6-1-2014

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Recommended Citation

G. U. Varieschi, "Kerr metric, geodesic motion, and Flyby Anomaly in fourth-order ConformalGravity," Gen. Relativ. Gravit., 46, 1741, June 2014 doi: dx.doi.org/10.1007/s10714-014-1741-z

Kerr Metric, Geodesic Motion, and Flyby Anomaly in Fourth-Order Conformal Gravity

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Abstract

In this paper we analyze the Kerr geometry in the context of Conformal Gravity, an alternative theory of gravitation, which is a direct extension of General Relativity. Following previous studies in the literature, we introduce an explicit expression of the Kerr metric in Conformal Gravity, which naturally reduces to the standard General Relativity Kerr geometry in the absence of Conformal Gravity effects. As in the standard case, we show that the Hamilton-Jacobi equation governing geodesic motion in a space-time based on this geometry is indeed separable and that a fourth constant of motion—similar to Carter's constant—can also be introduced in Conformal Gravity. Consequently, we derive the fundamental equations of geodesic motion and show that the problem of solving these equations can be reduced to one of quadratures.

In particular, we study the resulting time-like geodesics in Conformal Gravity Kerr geometry by numerically integrating the equations of motion for Earth flyby trajectories of spacecraft. We then compare our results with the existing data of the Flyby Anomaly in order to ascertain whether Conformal Gravity corrections are possibly the origin of this gravitational anomaly. Although Conformal Gravity slightly affects the trajectories of geodesic motion around a rotating spherical object, we show that these corrections are minimal and are not expected to be the origin of the Flyby Anomaly, unless conformal parameters are drastically different from current estimates.

Therefore, our results confirm previous analyses, showing that modifications due to Conformal Gravity are not likely to be detected at the Solar System level, but might affect gravity at the galactic or cosmological scale.

PACS numbers: 04.50.Kd; 98.80.Jk; 95.55.Pe

Keywords: conformal gravity, Kerr metric, geodesics, flyby anomaly

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Contents

I. INTRODUCTION

In recent years alternative theories of gravity have become progressively more popular in the scientific literature due to their ability to account for astrophysical observations without resorting to dark matter (DM) and dark energy (DE). In fact, despite recent experimental and observational data supporting the case for DM (for example, see [1], [2]), there is still no conclusive evidence about the actual origin of this component of the Universe. Similarly, the observation of an accelerated expansion of the Universe prompted cosmologists to introduce a DE component within the framework of General Relativity (GR) and standard cosmology.

On the other hand, alternative theories of gravity (such as MOND [3]-[4], TeVeS [5], NGT [6], and others) have built the case for a possible paradigm shift by avoiding the exotic DM/DE components and by introducing possible modifications to standard gravity (for reviews see [7], [8], [9], and references therein). This proposed paradigm shift is not so different from what Einstein himself did in 1915 by extending Newtonian gravity into GR. For instance, Einstein's explanation of the anomalies in the rate of precession of the planet Mercury, using GR instead of Newton's law of gravitation, virtually eliminated a possible "dark matter" component in the Solar System (the proposed existence of an unknown planet—named Vulcan—between Mercury and the Sun, suggested by Le Verrier as the possible cause of the observed anomalies).

Following this line of thought, we have analyzed in previous publications ([10], [11], [12]) the theory of Conformal Gravity (CG), a fourth-order extension of Einstein's second-order General Relativity, as a possible solution to current cosmological puzzles, such as DM and DE. In particular, in this paper we want to study the possible implications for geodesic motion of the stationary, axially symmetric solution (CG extension of the Kerr metric).

In Sect. II, we will start by reviewing the main results of Conformal Gravity and the equivalent of the Schwarzschild metric in CG. In Sect. III, the main part of our paper, we will study the equivalent of the Kerr metric in CG and perform a separation of variables in the related Hamilton-Jacobi equation (similar to the procedure introduced by B. Carter for the GR case). In Sect. IV, we will study the time-like geodesics in fourth-order Kerr space-time and apply our findings to possible gravitational anomalies detected in our Solar System. In particular, the so-called Flyby Anomaly (FA) may be related to the fourth-order Kerr metric analyzed in this paper.

II. CONFORMAL GRAVITY AND THE STATIC, SPHERICALLY SYMMETRIC METRIC

The German mathematician Hermann Weyl pioneered Conformal Gravity in 1918 ([13], [14], [15]) by introducing the so-called *conformal* or *Weyl tensor*, a combination of the Riemann tensor $R_{\lambda\mu\nu\kappa}$, the Ricci tensor $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$, and the Ricci scalar $R = R^{\mu}{}_{\mu}$ (see [10] for full details):

$$C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} - \frac{1}{2}(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu}) + \frac{1}{6}R\ (g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}). \quad (1)$$

The particular form $C^{\lambda}{}_{\mu\nu\kappa}(x)$ of the Weyl tensor is invariant under the local transformation of the metric:

$$g_{\mu\nu}(x) \rightarrow \widehat{g}_{\mu\nu}(x) = e^{2\alpha(x)}g_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x), \quad (2)$$

where the factor $\Omega(x) = e^{\alpha(x)}$ represents the amount of local "stretching" of the geometry. Thus, the name "conformal" indicates a theory invariant under all possible local stretchings of the space-time.

This generalization of GR was found to be a fourth-order theory, as opposed to the standard second-order General Relativity: the field equations contain derivatives up to the fourth order of the metric, with respect to the space-time coordinates. Following work done by R. Bach [16], C. Lanczos [17], and others, CG was based on the conformal (or Weyl) action:

$$I_W = -\alpha_g \int d^4x \ (-g)^{1/2} \ C_{\lambda\mu\nu\kappa} \ C^{\lambda\mu\nu\kappa}, \tag{3}$$

where $g \equiv \det(g_{\mu\nu})$ and α_g is the CG coupling constant. I_W is the unique general coordinate scalar action that is completely locally conformal invariant. Bach [16] also introduced the CG field equations, in the presence of an energy-momentum tensor[1] $T_{\mu\nu}$:

$$W_{\mu\nu} = \frac{1}{4\alpha_g} \ T_{\mu\nu}, \tag{4}$$

similar to Einstein's equations,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \ R = -\frac{8\pi G}{c^3} \ T_{\mu\nu}. \tag{5}$$

The "Bach tensor" $W_{\mu\nu}$ in Eq. (4) is analogous to Einstein's curvature tensor $G_{\mu\nu}$, on the left-hand side of Eq. (5), but has a much more complex structure, being defined as:

$$\begin{aligned} W_{\mu\nu} = & -\frac{1}{6} g_{\mu\nu} \ R^{;\lambda}{}_{;\lambda} + \frac{2}{3} R_{;\mu;\nu} + R_{\mu\nu}{}^{;\lambda}{}_{;\lambda} - R_{\mu}{}^{\lambda}{}_{;\nu;\lambda} - R_{\nu}{}^{\lambda}{}_{;\mu;\lambda} \\ & + \frac{2}{3} R \ R_{\mu\nu} - 2 R_{\mu}{}^{\lambda} \ R_{\lambda\nu} + \frac{1}{2} g_{\mu\nu} \ R_{\lambda\rho} \ R^{\lambda\rho} - \frac{1}{6} g_{\mu\nu} \ R^2, \end{aligned} \tag{6}$$

and including derivatives up to the fourth order of the metric with respect to space-time coordinates.

In 1989, Mannheim and Kazanas ([18], [19]) derived the exact and complete exterior solution for a static, spherically symmetric source in CG, i.e., the fourth-order analogue of the Schwarzschild exterior solution in GR. Other exact solutions, such as the equivalent Reissner-Nordström, Kerr, and Kerr-Newman in CG, were also derived later by Mannheim and Kazanas [20] (MK solutions in the following).

The MK solution for a static, spherically symmetric source, in the case $T_{\mu\nu} = 0$ (exterior solution), takes the form

$$ds^2 = -B(r) \ c^2 dt^2 + \frac{dr^2}{B(r)} + r^2 (d\theta^2 + \sin^2\theta \ d\phi^2), \tag{7}$$

[1] We follow here the convention [7] of introducing the energy-momentum tensor $T_{\mu\nu}$ so that the quantity cT_{00} has dimensions of an energy density.

with

$$B(r) = 1 - 3\beta\gamma - \frac{\beta(2-3\beta\gamma)}{r} + \gamma r - \kappa r^2. \tag{8}$$

The parameters in Eq. (8) are defined as follows: $\beta = \frac{GM}{c^2}$ (cm) is the geometrized mass, where M is the mass of the (spherically symmetric) source and G is the universal gravitational constant; two other parameters, γ (cm^{-1}) and κ (cm^{-2}), are required by CG. The standard Schwarzschild solution is recovered as $\gamma, \kappa \to 0$ in the equations above. These two new parameters are interpreted by MK [18] in the following way: κ, and the corresponding term $-\kappa r^2$, indicates a background De Sitter space-time, which is important only over cosmological distances, since κ has a very small value. Similarly, γ measures the departure from the Schwarzschild metric at smaller distances since the γr term becomes significant over galactic distance scales.

For values of $\gamma \sim 10^{-28}$–10^{-30} cm^{-1} the standard Newtonian $\frac{1}{r}$ term dominates at smaller distances so that CG yields the same results for geodesic motion as those of standard GR at the scale of the Solar System. At larger galactic distances, the additional γr term might explain the flat galactic rotation curves without the need of dark matter ([7], [21], [22]). At even bigger distances, the quadratic term $-\kappa r^2$ plays a role in the dynamics of stars rotating at the largest possible distances from galactic centers, as recently established by Mannheim and O'Brien ([23], [24], [25], [26]).

An equivalent way to illustrate the interplay of the different terms in Eq. (8) is to write a classical gravitational potential (multiplying by $c^2/2$ all the terms that contain gravitational parameters in that equation):

$$V(r) = \frac{c^2}{2}\left[-3\beta\gamma - \frac{\beta(2-3\beta\gamma)}{r} + \gamma r - \kappa r^2\right], \tag{9}$$

or by considering the potential energy $U(r) = mV(r)$, for a body of mass m in the gravitational field due to the source mass $M = \frac{\beta c^2}{G}$. From this potential energy we can obtain the equivalent classical central force:

$$F(r) = -\frac{dU}{dr} = -m\frac{dV}{dr} = -\frac{GMm(2-3\beta\gamma)}{2r^2} - \frac{mc^2\gamma}{2} + mc^2\kappa r, \tag{10}$$

where the right-hand side of the equation reduces to the standard attractive Newtonian force for $\gamma, \kappa \to 0$. The two additional force terms, due to CG, represent respectively a constant attractive force and a linear repulsive force, which might be the origin of the almost flat galactic rotation curves (due mainly to the constant force term) and of the accelerated expansion of the Universe (attributable to the repulsive linear term).

