8-1-1990

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SUMS OF SEMIPRIME, z, AND d l-IDEALS IN A CLASS OF f-RINGS

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(Communicated by Louis J. Ratliff, Jr.)

ABSTRACT. In this paper it is shown that there is a large class of f-rings in which the sum of any two semiprime l-ideals is semiprime. This result is used to give a class of commutative f-rings with identity element in which the sum of any two z-ideals which are l-ideals is a z-ideal and the sum of any two d-ideals is a d-ideal.

INTRODUCTION

An l-ideal I of an f-ring A is called semiprime if \( a^2 \in I \) implies \( a \in I \). An ideal I of a commutative ring A with identity element is called a z-ideal if whenever \( a, b \in A \) are in the same set of maximal ideals and \( a \in I \), then \( b \in I \). Given an element \( a \) of an f-ring \( A \), let \( \{a\}^d = \{x \in A : xa = 0\} \) and \( \{a\}^{dd} = \{x \in A : xy = 0 \text{ for all } y \in \{a\}^d\} \). An ideal I of a commutative f-ring is called a d-ideal if \( a \in I \) implies \( \{a\}^{dd} \subseteq I \).

Several authors have studied the sums of semiprime l-ideals, z-ideals, and d-ideals in various classes of f-rings. In [2, 14.8] it is shown that the sum of two z-ideals in \( C(X) \), the f-ring of all real-valued continuous functions defined on the topological space \( X \), is a z-ideal; and in [11, 4.1 and 5.1], Rudd shows that in absolutely convex subrings of \( C(X) \), the sum of two semiprime l-ideals is semiprime and the sum of two z-ideals is a z-ideal. Mason studies sums of z-ideals in absolutely convex subrings of the ring of all continuous functions on a topological space and in more general rings in [10]. An example is given in [5, §7] of an f-ring in which there are two (semiprime) z-ideals whose sum is not a z-ideal or semiprime. Huijsmans and de Pagter show in [7, 4.4] that in a normal Riesz space, the sum of two d-ideals is a d-ideal.

In [4, 3.9], Henriksen gives a condition on two semiprime l-ideals of an f-ring which is necessary and sufficient for their sum to be semiprime. Henriksen also notes in [4] that this condition can be difficult to apply globally, and so it seems difficult to use this result to determine in what classes of f-rings are the sum of any two semiprime l-ideals semiprime.
In this note we show that there is a large class of \( f \)-rings, specifically those \( f \)-rings in which minimal prime \( l \)-ideals are square dominated, in which the sum of any two semiprime \( l \)-ideals is semiprime. We use this result to show that in a commutative \( f \)-ring with identity element in which minimal prime \( l \)-ideals are square dominated, if the sum of any two minimal prime \( l \)-ideals is a \( z \)-ideal (resp. \( d \)-ideal), then the sum of any two \( z \)-ideals which are \( l \)-ideals is a \( z \)-ideal (resp. the sum of any two \( d \)-ideals is a \( d \)-ideal). As a corollary we show that in a commutative semiprime normal \( f \)-ring with identity element, the sum of any two \( z \)-ideals which are \( l \)-ideals (resp. \( d \)-ideals) is a \( z \)-ideal (resp. \( d \)-ideal).

1. Preliminaries

An \( f \)-ring is a lattice-ordered ring which is a subdirect product of totally ordered rings. For general information on \( f \)-rings see [1]. Given an \( f \)-ring \( A \) and \( x \in A \), we let \( A^+ = \{a \in A: a \geq 0\} \), \( x^+ = x \vee 0 \), \( x^- = (-x) \vee 0 \), and \( |x| = x \vee (-x) \).

A ring ideal \( I \) of an \( f \)-ring \( A \) is an \( l \)-ideal if \( IxI < II \), \( y \in I \) implies \( x \in I \). Given any element \( a \in A \) there is a smallest \( l \)-ideal containing \( a \), and we denote this by \( \langle a \rangle \).

