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SUMS OF SEMIPRIME, z , AND d l -IDEALS IN A CLASS OF f -RINGS

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ABSTRACT. In this paper it is shown that there is a large class of f -rings in which the sum of any two semiprime l -ideals is semiprime. This result is used to give a class of commutative f -rings with identity element in which the sum of any two z -ideals which are l -ideals is a z -ideal and the sum of any two d -ideals is a d -ideal.

INTRODUCTION

An l -ideal I of an f -ring A is called semiprime if $a^2 \in I$ implies $a \in I$. An ideal I of a commutative ring A with identity element is called a z -ideal if whenever $a, b \in A$ are in the same set of maximal ideals and $a \in I$, then $b \in I$. Given an element a of an f -ring A , let $\{a\}^d = \{x \in A: xa = 0\}$ and $\{a\}^{dd} = \{x \in A: xy = 0 \text{ for all } y \in \{a\}^d\}$. An ideal I of a commutative f -ring is called a d -ideal if $a \in I$ implies $\{a\}^{dd} \subseteq I$.

Several authors have studied the sums of semiprime l -ideals, z -ideals, and d -ideals in various classes of f -rings. In [2, 14.8] it is shown that the sum of two z -ideals in $C(X)$, the f -ring of all real-valued continuous functions defined on the topological space X , is a z -ideal; and in [11, 4.1 and 5.1], Rudd shows that in absolutely convex subrings of $C(X)$, the sum of two semiprime l -ideals is semiprime and the sum of two z -ideals is a z -ideal. Mason studies sums of z -ideals in absolutely convex subrings of the ring of all continuous functions on a topological space and in more general rings in [10]. An example is given in [5, §7] of an f -ring in which there are two (semiprime) z -ideals whose sum is not a z -ideal or semiprime. Huijsmans and de Pagter show in [7, 4.4] that in a normal Riesz space, the sum of two d -ideals is a d -ideal.

In [4, 3.9], Henriksen gives a condition on two semiprime l -ideals of an f -ring which is necessary and sufficient for their sum to be semiprime. Henriksen also notes in [4] that this condition can be difficult to apply globally, and so it seems difficult to use this result to determine in what classes of f -rings are the sum of any two semiprime l -ideals semiprime.

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In this note we show that there is a large class of f -rings, specifically those f -rings in which minimal prime l -ideals are square dominated, in which the sum of any two semiprime l -ideals is semiprime. We use this result to show that in a commutative f -ring with identity element in which minimal prime l -ideals are square dominated, if the sum of any two minimal prime l -ideals is a z -ideal (resp. d -ideal), then the sum of any two z -ideals which are l -ideals is a z -ideal (resp. the sum of any two d -ideals is a d -ideal). As a corollary we show that in a commutative semiprime normal f -ring with identity element, the sum of any two z -ideals which are l -ideals (resp. d -ideals) is a z -ideal (resp. d -ideal).

1. PRELIMINARIES

An f -ring is a lattice-ordered ring which is a subdirect product of totally ordered rings. For general information on f -rings see [1]. Given an f -ring A and $x \in A$, we let $A^+ = \{a \in A : a \geq 0\}$, $x^+ = x \vee 0$, $x^- = (-x) \vee 0$, and $|x| = x \vee (-x)$.

A ring ideal I of an f -ring A is an l -ideal if $|x| \leq |y|$, $y \in I$ implies $x \in I$. Given any element $a \in A$ there is a smallest l -ideal containing a , and we denote this by $\langle a \rangle$.

Suppose A is an f -ring and I is an l -ideal of A . The l -ideal I is *semiprime* (*prime*) if $a^2 \in I$ ($ab \in I$) implies $a \in I$ ($a \in I$ or $b \in I$). It is well known that in an f -ring, an l -ideal is semiprime if and only if it is an intersection of prime l -ideals, and that all l -ideals containing a given prime l -ideal form a chain. In an f -ring a semiprime l -ideal that contains a prime l -ideal is a prime l -ideal as shown in [12, 2.5], [7, 4.2].