In line with this simplified, classical approach to CG, by equating the magnitude of the force $F(r)$ to the test body centripetal force mv^2/r, we readily obtain the circular velocity expression in CG:

$$v(r) = \sqrt{\frac{GM(2-3\beta\gamma)}{2r} + \frac{c^2\gamma r}{2} - c^2\kappa r^2}, \tag{11}$$

where, in addition to the Newtonian $1/r$ term, two other conformal terms appear, linear and quadratic in r, which can be used to model galactic rotation curves in Conformal Gravity.

Extensive data fitting of galactic rotation curves has been carried out by Mannheim et al. ([7], [21], [22], [23], [24], [25], [26]) without any DM contribution. The values of the conformal parameters were determined as follows [7]:

$$\gamma^* = 5.42 \times 10^{-41}\,\mathrm{cm}^{-1},\ \gamma_0 = 3.06 \times 10^{-30}\,\mathrm{cm}^{-1},\ \kappa = 9.54 \times 10^{-54}\,\mathrm{cm}^{-2}, \tag{12}$$

where the CG γ parameter is split into a γ^*, due to the contributions to the gravitational potential of the individual stars in the galaxy being studied (stars of reference mass equal to the solar mass $M_\odot$), and a cosmological parameter γ_0, due to the contributions of all the other galaxies in the Universe. The third parameter κ is considered to be of purely cosmological origin.

By performing a "kinematical approach" to CG, in previous publications ([10], [11]), we have shown a different way to compute the CG parameters, obtaining values which differ by a few orders of magnitude from those in Eq. (12):

$$\gamma = 1.94 \times 10^{-28}\,\mathrm{cm}^{-1},\ \kappa = 6.42 \times 10^{-48}\,\mathrm{cm}^{-2}\,. \tag{13}$$

Using either the values in Eq. (12) or those in Eq. (13), inside Eqs. (7)-(11), one always obtains very small corrections to the dynamics of celestial bodies moving within the Solar System. Therefore, CG corrections were always considered negligible [27] at the Solar System level, not affecting in any measurable way the motion of planets, satellites, etc. The effect of CG would only be appreciable at the galactic and cosmological scales, where CG effectively replaces DM and DE.

One could argue that since the CG potential in Eq. (9) contains both a linear and a quadratic term in r, the effects of distant stars, galaxies, etc., should be manifest in our Solar System, thus giving us a way to confirm or rule out Conformal Gravity as a viable gravitational theory. However, CG effects from distant sources only show themselves as "tidal forces" on solar-system bodies, and these tidal effects are completely undetectable in our Solar System ([27], [28]).

The above analysis suggests that CG corrections to the dynamics of the Solar System are essentially negligible and that CG effects are of importance only over larger distance scales. However, these considerations were only based on the static, spherically symmetric CG metric, while many gravitational sources in the Universe are rotating bodies that might even possess electric/magnetic charge.

In the rest of this paper we will concentrate our efforts on the Kerr metric in CG, i.e., on the case of a stationary, axially symmetric rotating system without any charge. The study of solutions for charged bodies goes beyond the purpose of the current work, also because astrophysical black holes are thought to be neutral to a very good approximation.

We conclude this section with some general comments about the relation between CG and GR solutions. It is easy to show that any exterior vacuum solution to Einstein's gravity is a vacuum solution to CG ([18], [20]), and thus that any vacuum solution in GR is conformal to a vacuum solution in CG. On the contrary, there is no general proof that the converse is also true [29], namely that any vacuum solution in CG is conformal to a vacuum solution in GR, although this happens to be the case at least for the static, spherically symmetric CG solution in Eqs. (7)-(8), as originally discovered by Mannheim and Kazanas [18].

In the literature, there is no similar connection between GR and CG solutions, in the case of a rotating source [29]. This is also due to the fact that the most general vacuum solution for rotating sources in CG is not known, since the uniqueness of the fourth-order Kerr solution has not been established yet. Therefore, the study of the exterior Kerr solution in CG is fully justified, although it might be argued that 'global' vacuum solutions in CG (as in the case of black holes, where the interior solution is causally disconnected due to the presence of the event horizon) might be truly conformally equivalent to their GR counterparts.

The situation is different if interior solutions are also considered, since in the interior regions GR and CG solutions are not conformal to each other [30] (the former is a solution to a second-order Poisson equation, while the latter is a solution to a fourth-order Poisson equation). Therefore, it is the matching of the exterior solution to the interior one, at the source surface, that fixes the conformal factor of the exterior solution, thus forcing the CG exterior solution to belong to a conformal sector which is not equivalent, in general, to the one to which the GR exterior solution belongs. Hence, again, the study of the exterior Kerr solution in CG is fully warranted, especially in the case analyzed later in Sect. IV, where the fourth-order Kerr solution will be used to model the external gravitational field of the Earth.

III. KERR METRIC IN CONFORMAL GRAVITY

The solutions to the Reissner-Nordström, Kerr, and Kerr-Newman problems in CG were first introduced by Mannheim and Kazanas in their 1991 paper [20]. In particular, considering only the Kerr metric, the general line element for this geometry can be written as ($c = 1$ in the following):

$$\begin{aligned} ds^2 &= A(x,y)\ dx^2 + 2E(x,y)\ dx\ dy + C(x,y)\ dy^2 \\ &+ D(x,y)\ d\phi^2 + 2F(x,y)\ d\phi\ dt - B(x,y)\ dt^2, \end{aligned} \tag{14}$$

where the coordinates x and y can be identified respectively with the radial coordinate r and $\cos\theta$.

Following the formalism introduced by B. Carter [31], the metric is rewritten as [20]:

$$ds^2 = (bf - ce)\left[\frac{dx^2}{a} + \frac{dy^2}{d}\right] + \frac{1}{(bf - ce)}\left[d\ (b\ d\phi - c\ dt)^2 - a\ (e\ d\phi - f\ dt)^2\right], \tag{15}$$

where a, b, and c are functions depending only on $x \equiv r$, while d, e, and f are functions depending only on $y \equiv \cos\theta$. Carter then obtained the standard GR exterior Kerr solution by setting:

$$\begin{aligned} &a(x) = a^2 - 2MGx + x^2\ ;\ b(x) = a^2 + x^2\ ;\ c(x) = a\ ; \\ &d(y) = 1 - y^2\ ;\ e(y) = a(1 - y^2)\ ;\ f(y) = 1 \end{aligned} \tag{16}$$

where M is the source mass, while the parameter a (also denoted by j or α in the literature[2]) is the angular momentum parameter of the rotating source, i.e., $a = J/M$, with J being the source angular momentum. Again, the speed of light is set to 1 and should not be confused with the function c defined in Eq. (16).

[2] The function $a(x)$ in Eq. (16) should not be confused with the angular momentum parameter a.

It is straightforward to transform the metric in Eqs. (15)-(16), using the equivalences $x \equiv r$ and $y \equiv \cos\theta$, into the more familiar expression of the Kerr metric in Boyer-Lindquist coordinates:

$$ds^2 = -\left[1 - \frac{2\beta r}{r^2 + a^2\cos^2\theta}\right] dt^2 - \left[\frac{4\beta r a \sin^2\theta}{r^2 + a^2\cos^2\theta}\right] dt\, d\phi + \left[\frac{r^2 + a^2\cos^2\theta}{r^2 - 2\beta r + a^2}\right] dr^2 \qquad (17)$$
$$+ \left[r^2 + a^2\cos^2\theta\right] d\theta^2 + \left[r^2 + a^2 + \frac{2\beta r a^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right]\sin^2\theta\, d\phi^2.$$

In the literature, the Kerr metric is also expressed in a more compact way by using the following functions:

$$\rho^2 \equiv r^2 + a^2\cos^2\theta\ ;\ \Delta \equiv r^2 - 2\beta r + a^2\ ;\ \Sigma^2 \equiv \left(r^2 + a^2\right)^2 - a^2\Delta\sin^2\theta; \qquad (18)$$

we will use this simplified notation later, in Sect. III B.

In the *weak field* ($\beta/r \ll 1$) and *slow rotation* ($a/r \ll 1$) limit, the Kerr metric can be simplified as follows:

$$ds^2 \approx -\left(1 - \frac{2\beta}{r}\right) dt^2 - \frac{4\beta a\sin^2\theta}{r} dt\, d\phi + \left(1 + \frac{2\beta}{r}\right) dr^2 + r^2\ \left(d\theta^2 + \sin^2\theta\ d\phi^2\right). \qquad (19)$$

These conditions are verified for rotating sources such as the Sun, or planets in our Solar System.

A. Kerr metric in fourth-order gravity

The MK solution for the Kerr geometry in CG [20] uses the metric in Eq. (15) and the same values for the functions b, c, e, and f in Eq. (16), but redefines the other two functions, a and d, as fourth-degree polynomials:

$$a(x) = a^2 + ux + px^2 + vx^3 - kx^4, \qquad (20)$$
$$d(y) = 1 + r'y - py^2 + sy^3 - a^2ky^4,$$

where the parameters u, v, r',[3] and s need to verify the CG condition:

$$uv - r's = 0. \qquad (21)$$

[3] The parameter r' used here should not be confused with the radial coordinate r.

Mannheim and Kazanas were able to prove that the combination of Eqs. (15), (20), (21), and the definitions of the functions b, c, e, and f given in Eq. (16) represent the most general, exact solution in CG for the Kerr problem, although they did not investigate this solution any further and did not prove its uniqueness.

We can analyze in more detail the fourth-order Kerr solution (i.e., the Mannheim-Kazanas solution described above) by investigating its connections with the other metrics. One would expect that the fourth-order Kerr solution should reduce to the standard second-order Kerr solution, for $\gamma, \kappa \to 0$, analogous to the second-order reduction of the fourth-order Schwarzschild metric. Similarly, the fourth-order Kerr solution (for a rotating source) should reduce to the fourth-order Schwarzschild solution[4] (for a non-rotating source) for $a \to 0$, just as it does for the second-order solutions. The parameters u, p, v, k, r', and s could then be chosen as appropriate functions of β, γ, and κ so that the connections just described are verified.