Suppose \( A \) is an \( f \)-ring and \( I \) is an \( l \)-ideal of \( A \). The \( l \)-ideal \( I \) is semiprime (prime) if \( a^2 \in I \) (\( ab \in I \)) implies \( a \in I \) (\( a \in I \) or \( b \in I \)). It is well known that in an \( f \)-ring, an \( l \)-ideal is semiprime if and only if it is an intersection of prime \( l \)-ideals, and that all \( l \)-ideals containing a given prime \( l \)-ideal form a chain. In an \( f \)-ring a semiprime \( l \)-ideal that contains a prime \( l \)-ideal is a prime \( l \)-ideal as shown in [12, 2.5], [7, 4.2].

An ideal \( I \) of a commutative ring \( A \) with identity element is a \( z \)-ideal if, whenever \( a, b \in A \) are contained in the same set of maximal ideals and \( a \in I \), then \( b \in I \).

In a commutative \( f \)-ring, let \( \{a\}^d = \{x \in A: ax = 0\} \) and \( \{a\}^{dd} = \{x \in A: xy = 0 \text{ for all } y \in \{a\}^d\} \). An ideal \( I \) of a commutative \( f \)-ring is called a \( d \)-ideal if \( a \in I \) implies \( \{a\}^{dd} \subseteq I \).

Henriksen, in [4] calls an \( l \)-ideal \( I \) of an \( f \)-ring \( A \) square dominated if \( I = \{a \in A: |a| \leq x^2 \text{ for some } x \in A \text{ such that } x^2 \in I\} \). Two characterizations now follow describing those commutative semiprime \( f \)-rings in which all minimal prime \( l \)-ideals are square dominated. Parts (1) and (2) of the following lemma are shown to be equivalent in [8, 2.1]. That part (3) of the following lemma is equivalent to part (1) follows easily from the equivalence of parts (1) and (2).

**Lemma 1.1.** Let \( A \) be a commutative semiprime \( f \)-ring. The following are equivalent:

1. Every minimal prime \( l \)-ideal of \( A \) is square dominated
2. For every \( a \in A^+ \) the \( l \)-ideal \( \{a\}^d \) is square dominated
(3) The $l$-ideal $O_P = \{a \in A: \text{ there exists a } b \notin P \text{ such that } ab = 0\}$ is square dominated for all prime $l$-ideals $P$ of $A$.

2.

We begin with two results that will be needed when showing that in an $f$-ring in which minimal prime $l$-ideals are square dominated, the sum of two semiprime $l$-ideals is semiprime.

**Theorem 2.1.** Let $A$ be an $f$-ring. In $A$, the sum of a semiprime $l$-ideal and a square-dominated semiprime $l$-ideal is semiprime.

**Proof.** Suppose that $I$, $J$ are semiprime $l$-ideals and that $J$ is square dominated. Let $a^2 \in I + J$ with $a \geq 0$. Then $a^2 \leq i + j$ for some $i \in I^+$, $j \in J^+$. Since $J$ is square dominated, $j \leq j_1^2$ for some $j_1 \in A^+$ with $j_1^2 \in J$. So $a^2 \leq i + j_1^2$. Let $x = a - (a \wedge j_1)$ and $y = a \wedge j_1$. Since $J$ is a semiprime $l$-ideal, $j_1 \in J$ and $y \in J$. Now for any positive elements $a$, $j_1$ of any totally ordered ring, $a \wedge j_1 = a$ or $a \wedge j_1 = j_1$. In the first case $(a - (a \wedge j_1))^2 = 0$, and in the second case $(a - (a \wedge j_1))^2 = (a - j_1)^2 = a^2 - a j_1 - j_1 a + j_1^2 \leq a^2 - 2j_1^2 + j_1^2 = a^2 - j_1^2$. Therefore in any totally ordered ring, $(a - (a \wedge j_1))^2 \leq 0 \vee (a^2 - j_1^2)$. This implies that in the $f$-ring $A$, $x^2 = (a - (a \wedge j_1))^2 \leq 0 \vee (a^2 - j_1^2) \leq i$. Thus, $x^2 \in I$ and hence $x \in I$. We have $a = x + y$ with $x \in I$ and $y \in J$. Therefore $a \in I + J$. □

