum of any two prime l-ideals not in a chain is a z-ideal [10, 3.2].

One can easily mimic the proofs to (β) and (γ) to show analogous results about d-ideals.

(β') If, in a commutative f-ring with identity element, P is minimal in the class of prime l-ideals containing a d-ideal I, then P is also a d-ideal. In particular, minimal prime l-ideals are d-ideals.

(γ') If, in a commutative f-ring with identity element, the sum of any two minimal prime l-ideals is a prime d-ideal, then the sum of any two prime l-ideals which are not in a chain is a d-ideal.

Theorem 2.5. *Let A be a commutative f-ring with identity element in which minimal prime l-ideals are square dominated.*

(1) *If the sum of any two minimal prime l-ideals of A is a z-ideal, then the sum of any two z-ideals which are l-ideals of A is a z-ideal.*

(2) *If the sum of any two minimal prime l-ideals of A is a d-ideal, then the sum of any two d-ideals of A is a d-ideal.*

Proof. We first show part (1). Suppose I, J are z-ideals which are l-ideals. Then I, J are semiprime l-ideals by (α), and by Theorem 2.3, $I + J$ is a semiprime l-ideal. We will show that $I + J$ is the intersection of z-ideals. To do so, we let $z \in A$ such that $z \notin I + J$, and we will show there is a z-ideal containing $I + J$ but not z. Since $I + J$ is a semiprime l-ideal, it is the intersection of prime l-ideals. So there is a prime l-ideal P containing $I + J$ but not z. Let $P_1, P_2 \subseteq P$ be prime l-ideals minimal with respect to containing I, J respectively. By (β), P_1, P_2 are prime z-ideals. It follows from (γ) that $P_1 + P_2$ is a z-ideal. Also, $I + J \subseteq P_1 + P_2$ and $z \notin (P_1 + P_2)$ since $P_1 + P_2 \subseteq P$.

The proof of part (2) is analogous. □

Recall that for any element a of an f-ring, $\{a\}^d$ is a z-ideal and a d-ideal. Recall also that a prime l-ideal P of a commutative semiprime f-ring is minimal if and only if $a \in P$ implies there is a $b \notin P$ such that $ab = 0$.

Corollary 2.6. *Let A be a commutative semiprime f-ring with identity element in which minimal prime l-ideals are square dominated.*

(1) *If for every $a, b \in A^+$, $\{a\}^d + \{b\}^d$ is a z-ideal, then the sum of any two z-ideals which are l-ideals of A is a z-ideal.*
(2) *If for every $a, b \in A^+$, $\{a\}^d + \{b\}^d$ is a d-ideal, then the sum of any two d-ideals of A is a d-ideal.*

Proof. To show part (1), we need only show that the sum of any two minimal prime l-ideals is a z-ideal. Let P, Q be minimal prime l-ideals. Suppose a, b are in the same set of maximal ideals and $b \in P+Q$. Then $b = p+q$ for some $p \in P$, $q \in Q$. Also, there is $p_1, q_1 \in A^+$ such that $p_1 \notin P$, $q_1 \notin Q$, and $pp_1 = 0$, $qq_1 = 0$. So $b = p+q \in \{p_1\}^d + \{q_1\}^d$. By hypothesis, $\{p_1\}^d + \{q_1\}^d$ is a z-ideal. So $a \in \{p_1\}^d + \{q_1\}^d \subseteq P+Q$.

The proof of part (2) is analogous. □

An f-ring (and more generally a Riesz space) A is called *normal* if $A = \{a^+\}^d + \{a^-\}^d$ for all $a \in A$, or equivalently if $a \wedge b = 0$ implies $A = \{a\}^d + \{b\}^d$. In [8, 2.5] it is shown that in a commutative, semiprime normal f-ring with identity element, every minimal prime l-ideal is square dominated.

Corollary 2.7. *Let A be a commutative semiprime normal f-ring with identity element. In A, the sum of any two z-ideals which are l-ideals is a z-ideal and the sum of any two d-ideals is a d-ideal.*

Proof. In view of the fact that minimal prime l-ideals of A are square dominated and in light of Theorem 2.5, we need only show that the sum of any two minimal prime l-ideals is a z-ideal and a d-ideal. So let P, Q be minimal prime l-ideals. We will show that if $P \neq Q$, then $P+Q = A$. If $P \neq Q$, then there is an element $p \in P \backslash Q$. Since P is a minimal prime, there is an element $q \notin P$ such that $pq = 0$. Then $p \wedge q = 0$, and $\{p\}^d + \{q\}^d = A$. But $\{p\}^d \subseteq Q$, $\{q\}^d \subseteq P$. So $A = P+Q$. □

Huijsmans and de Pagter show in [6, 4.4] that in a normal Riesz space the sum of two d-ideals is a d-ideal, so the d-ideal portion of the previous corollary is known.

We conclude with an example showing that in a commutative f-ring with identity element in which minimal prime l-ideals are square dominated, the sum of two minimal prime l-ideals is not necessarily a d-ideal or z-ideal. So the hypothesis that the sum of any two minimal prime l-ideals is a z-ideal or a d-ideal cannot be omitted from Theorem 2.5. The example makes use of a construction of Henriksen and Smith which appears in [5].

Example 2.8. In $C([0, 1])$, let i denote the function defined by $i(x) = x$ and let $I = \{f \in C([0, 1]) : |f^n| \leq mi \text{ for some } n, m \in \mathbf{N}\}$. Then I is a semiprime l-ideal of $C([0, 1])$. Let $A = \{(f, g) \in C([0, 1]) \times C([0, 1]) : f-g \in I\}$. Then as shown in [5], A is a commutative semiprime f-ring with identity element.

As shown in [5, §3], the minimal prime l-ideals of A have the form $\{(f+g, f): f \in P, g \in I\}$ or $\{(f, f+g): f \in P, g \in I\}$ for some minimal prime l-ideal P of $C([0, 1])$. Using this fact, it is not hard to show that minimal prime l-ideals of A are square dominated.

Define a function h by $h(x) = \sum_{n=1}^{\infty} 1/2^n x^{1/n}$. Then $h \in C([0, 1])$ and $h \notin I$. Let P be a prime l-ideal such that $I \subseteq P$ and $h \notin P$, and let P_1 be a minimal prime l-ideal contained in P. In A, let $Q_1 = \{(f+g, f): f \in P_1, g \in I\}$ and $Q_2 = \{(f, f+g): f \in P_1, g \in I\}$. Then Q_1 and Q_2 are minimal prime l-ideals of A and so are z-ideals and d-ideals. But $Q_1 + Q_2$ is not a z-ideal, since the only maximal ideal (h, h) or (i, i) is contained in is $M = \{(f, g): f(0) = g(0) = 0\}$ and yet $(i, i) \in Q_1 + Q_2$ but $(h, h) \notin Q_1 + Q_2$. To see directly that $Q_1 + Q_2$ is not a d-ideal, note that $\{(i, i)\}^{dd} = A$ and $A \nsubseteq Q_1 + Q_2$.

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