However ([20], [32]), the CG fourth-order Kerr solution (rotating source) is actually *not compatible* with the (non-rotating) CG fourth-order Schwarzschild metric, for $a \to 0$, while it is possible to reduce it to the second-order Kerr metric for $\gamma, \kappa \to 0$. Despite this problem, it was shown by MK ([20], Sect. V) that, with the following choice of parameters and functions:

$$\begin{aligned} a = 0 \ ; \ p = 1 \ ; \ r' = s = v = 0 \\ a(x) = ux + x^2 - kx^4 \\ d(y) = 1 - y^2, \end{aligned} \tag{22}$$

the CG fourth-order Kerr solution reduces to a simple second-order Schwarzschild-de Sitter metric which, under an appropriate conformal transformation, can be brought to the form of the CG fourth-order Schwarzschild metric. Thus, the CG fourth-order Kerr solution is conformally equivalent to the CG fourth-order Schwarzschild metric, for $a = 0$, and this is sufficient to prove that the two solutions are related as expected [32].

Following these considerations, we have investigated this connection further in order to

[4] Here we are improperly naming the fourth-order CG solutions as "fourth-order Kerr," and "fourth-order Schwarzschild" solutions. All these solutions were in fact introduced by Mannheim and Kazanas. By using these names we simply mean the equivalent CG fourth-order solutions for the Kerr and Schwarzschild geometries.

obtain an explicit solution for the CG fourth-order Kerr metric in terms of the conformal parameters γ and κ. Expanding the analysis in Sect. V of Ref. [20], with some additional algebra, it is easy to show that the metric described by the parameters in Eq. (22) also requires:

$$
\begin{aligned}
u &= -\beta(2-3\beta\gamma) \\
k &= \kappa + \frac{\gamma^2(1-\beta\gamma)}{(2-3\beta\gamma)^2}
\end{aligned}
\tag{23}
$$

in order to be conformally equivalent to the MK solution in Eqs. (7)-(8).[5]

Therefore, the full CG fourth-order solution for the Kerr geometry can be written as in Eq. (15), with the functions:

$$
\begin{aligned}
& a(x) = a^2 + ux + x^2 - kx^4 \ ; \ b(x) = a^2 + x^2 \ ; \ c(x) = a \ ; \\
& d(y) = 1 - y^2 - a^2 k y^4 \ ; \ e(y) = a(1-y^2) \ ; \ f(y) = 1
\end{aligned}
\tag{24}
$$

with u and k expressed in terms of the original conformal parameters γ and κ, as in Eq. (23).[6]

A more practical expression for the CG fourth-order Kerr metric can be obtained by recasting the previous solution into the more familiar Boyer-Lindquist coordinates. After some algebra, we obtain:

$$
\begin{aligned}
ds^2 = & -\left[1 + \frac{ur}{r^2 + a^2\cos^2\theta} - k\left(r^2 - a^2\cos^2\theta\right)\right] \ dt^2 \\
& + 2\left[\frac{ura\sin^2\theta + ka\left(a^2(r^2+a^2)\cos^4\theta - r^4\sin^2\theta\right)}{r^2 + a^2\cos^2\theta}\right] \ dt\, d\phi \\
& + \left[\frac{r^2 + a^2\cos^2\theta}{r^2 + ur + a^2 - kr^4}\right] \ dr^2 + \left[\frac{r^2 + a^2\cos^2\theta}{1 - ka^2\cos^2\theta\cot^2\theta}\right] \ d\theta^2 \\
& + \left[\left(r^2 + a^2 - \frac{ura^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right)\sin^2\theta + ka^2\frac{\left(r^4\sin^4\theta - (r^2+a^2)^2\cos^4\theta\right)}{r^2 + a^2\cos^2\theta}\right] \ d\phi^2,
\end{aligned}
\tag{25}
$$

where u and k are again defined through Eq. (23). Since $u \to -2\beta$ and $k \to 0$, for $\gamma, \kappa \to 0$, it is easy to confirm that the fourth-order Kerr metric in Eq. (25) correctly reduces to the

[5] Following the discussion in Sect. V of Ref. [20], this requires us to set as integration constant $c = \gamma/(2-3\beta\gamma)$, which defines the transformation $r = \rho/(1-\rho c)$ between the radial coordinates, r and ρ, of the two equivalent metrics. This choice also sets the function $p(r) = r/(1+cr)$, which determines the conformal factor $\Omega(r) = p(r)/r$ connecting the two metrics.

[6] Due to the numerical values of γ and κ in Eq. (12), or in Eq. (13), we have $k \simeq \kappa$ in Eq. (23). Therefore, the parameters k and κ are practically equivalent but different in principle and should not be confused.

second-order Kerr metric of Eq. (17), when the CG parameters γ and κ are set to zero. On the other hand, setting $a = 0$, i.e., for a non-rotating source, we obtain from Eq. (25) the already mentioned Schwarzschild-de Sitter metric, which is conformally equivalent to the MK solution in Eqs. (7)-(8).

Therefore, the metric in Eq. (25) represents the exact CG fourth-order solution for the Kerr geometry, expressed explicitly in terms of the conformal parameters γ and κ. This solution will be used in the rest of the paper. As in the original study by Mannheim and Kazanas [20], we will not attempt to prove that this solution is also unique.

Just as it was done for the second-order solution, we can also consider the *weak field* ($u/r \ll 1$), *slow rotation* ($a/r \ll 1$) limit of the previous metric (assuming also $kr^2 \ll 1$, i.e., for values of the radial distance not too large, so that the quantity kr^2 is small and comparable with the other two quantities above):

$$\begin{aligned} ds^2 \approx & -\left(1 + \frac{u}{r} - kr^2\right)\, dt^2 + 2\left(\frac{u}{r} - kr^2\right) a \sin^2\theta \; dt\; d\phi \\ & + \left(1 - \frac{u}{r} + kr^2\right)\, dr^2 + r^2\left(d\theta^2 + \sin^2\theta \; d\phi^2\right), \end{aligned} \tag{26}$$

which reduces to the *weak field*, *slow rotation* Kerr second-order expression in Eq. (19), for $u \to -2\beta$ and $k \to 0$, in the non-conformal gravity case. We can also remark that, from the asymptotic behaviors (i.e., in the limit $r \to \infty$) of the metric coefficients in both Eqs. (25) and (26), the fourth-order Kerr metric does not asymptotically reduce to flat space-time, as in the second-order Kerr solution. This is, of course, a general feature of all CG solutions, including the original MK metric for the static, spherically symmetric source in Eqs. (7) and (8).

In the following sections, we will continue to use the full, exact solution in Eq. (25), without any approximation. The correctness of this expression has been tested with the aid of a Mathematica program, which was developed for other studies in CG [33]. This Mathematica routine is able to compute and manipulate symbolically all relevant tensors in both GR and CG. In particular, for a given metric such as the one in Eq. (25), our program can compute the full expression of the Bach tensor $W_{\mu\nu}$ in Eq. (6) and verify that $W_{\mu\nu} = 0$, as follows from Eq. (4), for all exterior solutions where $T_{\mu\nu} = 0$. Therefore, our fundamental solution in Eq. (25) agrees with the field equations of Conformal Gravity, for a Kerr geometry.

B. The separability of the Hamilton-Jacobi equation

Since its introduction in 1963, the Kerr metric [34] in Eq. (17) has played an essential role in the description of rotating black holes in General Relativity. Another fundamental theoretical advance in this field of GR was achieved in 1968 by B. Carter ([35], [36]) who demonstrated the separability of the related Hamilton-Jacobi equation and deduced the existence of an additional conserved quantity: Carter's constant $\mathcal{Q}$ (or $\mathcal{K}$).

This discovery essentially solved the problem of geodesic motion in Kerr space-time: in addition to the energy, the angular momentum about the symmetry axis, and the norm of the four-velocity, Carter's constant $\mathcal{Q}$ provided a fourth conserved quantity which reduced the problem of solving the equations of geodesic motion to one involving only quadratures. The complete analysis of the Kerr space-time, including the separability of the Hamilton-Jacobi equation and the study of geodesic motion, is presented in detail in the classic book by S. Chandrasekhar ([37], Chapters 6-7). In this section we will draw a parallel between the discussion by Chandrasekhar (especially the analysis in sections 62 and 64 of Chapter 7) and the similar discussion of the problem in CG fourth-order Kerr space-time.

We begin by adopting the same notation used in Chandrasekhar's book ($c = G = 1$ in the following), which is slightly different from the one used so far, for a better comparison of the GR and CG Kerr space-time formalism. The Kerr metric in Eqs. (17)-(18) can be rewritten as follows ([37], Chapter 6):

$$ds^2 = -\rho^2 \frac{\Delta}{\Sigma^2}\, dt^2 + \frac{\Sigma^2}{\rho^2} \left[d\phi - \frac{2aMr}{\Sigma^2} dt \right]^2 \sin^2\theta \; + \frac{\rho^2}{\Delta}\, dr^2 + \rho^2 d\theta^2 \tag{27}$$

with

$$\rho^2 \equiv r^2 + a^2 \cos^2\theta \; ; \; \Delta \equiv r^2 - 2Mr + a^2 \; ; \; \Sigma^2 \equiv \left(r^2 + a^2\right)^2 - a^2 \Delta \sin^2\theta. \tag{28}$$

In this new notation a still represents the angular momentum parameter ($a = J/M$), while M, in Eqs. (27)-(28), directly indicates the geometrized mass, i.e., $M \equiv \beta$ (β was equal to $\frac{GM}{c^2}$ in previous equations, where M was the non-geometrized mass). It is easy to check that Eq. (27) is the same metric of Eq. (17). We use a space-like convention for all the metrics in this paper, while Chandrasekhar adopted a time-like convention for the metric signature in his book.

Our CG fourth-order Kerr metric in Eq. (25) can also be recast in this notation:[7]

$$ds^2 = -\rho^2 \frac{\widetilde{\Delta}_r\,\widetilde{\Delta}_\theta}{\widetilde{\Sigma}^2}\,dt^2 + \frac{\widetilde{\Sigma}^2}{\rho^2}\left[d\phi + \frac{\widetilde{\Delta}_r - (r^2+a^2)\,\widetilde{\Delta}_\theta}{\widetilde{\Sigma}^2} a\;dt\right]^2 \sin^2\theta\; + \frac{\rho^2}{\widetilde{\Delta}_r}\,dr^2 + \frac{\rho^2}{\widetilde{\Delta}_\theta} d\theta^2, \quad (29)$$

with extended definitions for the CG auxiliary quantities and functions:

$$\widetilde{M} \equiv M\left(1 - \frac{3}{2}\,M\gamma\right)\;;\; \widetilde{\Delta}_r \equiv r^2 - 2\widetilde{M}r + a^2 - kr^4\;; \quad (30)$$
$$\widetilde{\Delta}_\theta \equiv 1 - ka^2\cos^2\theta\cot^2\theta\;;\; \widetilde{\Sigma}^2 \equiv \widetilde{\Delta}_\theta\left(r^2+a^2\right)^2 - a^2\widetilde{\Delta}_r\sin^2\theta\;.$$

It is easy to check that the CG auxiliary quantities in the last equation[8] reduce to those in Eq. (28) in the non-conformal gravity case, i.e., for $\gamma, \kappa \to 0$ ($\widetilde{\Delta}_\theta \to 1$, in the non-CG case). Therefore, the CG metric in Eq. (29) reduces to the GR metric in Eq. (27), when the non-conformal case is considered.