**Lemma 2.2.** Let $A$ be an $f$-ring in which minimal prime $l$-ideals are square dominated. In $A$, the sum of any two prime $l$-ideals is prime.

**Proof.** Let $I$ and $J$ be prime $l$-ideals of $A$. Let $I_1$, $J_1$ be minimal prime $l$-ideals contained in $I$, $J$ respectively. We will show $I + J$ is an intersection of prime $l$-ideals. To do so, we let $z \in A$ such that $z \notin I + J$ and we will show there is a prime $l$-ideal containing $I + J$ but not $z$. The $l$-ideal $I_1 + J_1$ is prime, and the prime $l$-ideals containing it form a chain. By the maximal principle, there is a prime $l$-ideal $Q$ containing $I_1 + J_1$ which is maximal with respect to not containing $z$. By the previous theorem, $I + J_1$ is semiprime. It also contains a prime $l$-ideal and is therefore prime. Similarly, $I_1 + J$ is prime. Thus $I \subseteq I + J_1 \subseteq Q$ and $J \subseteq I_1 + J \subseteq Q$. This implies that $I + J \subseteq Q$ and $z \notin Q$. Therefore $I + J$ is an intersection of prime $l$-ideals. So it is semiprime. It also contains a prime $l$-ideal and is therefore prime. □

In [3, 4.7], Gillman and Kohls show that in $C(X)$, the $f$-ring of all real-valued continuous functions defined on the topological space $X$, an $l$-ideal is an intersection of $l$-ideals, each of which contains a prime $l$-ideal. Their proof easily generalizes to prove that in an $f$-ring, an $l$-ideal which contains all nilpotent elements of the $f$-ring is an intersection of $l$-ideals, each of which contains a prime $l$-ideal. We will make use of this result in the proof of the following theorem.
Theorem 2.3. Let $A$ be an $f$-ring in which minimal prime $l$-ideals are square dominated. In $A$, the sum of any two semiprime $l$-ideals is semiprime.

Proof. Let $I$, $J$ be semiprime $l$-ideals. We will show $I + J$ is an intersection of prime $l$-ideals. To do so, we let $z \in A$ such that $z \notin I + J$ and we show that there is a prime $l$-ideal containing $I + J$ but not $z$. By Gillman and Kohl's result mentioned above, there is an $l$-ideal $Q$ containing $I + J$ and containing a prime $l$-ideal but not containing $z$. Let $P$ be a minimal prime $l$-ideal contained in $Q$. By Theorem 2.1, $P + I$ is semiprime. Also, it contains a prime $l$-ideal and so is prime. Similarly, $P + J$ is prime. Then by the previous lemma, $(P + I) + (P + J)$ is prime. Since $(P + I) + (P + J) \subseteq Q$, $z \notin (P + I) + (P + J)$ and $I + J \subseteq (P + I) + (P + J)$. □

The converse of the previous theorem does not hold, as we show next.

Example 2.4. In $C([0, 1])$, denote by $i$ the function $i(x) = x$, and by $e$ the function $e(x) = 1$. Let $A = \{f \in C([0, 1]) : f = ae + g \text{ where } a \in \mathbb{R}, g \in (i)\}$ with coordinate operations. Then $A$ is a commutative semiprime $f$-ring.

