8-1990

# Sums of Semiprime, Z, and D L-Ideals in a Class of F-Rings

Suzanne Larson
*Loyola Marymount University*, slarson@lmu.edu

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 109, Number 4, August 1990

# SUMS OF SEMIPRIME, $z$, AND $d$ $l$-IDEALS IN A CLASS OF $f$-RINGS

SUZANNE LARSON

(Communicated by Louis J. Ratliff, Jr.)

ABSTRACT. In this paper it is shown that there is a large class of $f$-rings in which the sum of any two semiprime $l$-ideals is semiprime. This result is used to give a class of commutative $f$-rings with identity element in which the sum of any two $z$-ideals which are $l$-ideals is a $z$-ideal and the sum of any two $d$-ideals is a $d$-ideal.

## INTRODUCTION

An $l$-ideal $I$ of an $f$-ring $A$ is called semiprime if $a^2 \in I$ implies $a \in I$. An ideal $I$ of a commutative ring $A$ with identity element is called a $z$-ideal if whenever $a, b \in A$ are in the same set of maximal ideals and $a \in I$, then $b \in I$. Given an element $a$ of an $f$-ring $A$, let $\{a\}^d = \{x \in A: xa = 0\}$ and $\{a\}^{dd} = \{x \in A: xy = 0$ for all $y \in \{a\}^d\}$. An ideal $I$ of a commutative $f$-ring is called a $d$-ideal if $a \in I$ implies $\{a\}^{dd} \subseteq I$.

Several authors have studied the sums of semiprime $l$-ideals, $z$-ideals, and $d$-ideals in various classes of $f$-rings. In [2, 14.8] it is shown that the sum of two $z$-ideals in $C(X)$, the $f$-ring of all real-valued continuous functions defined on the topological space $X$, is a $z$-ideal; and in [11, 4.1 and 5.1], Rudd shows that in absolutely convex subrings of $C(X)$, the sum of two semiprime $l$-ideals is semiprime and the sum of two $z$-ideals is a $z$-ideal. Mason studies sums of $z$-ideals in absolutely convex subrings of the ring of all continuous functions on a topological space and in more general rings in [10]. An example is given in [5, §7] of an $f$-ring in which there are two (semiprime) $z$-ideals whose sum is not a $z$-ideal or semiprime. Huijsmans and de Pagter show in [7, 4.4] that in a normal Riesz space, the sum of two $d$-ideals is a $d$-ideal.

In [4, 3.9], Henriksen gives a condition on two semiprime $l$-ideals of an $f$-ring which is necessary and sufficient for their sum to be semiprime. Henriksen also notes in [4] that this condition can be difficult to apply globally, and so it seems difficult to use this result to determine in what classes of $f$-rings are the sum of any two semiprime $l$-ideals semiprime.

Received by the editors June 5, 1989 and, in revised form, September 27, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 06F25.

In this note we show that there is a large class of $f$-rings, specifically those $f$-rings in which minimal prime $l$-ideals are square dominated, in which the sum of any two semiprime $l$-ideals is semiprime. We use this result to show that in a commutative $f$-ring with identity element in which minimal prime $l$-ideals are square dominated, if the sum of any two minimal prime $l$-ideals is a $z$-ideal (resp. $d$-ideal), then the sum of any two $z$-ideals which are $l$-ideals is a $z$-ideal (resp. the sum of any two $d$-ideals is a $d$-ideal). As a corollary we show that in a commutative semiprime normal $f$-ring with identity element, the sum of any two $z$-ideals which are $l$-ideals (resp. $d$-ideals) is a $z$-ideal (resp. $d$-ideal).

## 1. Preliminaries

An $f$-ring is a lattice-ordered ring which is a subdirect product of totally ordered rings. For general information on $f$-rings see [1]. Given an $f$-ring $A$ and $x \in A$, we let $A^+ = \{a \in A: a \geq 0\}$, $x^+ = x \vee 0$, $x^- = (-x) \vee 0$, and $|x| = x \vee (-x)$.

A ring ideal $I$ of an $f$-ring $A$ is an $l$-ideal if $|x| \leq |y|$, $y \in I$ implies $x \in I$. Given any element $a \in A$ there is a smallest $l$-ideal containing $a$, and we denote this by $\langle a \rangle$.

Suppose $A$ is an $f$-ring and $I$ is an $l$-ideal of $A$. The $l$-ideal $I$ is *semiprime* (*prime*) if $a^2 \in I$ $(ab \in I)$ implies $a \in I$ $(a \in I$ or $b \in I)$. It is well known that in an $f$-ring, an $l$-ideal is semiprime if and only if it is an intersection of prime $l$-ideals, and that all $l$-ideals containing a given prime $l$-ideal form a chain. In an $f$-ring a semiprime $l$-ideal that contains a prime $l$-ideal is a prime $l$-ideal as shown in [12, 2.5], [7, 4.2].

An ideal $I$ of a commutative ring $A$ with identity element is a $z$-ideal if, whenever $a, b \in A$ are contained in the same set of maximal ideals and $a \in I$, then $b \in I$.

In a commutative $f$-ring, let $\{a\}^d = \{x \in A: ax = 0\}$ and $\{a\}^{dd} = \{x \in A: xy = 0$ for all $y \in \{a\}^d\}$. An ideal $I$ of a commutative $f$-ring is called a $d$-ideal if $a \in I$ implies $\{a\}^{dd} \subseteq I$.