It is beyond the scope of this work to analyze in detail the singularities and other properties of the metric in Eq. (25) or Eq. (29). We just remark that our fourth-order metric is singular for $\widetilde{\Delta}_r = 0$, $\widetilde{\Delta}_\theta = 0$, and $\rho^2 = 0$ in a similar way to the second-order metric, which is singular for $\Delta = 0$ and $\rho^2 = 0$. A limited analysis of the curvature invariants of the fourth-order Kerr metric shows that the latter singularity, for $\rho^2 = 0$, is also a curvature singularity in the fourth-order case, while $\widetilde{\Delta}_r \equiv r^2 - 2\widetilde{M}r + a^2 - kr^4 = 0$ is likely to be the coordinate singularity related to the black hole horizon, but with a more complex structure due to the presence of the $-kr^4$ term. An additional singularity, for $\widetilde{\Delta}_\theta \equiv 1 - ka^2\cos^2\theta\cot^2\theta = 0$, appears in the fourth-order case: since the k parameter is very small, this singularity corresponds to $\theta \simeq 0,\ \pi$. This is also likely to correspond to a coordinate singularity.

Following the treatment outlined in [37] (Chapter 7, §62), the Hamilton-Jacobi equation for geodesic motion in a space-time with metric tensor $g^{\mu\nu}$ is

$$2\frac{\partial S}{\partial \tau} = g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}, \quad (31)$$

[7] In the form of Eq. (29) our metric looks similar to the well-known Kerr-AdS_4 black hole metric [38] for an asymptotically anti-de Sitter space. However, our solution is different and fully satisfies Eq. (4) of Conformal Gravity with $T_{\mu\nu} = 0$. It should also be noted that, since our fourth-order Kerr metric is not asymptotically flat, it might be also superradiantly unstable (as in the case of standard general relativity, see [39] or [40] for a review).

[8] In the metric of Eq. (25) we used the quantity $u = -\beta(2-3\beta\gamma)$, also defined in Eq. (23) with $\beta = GM/c^2$. Replacing β with the geometrized mass M yields: $u = -M(2-3M\gamma) = -2M(1-\frac{3}{2}M\gamma) = -2\widetilde{M}$, which explains the definition of $\widetilde{M}$ in Eq. (30).

where τ is an affine parameter along the geodesic, which can be identified with proper time for time-like geodesics, and S is Hamilton's principal function. The Lagrangian[9] $\mathcal{L}$ related to the metric in Eq. (29) is:

$$
\begin{aligned}
2\mathcal{L} = -g_{\mu\nu}\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial\tau} &= \left[1 - \frac{2\widetilde{M}r}{\rho^2} - k\left(r^2 - a^2\cos^2\theta\right)\right]\dot{t}^2 \\
&+ \frac{4a\widetilde{M}r\sin^2\theta - 2ka\left[a^2(r^2+a^2)\cos^4\theta - r^4\sin^2\theta\right]}{\rho^2}\dot{t}\,\dot{\phi} - \frac{\rho^2}{\widetilde{\Delta}_r}\dot{r}^2 - \frac{\rho^2}{\widetilde{\Delta}_\theta}\dot{\theta}^2 \\
&- \left[\left(r^2 + a^2 + \frac{2a^2\widetilde{M}r\sin^2\theta}{\rho^2}\right)\sin^2\theta + ka^2\frac{\left(r^4\sin^4\theta - (r^2+a^2)^2\cos^4\theta\right)}{\rho^2}\right]\dot{\phi}^2,
\end{aligned}
\tag{32}
$$

where the dot over the variables indicates the derivative with respect to the affine parameter τ.

The energy and the angular momentum integrals are easily computed:

$$
\begin{aligned}
p_t = \frac{\partial\mathcal{L}}{\partial\dot{t}} &= \left[1 - \frac{2\widetilde{M}r}{\rho^2} - k\left(r^2 - a^2\cos^2\theta\right)\right]\dot{t} \\
&+ \frac{2a\widetilde{M}r\sin^2\theta - ka\left[a^2(r^2+a^2)\cos^4\theta - r^4\sin^2\theta\right]}{\rho^2}\dot{\phi} \\
&= E = constant
\end{aligned}
\tag{33}
$$

and

$$
\begin{aligned}
-p_\phi = -\frac{\partial\mathcal{L}}{\partial\dot{\phi}} &= \frac{-2a\widetilde{M}r\sin^2\theta + ka\left[a^2(r^2+a^2)\cos^4\theta - r^4\sin^2\theta\right]}{\rho^2}\dot{t} \\
&+ \left[\left(r^2 + a^2 + \frac{2a^2\widetilde{M}r\sin^2\theta}{\rho^2}\right)\sin^2\theta + ka^2\frac{\left(r^4\sin^4\theta - (r^2+a^2)^2\cos^4\theta\right)}{\rho^2}\right]\dot{\phi} \\
&= L_z = constant.
\end{aligned}
\tag{34}
$$

These last three equations are the CG equivalent of Eqs. (147)-(149) in Chapter 7 of Ref. [37], where E and L_z are interpreted respectively as energy per unit mass and angular

[9] The negative sign appearing in Eq. (32), after the first equality sign, is due to our choice of the metric signature (different from the one adopted by Chandrasekhar), so that all the following equations can be compared directly with those in Ref. [37]. It should also be noted that the Lagrangian used to study geodesics is not conformally invariant, even in the non-rotating black hole case [41].

momentum—in the axial direction—per unit mass. The other two canonical momenta

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = -\frac{\rho^2}{\widetilde{\Delta}_r}\dot{r} \tag{35}$$
$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = -\frac{\rho^2}{\widetilde{\Delta}_\theta}\dot{\theta}$$

are obviously non-conserved quantities.

The third integral of motion is related to the conservation of the rest mass, which can be expressed as the constancy of the norm of the four-velocity $\mathbf{k}$:

$$|\mathbf{k}|^2 = \delta_1 = \begin{Bmatrix} 1 \ ; \ \text{for time-like geodesics} \\ 0 \ ; \ \text{for null geodesics} \end{Bmatrix}. \tag{36}$$

The fourth integral of motion is obtained by separation of variables in the Hamilton-Jacobi equation, by seeking a solution to Eq. (31) of the form:

$$S = \frac{1}{2}\delta_1 \tau - Et + L_z \phi + S_r(r) + S_\theta(\theta), \tag{37}$$

where S_r and S_θ are functions only of r and θ, respectively.

The procedure for the separation of variables follows closely the one described in pages 344-347 of Ref. [37], although the related algebra is more challenging, due to the additional terms of CG. For brevity, we will present here the final results of these algebraic computations, which were first derived by hand and then checked with our symbolic Mathematica routines for accuracy.

After various algebraic transformations, the Hamilton-Jacobi equation can be rewritten as

$$\left\{ \widetilde{\Delta}_r \left(\frac{dS_r}{dr}\right)^2 - \frac{1}{\widetilde{\Delta}_r}\left[\left(r^2+a^2\right)E - aL_z\right]^2 + \delta_1 r^2 \right\} \tag{38}$$
$$+ \left\{ \widetilde{\Delta}_\theta \left(\frac{dS_\theta}{d\theta}\right)^2 + \frac{1}{\widetilde{\Delta}_\theta}(aE\sin^2\theta - L_z)^2\csc^2\theta + \delta_1 a^2\cos^2\theta \right\} = 0$$

and the equation can now be separated by introducing a separation constant $\widetilde{\mathcal{K}}$, which is related to Carter's constant $\widetilde{\mathcal{Q}}$ as in the standard second-order analysis:[10]

$$\widetilde{\mathcal{K}} = \widetilde{\mathcal{Q}} + (L_z - aE)^2. \tag{39}$$

[10] As for the other quantities, in the following we will denote the fourth-order constants $\widetilde{\mathcal{K}}$ (or $\widetilde{\mathcal{Q}}$) with a tilde ($\tilde{}$) superscript to distinguish them from their second-order equivalents, $\mathcal{K}$ (or $\mathcal{Q}$).

By equating the two parts of Eq. (38) respectively to $-\widetilde{\mathcal{K}}$ and $+\widetilde{\mathcal{K}}$, we obtain the two separate equations:

$$\begin{aligned}
\widetilde{\Delta}_r^2\left(\frac{dS_r}{dr}\right)^2 &= \left[\left(r^2+a^2\right)E - aL_z\right]^2 - \widetilde{\Delta}_r\left(\widetilde{\mathcal{K}}+\delta_1 r^2\right) \\
\widetilde{\Delta}_\theta^2\left(\frac{dS_\theta}{d\theta}\right)^2 &= -(aE\sin^2\theta - L_z)^2\csc^2\theta + \widetilde{\Delta}_\theta\left(\widetilde{\mathcal{K}}-\delta_1 a^2\cos^2\theta\right),
\end{aligned} \tag{40}$$

which are similar to Eqs. (172)-(173) in Chapter 7 of Ref. [37], apart from a different arrangement of the terms in the two equations due to the presence of the additional conformal factor $\widetilde{\Delta}_\theta$.

Setting $\gamma, \kappa = 0$, and thus $\widetilde{\Delta}_r \to \Delta$, $\widetilde{\Delta}_\theta \to 1$, $\widetilde{M} \to M$, $\widetilde{\mathcal{K}} \to \mathcal{K}$, etc., we can indeed obtain the second-order equations of Ref. [37] by using the identity $(aE\sin^2\theta - L_z)^2\csc^2\theta = (L_z^2\csc^2\theta - a^2E^2)\cos^2\theta + (L_z - aE)^2$ and reinserting $\mathcal{Q} = \mathcal{K} - (L_z - aE)^2$ in the equations. Since the fourth-order equations are more easily written using $\widetilde{\mathcal{K}}$, instead of $\widetilde{\mathcal{Q}}$, we will adopt this notation in the following.