We will show that the sum of two semiprime $l$-ideals of $A$ is semiprime. So suppose $I, J$ are semiprime $l$-ideals. If $I$ or $J$ contains an element $f = ae + g$ such that $a \neq 0$, then it can be shown that $I$ or $J$ is square dominated. Then by Theorem 2.1, $I + J$ is semiprime. So we may now suppose that both $I, J \subseteq (i)$. If $f^2 \in I + J$, then there is $i_1 \in I^+$, $j_1 \in J^+$ such that $f^2 = i_1 + j_1$. Also, $f \in (i)$ which implies $|f| \leq ni$ and $f^2 \leq n^2i^2$ for some $n \in \mathbb{N}$. So $i_1 \leq n^2i^2$ and $j_1 \leq n^2i^2$. Therefore $\sqrt{i_1} \leq ni$ and $\sqrt{j_1} \in A$. Since $I$ is semiprime, $\sqrt{i_1} \in I$. Similarly, $\sqrt{j_1} \in J$. So $f \leq \sqrt{i_1} + \sqrt{j_1}$ implies $f \in I + J$. Thus $I + J$ is semiprime.

Next we show that not every minimal prime $l$-ideal of $A$ is square dominated. Let $f$ be a function such that $0 \leq f \leq i$, $f(x) = 0$ for all $x \in [1/4, 1]$, $f(x) = 0$ for all $x \in [1/(4n + 2), 1/4n]$, and $f(1/(4n + 3)) = 1/(4n + 3)$ for all $n \in \mathbb{N}$. Also, let $g$ be a function such that $0 \leq g \leq i$, $g(x) = 0$ for all $x \in [1/4, 1]$, $g(1/(4n + 1)) = 1/(4n + 1)$, and $g(x) = 0$ for all $x \in [1/(4n + 4), 1/(4n + 2)]$ for all $n \in \mathbb{N}$. Then $g \in \{f\}^d$, and there is no element $h \in A$ which satisfies $g \leq h^2$ and $h^2 \in \{f\}^d$. So $\{f\}^d$ is not square dominated, and Lemma 1.1 implies that not every minimal prime $l$-ideal of $A$ is square dominated.

Next we turn our attention to the sum of two $z$-ideals which are $l$-ideals and to the sum of two $d$-ideals. Note that in a commutative $f$-ring with identity element, a $z$-ideal is not always an $l$-ideal. However it can easily be seen that in a commutative $f$-ring with identity element, if every maximal ideal is an $l$-ideal (or equivalently if for all $x \geq 1$, $x^{-1}$ exists), then a $z$-ideal is always an $l$-ideal. In a commutative $f$-ring with identity element, every $d$-ideal is an $l$-ideal. G. Mason has established three results concerning $z$-ideals which we will use in the proof of the next theorem. The first is as follows.
(α) In a commutative ring with identity element, every \( z \)-ideal is semiprime [9, 1.0].

The second was proven for a commutative ring with identity element, and the third was proven for a commutative ring with identity element in which the prime ideals containing a given prime form a chain. With only very slight modifications to the proofs, these results can be given in the context of \( f \)-rings.

(β) If, in a commutative \( f \)-ring with identity element, \( P \) is minimal in the class of prime \( l \)-ideals containing a \( z \)-ideal \( J \) which is an \( l \)-ideal, then \( P \) is also a \( z \)-ideal. In particular, minimal prime \( l \)-ideals are \( z \)-ideals [9, 1.1].

(γ) If, in a commutative \( f \)-ring with identity element, the sum of any two minimal prime \( l \)-ideals is a prime \( z \)-ideal, then the sum of any two prime \( l \)-ideals not in a chain is a \( z \)-ideal [10, 3.2].

One can easily mimic the proofs to (β) and (γ) to show analogous results about \( d \)-ideals.

(β′) If, in a commutative \( f \)-ring with identity element, \( P \) is minimal in the class of prime \( l \)-ideals containing a \( d \)-ideal \( I \), then \( P \) is also a \( d \)-ideal. In particular, minimal prime \( l \)-ideals are \( d \)-ideals.