An ideal I of a commutative ring A with identity element is a z -ideal if, whenever $a, b \in A$ are contained in the same set of maximal ideals and $a \in I$, then $b \in I$.

In a commutative f -ring, let $\{a\}^d = \{x \in A : ax = 0\}$ and $\{a\}^{dd} = \{x \in A : xy = 0 \text{ for all } y \in \{a\}^d\}$. An ideal I of a commutative f -ring is called a d -ideal if $a \in I$ implies $\{a\}^{dd} \subseteq I$.

Henriksen, in [4] calls an l -ideal I of an f -ring A *square dominated* if $I = \{a \in A : |a| \leq x^2 \text{ for some } x \in A \text{ such that } x^2 \in I\}$. Two characterizations now follow describing those commutative semiprime f -rings in which all minimal prime l -ideals are square dominated. Parts (1) and (2) of the following lemma are shown to be equivalent in [8, 2.1]. That part (3) of the following lemma is equivalent to part (1) follows easily from the equivalence of parts (1) and (2).

Lemma 1.1. *Let A be a commutative semiprime f -ring. The following are equivalent:*

- (1) *Every minimal prime l -ideal of A is square dominated*
- (2) *For every $a \in A^+$ the l -ideal $\{a\}^d$ is square dominated*

- (3) The l -ideal $O_p = \{a \in A: \text{there exists } a, b \notin P \text{ such that } ab = 0\}$ is square dominated for all prime l -ideals P of A .

2.

We begin with two results that will be needed when showing that in an f -ring in which minimal prime l -ideals are square dominated, the sum of two semiprime l -ideals is semiprime.

Theorem 2.1. *Let A be an f -ring. In A , the sum of a semiprime l -ideal and a square-dominated semiprime l -ideal is semiprime.*

Proof. Suppose that I, J are semiprime l -ideals and that J is square dominated. Let $a^2 \in I + J$ with $a \geq 0$. Then $a^2 \leq i + j$ for some $i \in I^+, j \in J^+$. Since J is square dominated, $j \leq j_1^2$ for some $j_1 \in A^+$ with $j_1^2 \in J$. So $a^2 \leq i + j_1^2$. Let $x = a - (a \wedge j_1)$ and $y = a \wedge j_1$. Since J is a semiprime l -ideal, $j_1 \in J$ and $y \in J$. Now for any positive elements a, j_1 of any totally ordered ring, $a \wedge j_1 = a$ or $a \wedge j_1 = j_1$. In the first case $(a - (a \wedge j_1))^2 = 0$, and in the second case $(a - (a \wedge j_1))^2 = (a - j_1)^2 = a^2 - aj_1 - j_1a + j_1^2 \leq a^2 - 2j_1^2 + j_1^2 = a^2 - j_1^2$. Therefore in any totally ordered ring, $(a - (a \wedge j_1))^2 \leq 0 \vee (a^2 - j_1^2)$. This implies that in the f -ring A , $x^2 = (a - (a \wedge j_1))^2 \leq 0 \vee (a^2 - j_1^2) \leq i$. Thus, $x^2 \in I$ and hence $x \in I$. We have $a = x + y$ with $x \in I$ and $y \in J$. Therefore $a \in I + J$. \square

Lemma 2.2. *Let A be an f -ring in which minimal prime l -ideals are square dominated. In A , the sum of any two prime l -ideals is prime.*

Proof. Let I and J be prime l -ideals of A . Let I_1, J_1 be minimal prime l -ideals contained in I, J respectively. We will show $I + J$ is an intersection of prime l -ideals. To do so, we let $z \in A$ such that $z \notin I + J$ and we will show there is a prime l -ideal containing $I + J$ but not z . The l -ideal $I_1 + J_1$ is prime, and the prime l -ideals containing it form a chain. By the maximal principle, there is a prime l -ideal Q containing $I_1 + J_1$ which is maximal with respect to not containing z . By the previous theorem, $I + J_1$ is semiprime. It also contains a prime l -ideal and is therefore prime. Similarly, $I_1 + J$ is prime. Thus $I \subseteq I + J_1 \subseteq Q$ and $J \subseteq I_1 + J \subseteq Q$. This implies that $I + J \subseteq Q$ and $z \notin Q$. Therefore $I + J$ is an intersection of prime l -ideals. So it is semiprime. It also contains a prime l -ideal and is therefore prime. \square

In [3, 4.7], Gillman and Kohls show that in $C(X)$, the f -ring of all real-valued continuous functions defined on the topological space X , an l -ideal is an intersection of l -ideals, each of which contains a prime l -ideal. Their proof easily generalizes to prove that in an f -ring, an l -ideal which contains all nilpotent elements of the f -ring is an intersection of l -ideals, each of which contains a prime l -ideal. We will make use of this result in the proof of the following theorem.