The solution procedure then continues in a way similar to the second-order case (see again Ref. [37]). With the abbreviations:

$$\begin{aligned}
\widetilde{R}(r) &\equiv \left[\left(r^2+a^2\right)E - aL_z\right]^2 - \widetilde{\Delta}_r\left(\widetilde{\mathcal{K}}+\delta_1 r^2\right) \\
\widetilde{\Theta}(\theta) &\equiv -(aE\sin^2\theta - L_z)^2\csc^2\theta + \widetilde{\Delta}_\theta\left(\widetilde{\mathcal{K}}-\delta_1 a^2\cos^2\theta\right) \\
&= -\left[(L_z^2\csc^2\theta - a^2E^2)\cos^2\theta + (L_z - aE)^2\right] + \widetilde{\Delta}_\theta\left(\widetilde{\mathcal{K}}-\delta_1 a^2\cos^2\theta\right),
\end{aligned} \tag{41}$$

similar to the second-order expressions (Eqs. (174)-(175) in Chapter 7 of Ref. [37]):

$$\begin{aligned}
R(r) &\equiv \left[\left(r^2+a^2\right)E - aL_z\right]^2 - \Delta\left[\mathcal{Q}+(L_z - aE)^2 + \delta_1 r^2\right] \\
&= \left[\left(r^2+a^2\right)E - aL_z\right]^2 - \Delta\left(\mathcal{K}+\delta_1 r^2\right) \\
\Theta(\theta) &\equiv \mathcal{Q} - \left[a^2(\delta_1 - E^2) + L_z^2\csc^2\theta\right]\cos^2\theta \\
&= -\left[(L_z^2\csc^2\theta - a^2E^2)\cos^2\theta + (L_z - aE)^2\right] + \left(\mathcal{K}-\delta_1 a^2\cos^2\theta\right),
\end{aligned} \tag{42}$$

we can write the solution for S, from Eq. (37), as

$$S = \frac{1}{2}\delta_1\tau - Et + L_z\phi + \int^r \frac{\sqrt{\widetilde{R}(r)}}{\widetilde{\Delta}_r}dr + \int^\theta \frac{\sqrt{\widetilde{\Theta}(\theta)}}{\widetilde{\Delta}_\theta}d\theta. \tag{43}$$

To obtain the equations of motion, we then set to zero the partial derivatives of S with respect to the constants of motion, $\widetilde{\mathcal{K}}$ (or $\widetilde{\mathcal{Q}}$), δ_1, E, and L_z, following again a procedure similar to the second-order case. We will just report here the final results for our CG fourth-order case and compare them directly with the GR second-order case (see Eqs. (183)-(186) in Chapter 7 of Ref. [37]):

$$\dot{r}^2 = \left(\frac{dr}{d\tau}\right)^2 = \left\{\begin{array}{l} \frac{\widetilde{R}(r)}{\rho^4} = \frac{\widetilde{\Delta}_r{}^2}{\rho^4} p_r^2 \ ; \ \text{4th-order} \\ \frac{R(r)}{\rho^4} = \frac{\Delta^2}{\rho^4} p_r^2 \ ; \ \text{2nd-order} \end{array}\right\}, \tag{44}$$

$$\dot{\theta}^2 = \left(\frac{d\theta}{d\tau}\right)^2 = \left\{\begin{array}{l} \frac{\widetilde{\Theta}(\theta)}{\rho^4} = \frac{\widetilde{\Delta}_\theta^2}{\rho^4} p_\theta^2 \ ; \ \text{4th-order} \\ \frac{\Theta(\theta)}{\rho^4} = \frac{1}{\rho^4} p_\theta^2 \ ; \ \text{2nd-order} \end{array}\right\}, \tag{45}$$

$$\dot{\phi} = \left(\frac{d\phi}{d\tau}\right) = \left\{\begin{array}{c} \frac{1}{\rho^2 \widetilde{\Delta}_r \widetilde{\Delta}_\theta} \left\{\left[\left(r^2+a^2\right)\widetilde{\Delta}_\theta - \widetilde{\Delta}_r\right] aE + \left[\widetilde{\Delta}_\theta \rho^2 + \widetilde{\Delta}_r - \left(r^2+a^2\right)\widetilde{\Delta}_\theta\right] L_z \csc^2\theta\right\} \\ = \frac{1}{\rho^2}\left\{\frac{a\left[\left(r^2+a^2\right)E - aL_z\right]}{\widetilde{\Delta}_r} + \frac{\left(L_z \csc^2\theta - aE\right)}{\widetilde{\Delta}_\theta}\right\} \ ; \ \text{4th-order} \\ \frac{1}{\rho^2\Delta}\left[2aMrE + (\rho^2 - 2Mr)L_z \csc^2\theta\right] \\ = \frac{1}{\rho^2}\left\{\frac{a\left[\left(r^2+a^2\right)E - aL_z\right]}{\Delta} + \left(L_z\csc^2\theta - aE\right)\right\} \ ; \ \text{2nd-order} \end{array}\right\}, \tag{46}$$

$$\dot{t} = \left(\frac{dt}{d\tau}\right) = \left\{\begin{array}{c} \frac{1}{\rho^2 \widetilde{\Delta}_r \widetilde{\Delta}_\theta} \left(\widetilde{\Sigma}^2 E + \left[\widetilde{\Delta}_r - \left(r^2+a^2\right)\widetilde{\Delta}_\theta\right] aL_z\right) \\ = \frac{1}{\rho^2}\left\{\frac{\left(r^2+a^2\right)\left[\left(r^2+a^2\right)E - aL_z\right]}{\widetilde{\Delta}_r} + \frac{a\sin^2\theta\left(L_z\csc^2\theta - aE\right)}{\widetilde{\Delta}_\theta}\right\} \ ; \ \text{4th-order} \\ \frac{1}{\rho^2\Delta}\left(\Sigma^2 E - 2aMrL_z\right) \\ = \frac{1}{\rho^2}\left\{\frac{\left(r^2+a^2\right)\left[\left(r^2+a^2\right)E - aL_z\right]}{\Delta} + a\sin^2\theta\left(L_z\csc^2\theta - aE\right)\right\} \ ; \ \text{2nd-order} \end{array}\right\}, \tag{47}$$

where the functions $\widetilde{R}(r)$, $R(r)$, $\widetilde{\Theta}(\theta)$, and $\Theta(\theta)$ are defined in Eqs. (41)-(42), while the auxiliary quantities and functions are described in Eqs. (28) and (30).[11]

It should be noted that our fourth-order equations of motion above are very similar to the analogous equations of motion in Kerr-de Sitter spacetimes ([42], [43], [44]), once the CG parameter k is connected to the cosmological constant Λ by setting $k = \Lambda/3$. In fact, our Eqs. (44)-(47) have the same structure, for example, of Eqs. (14)-(17) in Ref. [43], since the procedure used for the separation of variables is the same. However, these two groups of equations differ in the definition of the auxiliary functions, $\widetilde{\Delta}_r$, $\widetilde{\Delta}_\theta$ and Δ_r, Δ_θ, respectively

[11] Comparing the fourth-order equations with the respective second-order equations of the standard theory, it can be noted that the CG equations are obtained from the GR equations, by performing the following substitutions: $M \to \widetilde{M}$, $\Delta \to \widetilde{\Delta}_r$, $\Sigma \to \widetilde{\Sigma}$, $\pm 2Mr \to \mp\left[\widetilde{\Delta}_r - (r^2 - a^2)\widetilde{\Delta}_\theta\right]$ and, in some appropriate places, by replacing a factor of 1 with $\widetilde{\Delta}_\theta$.

(and the additional function $\chi = 1 + \frac{a^2\Lambda}{3}$ is used only in the Kerr-de Sitter case), so they are not completely equivalent.

As in the second-order case, following Eqs. (44)-(47), it is evident that the problem of solving the equations of geodesic motion has been reduced to one of quadratures. In the next section, we will apply directly our results to the particular case of the so-called *Flyby Anomaly* (FA) in order to check if our CG fourth-order solutions can explain this gravitational puzzle.

As a final point of this section, we note that the new, fourth-order Carter's constant $\widetilde{\mathcal{K}}$ (as well as the second-order $\mathcal{K}$) can be obtained explicitly, in terms of r or θ, by combining together Eqs. (41) and (42) with Eqs. (44) and (45):

$$\begin{aligned}
\widetilde{\mathcal{K}} &= -\widetilde{\Delta}_r p_r^2 + \frac{\left[\left(r^2+a^2\right)E - aL_z\right]^2}{\widetilde{\Delta}_r} - \delta_1 r^2 \\
&= \widetilde{\Delta}_\theta p_\theta^2 + \frac{\left(aE\sin\theta - L_z\csc\theta\right)^2}{\widetilde{\Delta}_\theta} + \delta_1 a^2\cos^2\theta \ ; \ \text{4th-order} \\
\mathcal{K} &= -\Delta p_r^2 + \frac{\left[\left(r^2+a^2\right)E - aL_z\right]^2}{\Delta} - \delta_1 r^2 \\
&= p_\theta^2 + \left(aE\sin\theta - L_z\csc\theta\right)^2 + \delta_1 a^2\cos^2\theta \ ; \ \text{2nd-order.}
\end{aligned} \tag{48}$$

The alternative form of Carter's constant ($\widetilde{\mathcal{Q}}$ or $\mathcal{Q}$) can be obtained using Eq. (39), or the equivalent second-order relation.

IV. GEODESIC MOTION AND THE FLYBY ANOMALY

The Flyby Anomaly is a small unexpected increase in the geocentric range-rate observed during Earth-flybys of some spacecraft (Galileo [45], NEAR [45], Rosetta [46]), as evidenced by both Doppler and ranging data. It is usually reported as an anomalous change $\Delta V_\infty \sim 1-10\ \mathrm{mm\,/\,s}$ in the osculating hyperbolic excess velocity V_∞ of the spacecraft (see [47], [48], [49], [50], [51], [52] for details and reviews), which is defined as:

$$V_\infty = \sqrt{v^2(r,\theta) - \frac{2\mu}{r}}, \tag{49}$$

where $v(r,\theta)$ is the spacecraft speed along its trajectory, which will be computed later in this section using the previous analysis of geodesic motion. In the last equation $\mu = 3.986004\times 10^{20}\ \mathrm{cm^3\,/\,s^2}$ is the universal gravitational constant times the Earth mass and r is the radial distance in a geocentric reference frame.