(γ′) If, in a commutative \( f \)-ring with identity element, the sum of any two minimal prime \( l \)-ideals is a prime \( d \)-ideal, then the sum of any two prime \( l \)-ideals which are not in a chain is a \( d \)-ideal.

Theorem 2.5. Let \( A \) be a commutative \( f \)-ring with identity element in which minimal prime \( l \)-ideals are square dominated.

(1) If the sum of any two minimal prime \( l \)-ideals of \( A \) is a \( z \)-ideal, then the sum of any two \( z \)-ideals which are \( l \)-ideals of \( A \) is a \( z \)-ideal.

(2) If the sum of any two minimal prime \( l \)-ideals of \( A \) is a \( d \)-ideal, then the sum of any two \( d \)-ideals of \( A \) is a \( d \)-ideal.

Proof. We first show part (1). Suppose \( I, J \) are \( z \)-ideals which are \( l \)-ideals. Then \( I, J \) are semiprime \( l \)-ideals by (α), and by Theorem 2.3, \( I + J \) is a semiprime \( l \)-ideal. We will show that \( I + J \) is the intersection of \( z \)-ideals. To do so, we let \( z \in A \) such that \( z \notin I + J \), and we will show there is a \( z \)-ideal containing \( I + J \) but not \( z \). Since \( I + J \) is a semiprime \( l \)-ideal, it is the intersection of prime \( l \)-ideals. So there is a prime \( l \)-ideal \( P \) containing \( I + J \) but not \( z \). Let \( P_1, P_2 \subseteq P \) be prime \( l \)-ideals minimal with respect to containing \( I, J \) respectively. By (β), \( P_1, P_2 \) are prime \( z \)-ideals. It follows from (γ) that \( P_1 + P_2 \) is a \( z \)-ideal. Also, \( I + J \subseteq P_1 + P_2 \) and \( z \notin (P_1 + P_2) \) since \( P_1 + P_2 \subseteq P \).

The proof of part (2) is analogous. □

Recall that for any element \( a \) of an \( f \)-ring, \( \{a\}^d \) is a \( z \)-ideal and a \( d \)-ideal. Recall also that a prime \( l \)-ideal \( P \) of a commutative semiprime \( f \)-ring is minimal if and only if \( a \in P \) implies there is a \( b \notin P \) such that \( ab = 0 \).
Corollary 2.6. Let $A$ be a commutative semiprime $f$-ring with identity element in which minimal prime $l$-ideals are square dominated.

(1) If for every $a, b \in A^+$, $(a)^d + (b)^d$ is a $z$-ideal, then the sum of any two $z$-ideals which are $l$-ideals of $A$ is a $z$-ideal.

(2) If for every $a, b \in A^+$, $(a)^d + (b)^d$ is a $d$-ideal, then the sum of any two $d$-ideals of $A$ is a $d$-ideal.

Proof. To show part (1), we need only show that the sum of any two minimal prime $l$-ideals is a $z$-ideal. Let $P, Q$ be minimal prime $l$-ideals. Suppose $a, b$ are in the same set of maximal ideals and $b \in P + Q$. Then $b = p + q$ for some $p \in P, q \in Q$. Also, there is $p_1, q_1 \in A^+$ such that $p_1 \notin P, q_1 \notin Q$, and $pp_1 = 0, qq_1 = 0$. So $b = p + q \in (p_1)^d + (q_1)^d$. By hypothesis, $(p_1)^d + (q_1)^d$ is a $z$-ideal. So $a \in (p_1)^d + (q_1)^d \subseteq P + Q$.

The proof of part (2) is analogous. □

An $f$-ring (and more generally a Riesz space) $A$ is called normal if $A = (a^+)^d + (a^-)^d$ for all $a \in A$, or equivalently if $a \land b = 0$ implies $A = (a)^d + (b)^d$. In [8, 2.5] it is shown that in a commutative, semiprime normal $f$-ring with identity element, every minimal prime $l$-ideal is square dominated.