Theorem 2.3. *Let A be an f -ring in which minimal prime l -ideals are square dominated. In A , the sum of any two semiprime l -ideals is semiprime.*

Proof. Let I, J be semiprime l -ideals. We will show $I + J$ is an intersection of prime l -ideals. To do so, we let $z \in A$ such that $z \notin I + J$ and we show that there is a prime l -ideal containing $I + J$ but not z . By Gillman and Kohl's result mentioned above, there is an l -ideal Q containing $I + J$ and containing a prime l -ideal but not containing z . Let P be a minimal prime l -ideal contained in Q . By Theorem 2.1, $P + I$ is semiprime. Also, it contains a prime l -ideal and so is prime. Similarly, $P + J$ is prime. Then by the previous lemma, $(P + I) + (P + J)$ is prime. Since $(P + I) + (P + J) \subseteq Q$, $z \notin (P + I) + (P + J)$ and $I + J \subseteq (P + I) + (P + J)$. \square

The converse of the previous theorem does not hold, as we show next.

Example 2.4. In $C([0, 1])$, denote by i the function $i(x) = x$, and by e the function $e(x) = 1$. Let $A = \{f \in C([0, 1]): f = ae + g \text{ where } a \in \mathbf{R}, g \in \langle i \rangle\}$ with coordinate operations. Then A is a commutative semiprime f -ring.

We will show that the sum of two semiprime l -ideals of A is semiprime. So suppose I, J are semiprime l -ideals. If I or J contains an element $f = ae + g$ such that $a \neq 0$, then it can be shown that I or J is square dominated. Then by Theorem 2.1, $I + J$ is semiprime. So we may now suppose that both $I, J \subseteq \langle i \rangle$. If $f^2 \in I + J$, then there is $i_1 \in I^+, j_1 \in J^+$ such that $f^2 = i_1 + j_1$. Also, $f \in \langle i \rangle$ which implies $|f| \leq ni$ and $f^2 \leq n^2 i^2$ for some $n \in \mathbf{N}$. So $i_1 \leq n^2 i^2$ and $j_1 \leq n^2 i^2$. Therefore $\sqrt{i_1} \leq ni$ and $\sqrt{i_1} \in A$. Since I is semiprime, $\sqrt{i_1} \in I$. Similarly, $\sqrt{j_1} \in J$. So $f \leq \sqrt{i_1} + \sqrt{j_1}$ implies $f \in I + J$. Thus $I + J$ is semiprime.

Next we show that not every minimal prime l -ideal of A is square dominated. Let f be a function such that $0 \leq f \leq i$, $f(x) = 0$ for all $x \in [1/4, 1]$, $f(x) = 0$ for all $x \in [1/(4n + 2), 1/4n]$, and $f(1/(4n + 3)) = 1/(4n + 3)$ for all $n \in \mathbf{N}$. Also, let g be a function such that $0 \leq g \leq i$, $g(x) = 0$ for all $x \in [1/4, 1]$, $g(1/(4n + 1)) = 1/(4n + 1)$, and $g(x) = 0$ for all $x \in [1/(4n + 4), 1/(4n + 2)]$ for all $n \in \mathbf{N}$. Then $g \in \{f\}^d$, and there is no element $h \in A$ which satisfies $g \leq h^2$ and $h^2 \in \{f\}^d$. So $\{f\}^d$ is not square dominated, and Lemma 1.1 implies that not every minimal prime l -ideal of A is square dominated.