In Ref. [52] an empirical formula was introduced which could approximate well the observed anomalies in at least three cases (Galileo first flyby, GLL-I; NEAR; and Rosetta) in terms of the incoming/outgoing declinations (δ_i and δ_o) of the asymptotic spacecraft velocities. Explicitly:

$$\frac{\Delta V_\infty}{V_\infty} = K\left(\cos\delta_i - \cos\delta_o\right), \tag{50}$$

with $K \equiv \frac{2\omega_E R_E}{c} = 3.099 \times 10^{-6}$, where ω_E and R_E indicate respectively Earth's angular rotational velocity and mean radius.

For three other Earth flybys (Galileo second flyby, GLL-II; Cassini; and MESSENGER) it was either not possible to detect a clear anomaly, or any anomaly at all. The recent Earth flyby of NASA's spacecraft Juno (October 9, 2013) is offering a new opportunity to observe the anomaly. While Juno's data are currently being analyzed by NASA/JPL, preliminary studies [53] already rule out standard physics explanations of the anomaly. Similarly, no convincing explanations exist for past flybys, coming from both gravitational and non-gravitational physics (see [48], [51], [53] and references therein).

Although it is still an open question whether the Kerr metric can be used for rotating, axially-symmetric astrophysical objects other than black holes,[12] we have nevertheless analyzed the geodesic motion in fourth-order Kerr geometry associated with Earth flybys to check if the reported anomalies can be explained by Conformal Gravity. In fact, the Kerr metric is a very good approximation to rotating spacetimes, as long as they rotate slowly—which is the case of Solar System applications ([54], [55], [56]). For other recent extensive studies on geodesic motion and other solutions in conformal Weyl gravity see Refs. [57], [41], [58], [59].

We also need to point out that, in view of the conformal equivalence between the fourth-order Kerr metric and the second-order Kerr-de Sitter metric (discussed in Sect. V of Ref. [20]) and of the similarities between the equations of motion in these two geometries (as discussed in the paragraph following Eq. (47) above), the geodesic motions in these two cases are comparable. Therefore, our treatment of geodesic motion in fourth-order Kerr geometry parallels in many respects the similar analyses of geodesic motion in Kerr-de Sitter spacetimes ([42], [43], [44]).

[12] The problem of finding a rotating perfect-fluid interior solution, which can be matched to a Kerr exterior solution, has not been solved yet.

First, we studied the general laws of geodesic motion in the fourth-order Kerr geometry, following the similar analysis for the second-order case in Ref. [37] (Chapter 7, §63-64). In particular, we considered the most general time-like geodesics by setting $\delta_1 = 1$ in previous equations of Sect. III B (similarly, $\delta_1 = 0$ would be used for null geodesics). We then considered the projection of the geodesics on the (r, θ)-plane:

$$\int_{r_i}^{r} \frac{dr}{\sqrt{\widetilde{R}(r)}} = \pm \int_{\theta_i}^{\theta} \frac{d\theta}{\sqrt{\widetilde{\Theta}(\theta)}}, \tag{51}$$

which follows from Eqs. (44)-(45). Here r_i and θ_i are some initial values for the two coordinates and a similar equation holds also for the second-order case, with $R(r)$ and $\Theta(\theta)$ instead of $\widetilde{R}(r)$ and $\widetilde{\Theta}(\theta)$. The double sign on the right-hand side of the equation allows for possible increase/decrease of the radial distance with increasing (or decreasing) values of the polar angle θ.

It is well known that these orbits, even in the GR second-order case, are in general non-planar, unless we restrict ourselves to purely equatorial orbits. However, in the case being analyzed here (spacecraft in the Earth's gravitational field), the classical unbound hyperbolic Newtonian orbits are a very good approximation. Therefore, for each of the spacecraft motions being analyzed, we used the reported Newtonian orbital parameters to calculate the values of the constants of motion (E, L_z, $\mathcal{K}$, and $\widetilde{\mathcal{K}}$), but then we computed the orbital motion using Eq. (51)—and the equivalent second-order orbit equation—in order to seek possible discrepancies between the two cases.

It is customary to minimize the parameters by setting:

$$\xi = L_z/E \; ; \; \eta = \mathcal{Q}/E^2 \; ; \; \widetilde{\eta} = \widetilde{\mathcal{Q}}/E^2 \tag{52}$$

and rewrite Eq. (51), and the equivalent second-order equation, in terms of modified functions (in italics): $\mathit{R}(r) = R(r)/E^2$, $\widetilde{\mathit{R}}(r) = \widetilde{R}(r)/E^2$, $\mathit{\Theta}(\theta) = \Theta(\theta)/E^2$, and $\widetilde{\mathit{\Theta}}(\theta) = \widetilde{\Theta}(\theta)/E^2$. The modified radial functions can then be written explicitly as:

$$\begin{aligned} \mathit{R}(r) &= r^4 + (a^2 - \xi^2 - \eta)r^2 + 2M[\eta + (\xi - a)^2]r - a^2\eta - r^2\Delta/E^2 \\ \widetilde{\mathit{R}}(r) &= [1 + k(\widetilde{\eta} + (\xi - a)^2)]r^4 + (a^2 - \xi^2 - \widetilde{\eta})r^2 + 2\widetilde{M}[\widetilde{\eta} + (\xi - a)^2]r - a^2\widetilde{\eta} - r^2\widetilde{\Delta}_r/E^2, \end{aligned} \tag{53}$$

while the angular-θ functions and related integrals are more easily computed in terms of $\mu = \cos\theta$ and using the additional parameter $\alpha^2 = a^2\,(1 - 1/E^2) > 0$ for unbound orbits ($E^2 > 1$).

With some algebra the modified angular functions, expressed in terms of μ, become:

$$\begin{aligned} \mathit{\Theta}_\mu &\equiv \Theta_\mu/E^2 = \eta - [\xi^2 + \eta - \alpha^2]\mu^2 - \alpha^2\mu^4 = \alpha^2[(\mu_-^2 - \mu^2)(\mu_+^2 - \mu^2)] \\ \widetilde{\mathit{\Theta}}_\mu &\equiv \widetilde{\Theta}_\mu/E^2 = \widetilde{\eta} - [\xi^2 + \widetilde{\eta} - \alpha^2]\mu^2 - [\alpha^2 + ka^2(\widetilde{\eta} + (\xi - a)^2)]\mu^4 - ka^2[\alpha^2 - a^2]\mu^6, \end{aligned} \tag{54}$$

where the first function (second-order GR case) can be expressed in terms of the two constants

$$\mu_\pm^2 = \frac{1}{2\alpha^2}\left\{[(\xi^2 + \eta - \alpha^2)^2 + 4\alpha^2\eta]^{1/2} \mp (\xi^2 + \eta - \alpha^2)\right\} \tag{55}$$

obtained by solving the quadratic equation associated with it: $\eta - [\xi^2 + \eta - \alpha^2]x - \alpha^2 x^2 = 0$. Similarly, by solving the cubic equation associated with the second function in Eq. (54), $\widetilde{\eta} - [\xi^2 + \widetilde{\eta} - \alpha^2]x - [\alpha^2 + ka^2(\widetilde{\eta} + (\xi - a)^2)]x^2 - ka^2[\alpha^2 - a^2]x^3 = 0$, we could also factorize this function (fourth-order CG case) in terms of three constants, μ_1^2, μ_2^2, and μ_3^2, whose expressions are rather cumbersome and will be omitted here.

The orbit equation (51) can be written in terms of the new functions in Eqs. (53)-(54), for both the GR and CG cases:

$$\begin{aligned} \int_{r_i}^{r} \frac{dr}{\sqrt{R(r)}} &= \mp \int_{\mu_i}^{\mu} \frac{d\mu}{\sqrt{\mathit{\Theta}_\mu}} \\ \int_{\widetilde{r}_i}^{\widetilde{r}} \frac{d\widetilde{r}}{\sqrt{\widetilde{R}(\widetilde{r})}} &= \mp \int_{\mu_i}^{\mu} \frac{d\mu}{\sqrt{\widetilde{\mathit{\Theta}}_\mu}}, \end{aligned} \tag{56}$$

where, from now on, we will also distinguish between the radial coordinates r, computed with second-order equations, and $\widetilde{r}$, computed with fourth-order equations, since in general $r \neq \widetilde{r}$ for a given value of the angle θ (or μ).

In the angular integrals on the right-hand sides of Eq. (56), the range of μ^2 is between zero and the smaller of the two roots μ_-^2 and μ_+^2 for the first equation, and between zero and the smallest of the three roots μ_1^2, μ_2^2, and μ_3^2 for the second equation.[13] Calling $\mu_{\min} = \min(|\mu_-|, |\mu_+|)$ and $\widetilde{\mu}_{\min} = \min(|\mu_1|, |\mu_2|, |\mu_3|)$, in the two different cases, the range for μ is therefore $\mu \in (-\mu_{\min}, \mu_{\min})$ and $\mu \in (-\widetilde{\mu}_{\min}, \widetilde{\mu}_{\min})$, respectively, in the two lines of the last equation. The initial radial value r_i (or $\widetilde{r}_i$) corresponds to μ_i, or $\theta_i = \arccos \mu_i$, for both cases, where a convenient value for μ_i, or θ_i, can be used. In this way, the resulting unbound

[13] This is actually true for the case of unbound orbits with $\eta > 0$ (or $\widetilde{\eta} > 0$), which is valid for the spacecraft motion related to the FA. See Ref. [37] (Chapter 7, §63-64) for a full discussion of all other possible cases.

orbits will intersect the equatorial plane and be confined within the cones $-\mu_{\min} < \mu < \mu_{\min}$ and $-\widetilde{\mu}_{\min} < \mu < \widetilde{\mu}_{\min}$.