Corollary 2.7. Let $A$ be a commutative semiprime normal $f$-ring with identity element. In $A$, the sum of any two $z$-ideals which are $l$-ideals is a $z$-ideal and the sum of any two $d$-ideals is a $d$-ideal.

Proof. In view of the fact that minimal prime $l$-ideals of $A$ are square dominated and in light of Theorem 2.5, we need only show that the sum of any two minimal prime $l$-ideals is a $z$-ideal and a $d$-ideal. So let $P, Q$ be minimal prime $l$-ideals. We will show that if $P \neq Q$, then $P + Q = A$. If $P \neq Q$, then there is an element $p \in P \setminus Q$. Since $P$ is a minimal prime, there is an element $q \notin P$ such that $pq = 0$. Then $p \land q = 0$, and $(p)^d + (q)^d = A$. But $(p)^d \subseteq Q, \ (q)^d \subseteq P$. So $A = P + Q$. □

Huijsmans and de Pagter show in [6, 4.4] that in a normal Riesz space the sum of two $d$-ideals is a $d$-ideal, so the $d$-ideal portion of the previous corollary is known.

We conclude with an example showing that in a commutative $f$-ring with identity element in which minimal prime $l$-ideals are square dominated, the sum of two minimal prime $l$-ideals is not necessarily a $d$-ideal or $z$-ideal. So the hypothesis that the sum of any two minimal prime $l$-ideals is a $z$-ideal or a $d$-ideal cannot be omitted from Theorem 2.5. The example makes use of a construction of Henriksen and Smith which appears in [5].

Example 2.8. In $C([0, 1])$, let $i$ denote the function defined by $i(x) = x$ and let $I = \{f \in C([0, 1]) : |f^n| \leq mi$ for some $n, m \in \mathbb{N}\}$. Then $I$ is a semiprime $l$-ideal of $C([0, 1])$. Let $A = \{(f, g) \in C([0, 1]) \times C([0, 1]) : f - g \in I\}$. Then as shown in [5], $A$ is a commutative semiprime $f$-ring with identity element.
As shown in [5, §3], the minimal prime \( l \)-ideals of \( A \) have the form \( \{(f + g, f) \mid f \in P, g \in I\} \) or \( \{(f, f + g) \mid f \in P, g \in I\} \) for some minimal prime \( l \)-ideal \( P \) of \( C([0, 1]) \). Using this fact, it is not hard to show that minimal prime \( l \)-ideals of \( A \) are square dominated.

Define a function \( h \) by \( h(x) = \sum_{n=1}^{\infty} 1/2^n x^{1/n} \). Then \( h \in C([0, 1]) \) and \( h \notin I \). Let \( P \) be a prime \( l \)-ideal such that \( I \subseteq P \) and \( h \notin P \), and let \( P_1 \) be a minimal prime \( l \)-ideal contained in \( P \). In \( A \), let \( Q_1 = \{(f + g, f) \mid f \in P_1, g \in I\} \) and \( Q_2 = \{(f, f + g) \mid f \in P_1, g \in I\} \). Then \( Q_1 \) and \( Q_2 \) are minimal prime \( l \)-ideals of \( A \) and so are \( z \)-ideals and \( d \)-ideals. But \( Q_1 + Q_2 \) is not a \( z \)-ideal, since the only maximal ideal \( (h, h) \) or \( (i, i) \) is contained in is \( M = \{(f, g) \mid f(0) = g(0) = 0\} \) and yet \( (i, i) \in Q_1 + Q_2 \) but \( (h, h) \notin Q_1 + Q_2 \).

To see directly that \( Q_1 + Q_2 \) is not a \( d \)-ideal, note that \( \{(i, i)\}_{dd} = A \) and \( A \notin Q_1 + Q_2 \).

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