Next we turn our attention to the sum of two z -ideals which are l -ideals and to the sum of two d -ideals. Note that in a commutative f -ring with identity element, a z -ideal is not always an l -ideal. However it can easily be seen that in a commutative f -ring with identity element, if every maximal ideal is an l -ideal (or equivalently if for all $x \geq 1$, x^{-1} exists), then a z -ideal is always an l -ideal. In a commutative f -ring with identity element, every d -ideal is an l -ideal. G. Mason has established three results concerning z -ideals which we will use in the proof of the next theorem. The first is as follows.

(α) In a commutative ring with identity element, every z -ideal is semiprime [9, 1.0].

The second was proven for a commutative ring with identity element, and the third was proven for a commutative ring with identity element in which the prime ideals containing a given prime form a chain. With only very slight modifications to the proofs, these results can be given in the context of f -rings.

(β) If, in a commutative f -ring with identity element, P is minimal in the class of prime l -ideals containing a z -ideal I which is an l -ideal, then P is also a z -ideal. In particular, minimal prime l -ideals are z -ideals [9, 1.1].

(γ) If, in a commutative f -ring with identity element, the sum of any two minimal prime l -ideals is a prime z -ideal, then the sum of any two prime l -ideals not in a chain is a z -ideal [10, 3.2].

One can easily mimic the proofs to (β) and (γ) to show analogous results about d -ideals.

(β') If, in a commutative f -ring with identity element, P is minimal in the class of prime l -ideals containing a d -ideal I , then P is also a d -ideal. In particular, minimal prime l -ideals are d -ideals.

(γ') If, in a commutative f -ring with identity element, the sum of any two minimal prime l -ideals is a prime d -ideal, then the sum of any two prime l -ideals which are not in a chain is a d -ideal.

Theorem 2.5. *Let A be a commutative f -ring with identity element in which minimal prime l -ideals are square dominated.*

- (1) *If the sum of any two minimal prime l -ideals of A is a z -ideal, then the sum of any two z -ideals which are l -ideals of A is a z -ideal.*
- (2) *If the sum of any two minimal prime l -ideals of A is a d -ideal, then the sum of any two d -ideals of A is a d -ideal.*

Proof. We first show part (1). Suppose I, J are z -ideals which are l -ideals. Then I, J are semiprime l -ideals by (α), and by Theorem 2.3, $I + J$ is a semiprime l -ideal. We will show that $I + J$ is the intersection of z -ideals. To do so, we let $z \in A$ such that $z \notin I + J$, and we will show there is a z -ideal containing $I + J$ but not z . Since $I + J$ is a semiprime l -ideal, it is the intersection of prime l -ideals. So there is a prime l -ideal P containing $I + J$ but not z . Let $P_1, P_2 \subseteq P$ be prime l -ideals minimal with respect to containing I, J respectively. By (β), P_1, P_2 are prime z -ideals. It follows from (γ) that $P_1 + P_2$ is a z -ideal. Also, $I + J \subseteq P_1 + P_2$ and $z \notin (P_1 + P_2)$ since $P_1 + P_2 \subseteq P$.

The proof of part (2) is analogous. \square

Recall that for any element a of an f -ring, $\{a\}^d$ is a z -ideal and a d -ideal. Recall also that a prime l -ideal P of a commutative semiprime f -ring is minimal if and only if $a \in P$ implies there is a $b \notin P$ such that $ab = 0$.

Corollary 2.6. *Let A be a commutative semiprime f -ring with identity element in which minimal prime l -ideals are square dominated.*

- (1) *If for every $a, b \in A^+$, $\{a\}^d + \{b\}^d$ is a z -ideal, then the sum of any two z -ideals which are l -ideals of A is a z -ideal.*
- (2) *If for every $a, b \in A^+$, $\{a\}^d + \{b\}^d$ is a d -ideal, then the sum of any two d -ideals of A is a d -ideal.*

Proof. To show part (1), we need only show that the sum of any two minimal prime l -ideals is a z -ideal. Let P, Q be minimal prime l -ideals. Suppose a, b are in the same set of maximal ideals and $b \in P + Q$. Then $b = p + q$ for some $p \in P, q \in Q$. Also, there is $p_1, q_1 \in A^+$ such that $p_1 \notin P, q_1 \notin Q$, and $pp_1 = 0, qq_1 = 0$. So $b = p + q \in \{p_1\}^d + \{q_1\}^d$. By hypothesis, $\{p_1\}^d + \{q_1\}^d$ is a z -ideal. So $a \in \{p_1\}^d + \{q_1\}^d \subseteq P + Q$.