The integrals in Eq. (56) can be evaluated analytically in terms of elliptic integrals of the first kind, or numerically with Mathematica routines. Once the orbit equations in the (r, θ) and $(\widetilde{r}, \theta)$ planes have been determined, at least numerically, the coordinate velocities of a test particle undergoing geodesic motion can be computed by combining the fundamental equations (44)-(47) into the following expressions:

$$\begin{Bmatrix} \widetilde{v}_r \\ v_r \end{Bmatrix} = \frac{\dot{r}}{\dot{t}} = \begin{Bmatrix} \dfrac{\sqrt{\widetilde{R}(\widetilde{r})}}{\frac{(\widetilde{r}^2+a^2)[(\widetilde{r}^2+a^2)E-aL_z]}{\widetilde{\Delta}_r}+\frac{a\sin^2\theta(L_z\csc^2\theta-aE)}{\widetilde{\Delta}_\theta}} \ ; \ \text{4th-order} \\ \dfrac{\sqrt{R(r)}}{\frac{(r^2+a^2)[(r^2+a^2)E-aL_z]}{\Delta}+a\sin^2\theta(L_z\csc^2\theta-aE)} \ ; \ \text{2nd-order} \end{Bmatrix}, \tag{57}$$

$$\begin{Bmatrix} \widetilde{v}_\theta \\ v_\theta \end{Bmatrix} = r\frac{\dot{\theta}}{\dot{t}} = \begin{Bmatrix} \dfrac{\widetilde{r}\sqrt{\widetilde{\Theta}(\theta)}}{\frac{(\widetilde{r}^2+a^2)[(\widetilde{r}^2+a^2)E-aL_z]}{\widetilde{\Delta}_r}+\frac{a\sin^2\theta(L_z\csc^2\theta-aE)}{\widetilde{\Delta}_\theta}} \ ; \ \text{4th-order} \\ \dfrac{r\sqrt{\Theta(\theta)}}{\frac{(r^2+a^2)[(r^2+a^2)E-aL_z]}{\Delta}+a\sin^2\theta(L_z\csc^2\theta-aE)} \ ; \ \text{2nd-order} \end{Bmatrix}, \tag{58}$$

$$\begin{Bmatrix} \widetilde{v}_\phi \\ v_\phi \end{Bmatrix} = r\sin\theta\frac{\dot{\phi}}{\dot{t}} = \begin{Bmatrix} \dfrac{\widetilde{r}\sin\theta\left\{\frac{a\left[\left(\widetilde{r}^2+a^2\right)E-aL_z\right]}{\widetilde{\Delta}_r}+\frac{(L_z\csc^2\theta-aE)}{\widetilde{\Delta}_\theta}\right\}}{\frac{(\widetilde{r}^2+a^2)[(\widetilde{r}^2+a^2)E-aL_z]}{\widetilde{\Delta}_r}+\frac{a\sin^2\theta(L_z\csc^2\theta-aE)}{\widetilde{\Delta}_\theta}} \ ; \ \text{4th-order} \\ \dfrac{r\sin\theta\left\{\frac{a\left[\left(r^2+a^2\right)E-aL_z\right]}{\Delta}+(L_z\csc^2\theta-aE)\right\}}{\frac{(r^2+a^2)[(r^2+a^2)E-aL_z]}{\Delta}+a\sin^2\theta(L_z\csc^2\theta-aE)} \ ; \ \text{2nd-order} \end{Bmatrix}. \tag{59}$$

Therefore, the particle speed $v(r, \theta) = \sqrt{v_r^2 + v_\theta^2 + v_\phi^2}$, or $\widetilde{v}(\widetilde{r}, \theta) = \sqrt{\widetilde{v}_r^2 + \widetilde{v}_\theta^2 + \widetilde{v}_\phi^2}$, for both cases, is obtained by combining the previous three equations with the orbit equation $r = r(\theta)$, or $\widetilde{r} = \widetilde{r}(\theta)$, computed from the integrals in Eq. (56). The osculating hyperbolic excess velocity V_∞ in Eq. (49) can be evaluated along the particle orbit, for both the GR and CG cases, and possible differences between the two situations can be postulated as a cause of the Flyby Anomaly. The anomalous change ΔV_∞ in the hyperbolic excess velocity can be due to the difference between the two values of V_∞ computed with CG and GR, i.e.:

$$\Delta V_\infty = V_\infty^{(CG)} - V_\infty^{(GR)}. \tag{60}$$

Once the orbit equations, $r = r(\theta)$ and $\widetilde{r} = \widetilde{r}(\theta)$, have been computed, it is also possible—at least numerically—to fully integrate the equations of motion (44)-(47) and determine all four coordinates r $(\widetilde{r})$, θ, ϕ, and t as functions of the affine parameter τ, or directly obtain the spherical coordinates r $(\widetilde{r})$, θ, ϕ as functions of time t, i.e., the classical equations of motion: $r = r(t)$, $\theta = \theta(t)$, $\phi = \phi(t)$. However, this approach would be unnecessary for the analysis of the FA in the context of the geodesics in the Kerr geometry since the variations of t and ϕ along the orbits do not reveal any additional information about the velocities which is not already included in the orbit equation. Therefore, we will not pursue the full integration of the equations of motion in the following but only consider the orbit equations and the related velocities.

Using the data reported in Table I of Ref. [52] and those retrieved directly from the HORIZONS Web-Interface by JPL/NASA,[14] we have obtained the values of the constants of motion, E, L_z, $\mathcal{K}$, and $\widetilde{\mathcal{K}}$, and computed the necessary integration parameters (r_i, μ_i, $\mu_{\min}$, $\widetilde{\mu}_{\min}$) for the three flybys with a clear anomaly detection (NEAR, Rosetta, GLL-I). Following the previous analysis, the orbit equation was numerically integrated for different angular values of θ in order to obtain the related radial coordinates r - $\widetilde{r}$, and the respective velocities along the orbits were computed with Eqs. (57)-(59). Our numerical routines also allowed us to change the two main conformal parameters γ and κ in order to study the effects of Conformal Gravity on the orbits and velocities.

Using the values of γ and κ in Eqs. (12)-(13), and standard values for Earth's mass, angular momentum, etc., we obtained estimates of the anomalous change ΔV_∞ along the orbits on the order of $\Delta V_\infty \sim 10^{-8} - 10^{-4}\ \mathrm{cm}\,/\,\mathrm{s}$, thus negligible compared to the actual magnitude of the anomaly ($\Delta V_\infty \sim 0.1 - 1.0\ \mathrm{cm}\,/\,\mathrm{s}$). Larger values of ΔV_∞ can be obtained with our simulations by increasing the conformal parameters.

In particular, since the effect of the γ parameter, in the equations of the previous sections, is simply to rescale the mass M into $\widetilde{M} = M\left(1 - \frac{3}{2}\ M\gamma\right)$, as in Eq. (30), its variations do not affect the results considerably. On the contrary, the value of the second parameter, κ, can affect the results significantly since it is present in both $\widetilde{\Delta}_r$ and $\widetilde{\Delta}_\theta$ terms.

Therefore, we studied how the results for ΔV_∞ are affected by possible variations of the parameter κ for the three considered flybys. Fig. 1 shows results of our numerical computations for values of the conformal parameters as in Eq. (13), i.e., for $\gamma = 1.94 \times 10^{-28}\,\mathrm{cm}^{-1}$ and $\kappa = 6.42 \times 10^{-48}\,\mathrm{cm}^{-2}$. For each case, ΔV_∞ is computed as a function of the polar angle θ (in degrees) along the spacecraft trajectory. For example, in the case of the NEAR spacecraft (red solid curve in the figure), the incoming declination $\delta_i = -20.76\,^\circ$ corresponds to an incoming polar angle $\theta_{inc} = 69.24\,^\circ$, which decreases to a minimum angle of

[14] JPL/NASA website at: http://ssd.jpl.nasa.gov/horizons.cgi

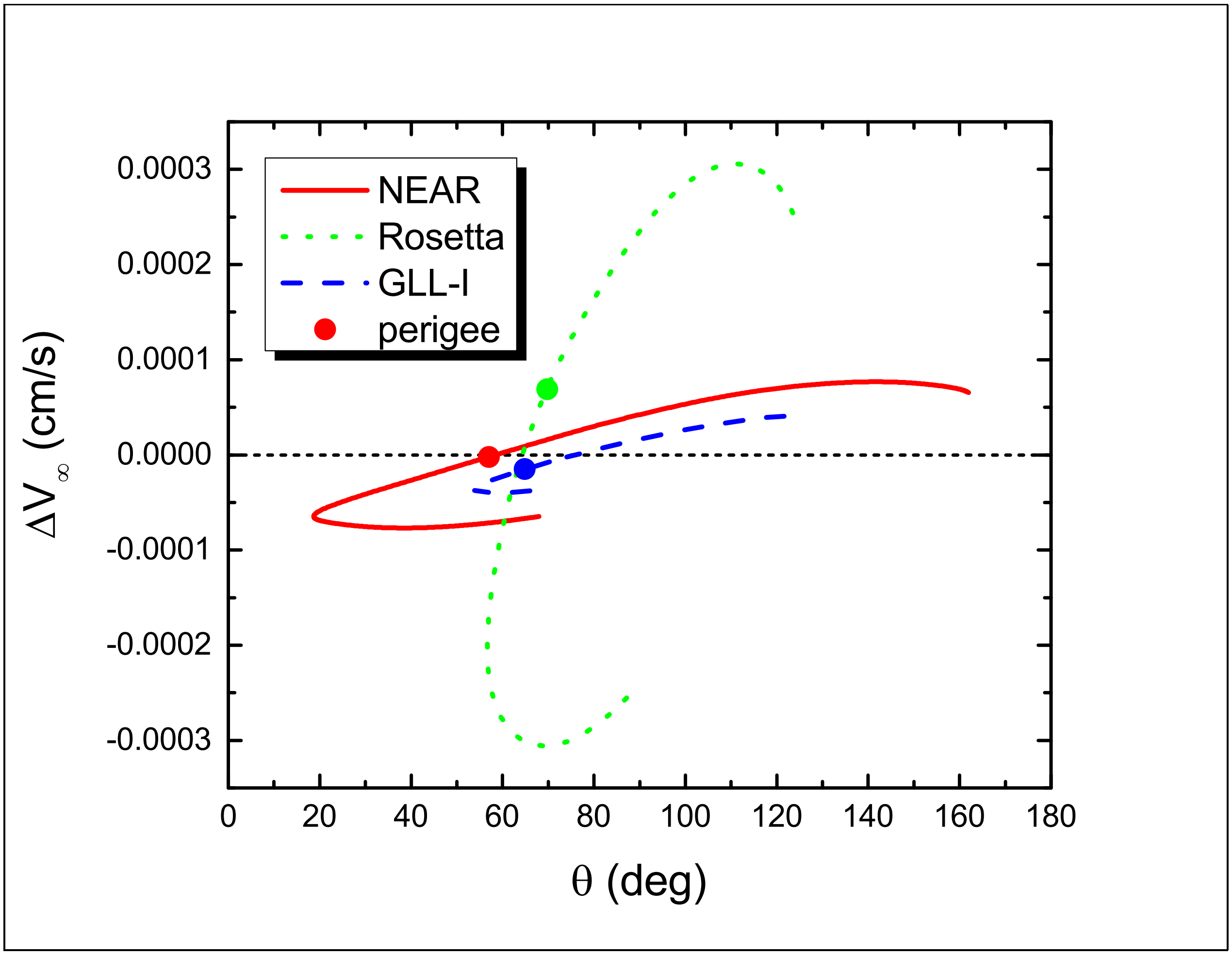

FIG. 1: CG results for the anomaly ΔV_{∞} are shown here, for the three cases examined, as a function of the polar angle θ. The computed anomaly values are very small, compared to the observed values, due to the choice of the second conformal parameter ($\kappa = 6.42 \times 10^{-48}\,\mathrm{cm}^{-2}$). The simulation initial conditions were set to give approximately zero anomaly at perigee angles (marked by solid circles in each case) so that the anomaly values could be positive as well as negative in the different parts of the orbits.

about 18°, then increases to the perigee angle ($\theta_{per} = 57^{\circ}$, marked by a red solid circle in the figure), and finally reaches the maximum outgoing angle $\theta_{out} = 161.96^{\circ}$, corresponding to the outgoing declination $\delta_o = -71.96^{\circ}$. Similar plots are obtained for Rosetta (green dotted curve) and Galileo GLL-I (blue dashed curve), where all the respective angles (incoming, outgoing, minimum, perigee) were obtained from the data in Table I of Ref. [52].