The proof of part (2) is analogous. \square

An f -ring (and more generally a Riesz space) A is called *normal* if $A = \{a^+\}^d + \{a^-\}^d$ for all $a \in A$, or equivalently if $a \wedge b = 0$ implies $A = \{a\}^d + \{b\}^d$. In [8, 2.5] it is shown that in a commutative, semiprime normal f -ring with identity element, every minimal prime l -ideal is square dominated.

Corollary 2.7. *Let A be a commutative semiprime normal f -ring with identity element. In A , the sum of any two z -ideals which are l -ideals is a z -ideal and the sum of any two d -ideals is a d -ideal.*

Proof. In view of the fact that minimal prime l -ideals of A are square dominated and in light of Theorem 2.5, we need only show that the sum of any two minimal prime l -ideals is a z -ideal and a d -ideal. So let P, Q be minimal prime l -ideals. We will show that if $P \neq Q$, then $P + Q = A$. If $P \neq Q$, then there is an element $p \in P \setminus Q$. Since P is a minimal prime, there is an element $q \notin P$ such that $pq = 0$. Then $p \wedge q = 0$, and $\{p\}^d + \{q\}^d = A$. But $\{p\}^d \subseteq Q, \{q\}^d \subseteq P$. So $A = P + Q$. \square

Huijsmans and de Pagter show in [6, 4.4] that in a normal Riesz space the sum of two d -ideals is a d -ideal, so the d -ideal portion of the previous corollary is known.

We conclude with an example showing that in a commutative f -ring with identity element in which minimal prime l -ideals are square dominated, the sum of two minimal prime l -ideals is not necessarily a d -ideal or z -ideal. So the hypothesis that the sum of any two minimal prime l -ideals is a z -ideal or a d -ideal cannot be omitted from Theorem 2.5. The example makes use of a construction of Henriksen and Smith which appears in [5].

Example 2.8. In $C([0, 1])$, let i denote the function defined by $i(x) = x$ and let $I = \{f \in C([0, 1]): |f^n| \leq mi \text{ for some } n, m \in \mathbf{N}\}$. Then I is a semiprime l -ideal of $C([0, 1])$. Let $A = \{(f, g) \in C([0, 1]) \times C([0, 1]): f - g \in I\}$. Then as shown in [5], A is a commutative semiprime f -ring with identity element.

As shown in [5, §3], the minimal prime l -ideals of A have the form $\{(f + g, f): f \in P, g \in I\}$ or $\{(f, f + g): f \in P, g \in I\}$ for some minimal prime l -ideal P of $C([0, 1])$. Using this fact, it is not hard to show that minimal prime l -ideals of A are square dominated.

Define a function h by $h(x) = \sum_{n=1}^{\infty} 1/2^n x^{1/n}$. Then $h \in C([0, 1])$ and $h \notin I$. Let P be a prime l -ideal such that $I \subseteq P$ and $h \notin P$, and let P_1 be a minimal prime l -ideal contained in P . In A , let $Q_1 = \{(f + g, f): f \in P_1, g \in I\}$ and $Q_2 = \{(f, f + g): f \in P_1, g \in I\}$. Then Q_1 and Q_2 are minimal prime l -ideals of A and so are z -ideals and d -ideals. But $Q_1 + Q_2$ is not a z -ideal, since the only maximal ideal (h, h) or (i, i) is contained in $M = \{(f, g): f(0) = g(0) = 0\}$ and yet $(i, i) \in Q_1 + Q_2$ but $(h, h) \notin Q_1 + Q_2$. To see directly that $Q_1 + Q_2$ is not a d -ideal, note that $\{(i, i)\}^{dd} = A$ and $A \not\subseteq Q_1 + Q_2$.

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