As seen in the figure, all the computed values for the anomaly are on the order of $\Delta V_{\infty} \sim 10^{-4}$ cm / s, thus negligible compared to the experimental values reported. In this case, the initial conditions for the integrals in Eq. (56) were set to give approximately zero anomaly at the perigee angle (marked by solid circles in each case) so that the resulting ΔV_∞ was negative during the first part of the orbit (from the incoming angle to the perigee) and positive during the second part (from perigee to the outgoing angle). It should also be noted that our simulations show the largest anomaly for the Rosetta case, while in the experimental data this spacecraft has the lowest anomaly value of the three.

Fig. 2 shows instead results for an increased value of $\kappa \simeq 5.00 \times 10^{-40}\,\mathrm{cm}^{-2}$ (while keeping the same value as before for γ), thus obtaining increased values for the anomaly ΔV_∞. The meanings of the curves for the three spacecraft are similar to those of Fig. 1, but we have adopted a different choice of the initial conditions for the integrals in Eq. (56), which yields to almost zero anomaly during the first part of the orbit and positive anomaly during the second part of the orbit. These settings are similar to the original experimental fitting ([45], [46], [52]) of the Doppler and ranging data, where the pre-encounter trajectory was fitted to the data (thus showing zero anomaly during the first part of the orbit), while post-encounter data (second part of the orbit) were showing positive residuals, therefore yielding positive values for the anomaly.

In Fig. 2, we also show the measured values of the anomaly, for the three cases examined, represented by the horizontal short-dashed lines (from [52]: respectively, $\Delta V_\infty = 1.346\ \mathrm{cm/s}$ for NEAR, $\Delta V_\infty = 0.180\ \mathrm{cm/s}$ for Rosetta, $\Delta V_\infty = 0.392\ \mathrm{cm/s}$ for Galileo GLL-I). In this figure, the value of $\kappa \simeq 5.00 \times 10^{-40}\,\mathrm{cm}^{-2}$ was chosen to fit the maximum value of the computed anomaly for NEAR (red solid curve) to the experimental value $\Delta V_\infty = 1.346\ \mathrm{cm/s}$ (red horizontal dashed line) since the NEAR anomaly is the most prominent. As in Fig. 1, the computed Rosetta anomaly (green dotted curve) appears to be much larger than the observed value, while the Galileo GLL-I anomaly (blue dashed curve) is consistent with the value of the measured anomaly (blue horizontal dashed line).

Therefore, our analysis shows that it is possible to obtain CG corrections to the geodesic motion of spacecraft executing Earth flybys, yielding values for the anomaly ΔV_∞ comparable to the observed ones, but only if the value of the conformal parameter κ is increased to about $\kappa \sim 10^{-40}\,\mathrm{cm}^{-2}$. Such a value is not supported by the current estimates of this parameter in Eqs. (12) and (13), corresponding to a range of $\kappa \sim 10^{-53} - 10^{-47}\,\mathrm{cm}^{-2}$. Our simulations also overestimate the value of the Rosetta anomaly, compared to the NEAR and GLL-I cases.

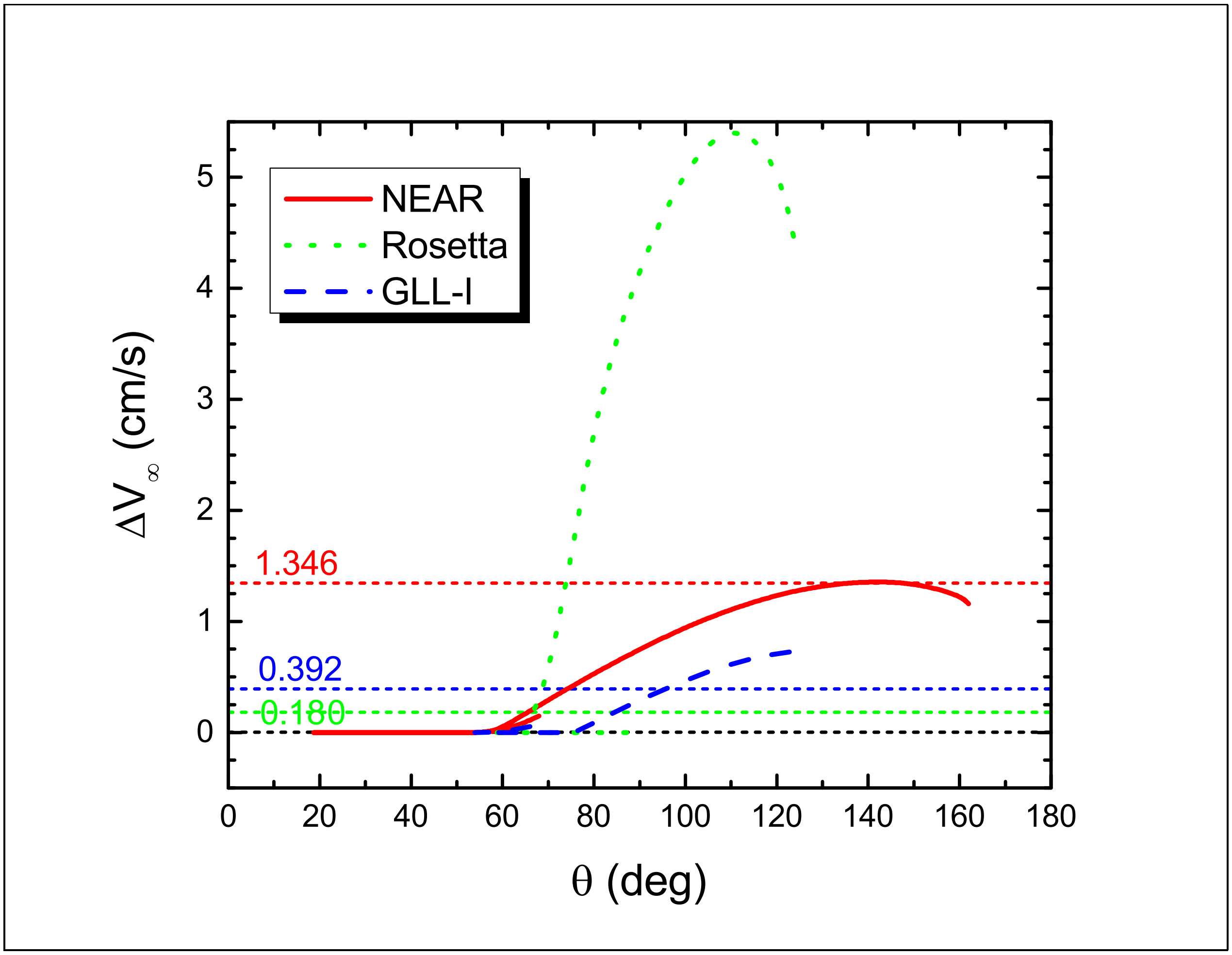

FIG. 2: CG results for the anomaly ΔV_∞ are shown here, for the three cases examined, as a function of the polar angle θ. The computed anomaly values are comparable to the observed values, due to an increased value of the second conformal parameter ($\kappa \simeq 5.00 \times 10^{-40}\,\mathrm{cm}^{-2}$). The simulation initial conditions were set to give approximately zero anomaly during the first part of each orbit, so that the anomaly values are now restricted to positive range in the second part of the orbits. Also shown are the experimental values of the anomaly (short-dashed horizontal lines and related numerical values) for each case.

If the κ parameter is strictly constrained by cosmological data to the above range of $\kappa \sim 10^{-53} - 10^{-47}\,\mathrm{cm}^{-2}$, then Conformal Gravity corrections to geodesic motion around Earth are essentially negligible and are unlikely to be the origin of the Flyby Anomaly. We leave further analysis of the FA, within the framework of CG, to future work, once the data of the recent Juno flyby are made available.

V. CONCLUSIONS

In this work we analyzed possible modifications to the Kerr geometry, geodesic motion, and the problem of the separability of the Hamilton-Jacobi equation, due to fourth-order Conformal Gravitational theory.

We obtained an explicit form of the equivalent Kerr metric in CG, expressed in terms of the supplemental conformal parameters γ and κ, thus characterizing the geometry of a Kerr black hole in fourth-order gravity.

Using this explicit metric, we were able to show that the related Hamilton-Jacobi equation is separable also in CG, and that an additional conserved quantity—similar to the original Carter's constant—can be introduced likewise in this case. As a consequence, geodesic motion in fourth-order Kerr geometry can be studied along the lines of the similar second-order GR case.

Assuming that the Kerr metric can be used as an exterior geometry for any rotating axially-symmetric body, we have performed a limited analysis of the geodesic motion of spacecraft executing flybys around Earth in order to assess the interpretation of CG as the origin of the Flyby Anomaly.

Our preliminary analysis shows that CG is not likely to be the origin of the FA, given the currently estimated values of the conformal parameters. However, CG modifications of the geodesic motion might yield effects comparable to the FA for increased values of the conformal parameters (in particular, of the second parameter κ). Further studies will be needed to investigate this possibility, in view of new FA data obtained from the recent Juno spacecraft flyby.

Acknowledgments

The author would like to thank Loyola Marymount University and the Seaver College of Science and Engineering for continued support and for granting a sabbatical leave of absence to the author, during which this work was completed. The author is also indebted to Ms.

Z. Burstein for helpful comments and for proofreading the original manuscript.

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