Henriksen, in [4] calls an $l$-ideal $I$ of an $f$-ring $A$ *square dominated* if $I = \{a \in A: |a| \leq x^2$ for some $x \in A$ such that $x^2 \in I\}$. Two characterizations now follow describing those commutative semiprime $f$-rings in which all minimal prime $l$-ideals are square dominated. Parts (1) and (2) of the following lemma are shown to be equivalent in [8, 2.1]. That part (3) of the following lemma is equivalent to part (1) follows easily from the equivalence of parts (1) and (2).

**Lemma 1.1.** *Let $A$ be a commutative semiprime $f$-ring. The following are equivalent*:

(1) *Every minimal prime $l$-ideal of $A$ is square dominated*
(2) *For every $a \in A^+$ the $l$-ideal $\{a\}^d$ is square dominated*

(3) *The l-ideal* $O_P = \{a \in A\colon$ *there exists a* $b \notin P$ *such that* $ab = 0\}$ *is square dominated for all prime l-ideals* $P$ *of* $A$.

## 2.

We begin with two results that will be needed when showing that in an $f$-ring in which minimal prime $l$-ideals are square dominated, the sum of two semiprime $l$-ideals is semiprime.

**Theorem 2.1.** *Let* $A$ *be an* $f$*-ring. In* $A$*, the sum of a semiprime l-ideal and a square-dominated semiprime l-ideal is semiprime.*

*Proof.* Suppose that $I$, $J$ are semiprime $l$-ideals and that $J$ is square dominated. Let $a^2 \in I+J$ with $a \geq 0$. Then $a^2 \leq i+j$ for some $i \in I^+$, $j \in J^+$. Since $J$ is square dominated, $j \leq j_1^2$ for some $j_1 \in A^+$ with $j_1^2 \in J$. So $a^2 \leq i+j_1^2$. Let $x = a-(a\wedge j_1)$ and $y = a\wedge j_1$. Since $J$ is a semiprime $l$-ideal, $j_1 \in J$ and $y \in J$. Now for any positive elements $a$, $j_1$ of any totally ordered ring, $a\wedge j_1 = a$ or $a\wedge j_1 = j_1$. In the first case $(a-(a\wedge j_1))^2 = 0$, and in the second case $(a-(a\wedge j_1))^2 = (a-j_1)^2 = a^2-aj_1-j_1a+j_1^2 \leq a^2-2j_1^2+j_1^2 = a^2-j_1^2$. Therefore in any totally ordered ring, $(a-(a\wedge j_1))^2 \leq 0\vee(a^2-j_1^2)$. This implies that in the $f$-ring $A$, $x^2 = (a-(a\wedge j_1))^2 \leq 0\vee(a^2-j_1^2) \leq i$. Thus, $x^2 \in I$ and hence $x \in I$. We have $a = x+y$ with $x \in I$ and $y \in J$. Therefore $a \in I+J$. □

**Lemma 2.2.** *Let* $A$ *be an* $f$*-ring in which minimal prime l-ideals are square dominated. In* $A$*, the sum of any two prime l-ideals is prime.*

*Proof.* Let $I$ and $J$ be prime $l$-ideals of $A$. Let $I_1$, $J_1$ be minimal prime $l$-ideals contained in $I$, $J$ respectively. We will show $I+J$ is an intersection of prime $l$-ideals. To do so, we let $z \in A$ such that $z \notin I+J$ and we will show there is a prime $l$-ideal containing $I+J$ but not $z$. The $l$-ideal $I_1+J_1$ is prime, and the prime $l$-ideals containing it form a chain. By the maximal principle, there is a prime $l$-ideal $Q$ containing $I_1+J_1$ which is maximal with respect to not containing $z$. By the previous theorem, $I+J_1$ is semiprime. It also contains a prime $l$-ideal and is therefore prime. Similarly, $I_1+J$ is prime. Thus $I \subseteq I+J_1 \subseteq Q$ and $J \subseteq I_1+J \subseteq Q$. This implies that $I+J \subseteq Q$ and $z \notin Q$. Therefore $I+J$ is an intersection of prime $l$-ideals. So it is semiprime. It also contains a prime $l$-ideal and is therefore prime. □

In [3, 4.7], Gillman and Kohls show that in $C(X)$, the $f$-ring of all real-valued continuous functions defined on the topological space $X$, an $l$-ideal is an interesection of $l$-ideals, each of which contains a prime $l$-ideal. Their proof easily generalizes to prove that in an $f$-ring, an $l$-ideal which contains all nilpotent elements of the $f$-ring is an intersection of $l$-ideals, each of which contains a prime $l$-ideal. We will make use of this result in the proof of the following theorem.

**Theorem 2.3.** *Let $A$ be an $f$-ring in which minimal prime $l$-ideals are square dominated. In $A$, the sum of any two semiprime $l$-ideals is semiprime.*

*Proof.* Let $I$, $J$ be semiprime $l$-ideals. We will show $I+J$ is an intersection of prime $l$-ideals. To do so, we let $z \in A$ such that $z \notin I+J$ and we show that there is a prime $l$-ideal containing $I+J$ but not $z$. By Gillman and Kohl's result mentioned above, there is an $l$-ideal $Q$ containing $I+J$ and containing a prime $l$-ideal but not containing $z$. Let $P$ be a minimal prime $l$-ideal contained in $Q$. By Theorem 2.1, $P+I$ is semiprime. Also, it contains a prime $l$-ideal and so is prime. Similarly, $P+J$ is prime. Then by the previous lemma, $(P+I)+(P+J)$ is prime. Since $(P+I)+(P+J) \subseteq Q$, $z \notin (P+I)+(P+J)$ and $I+J \subseteq (P+I)+(P+J)$. □

The converse of the previous theorem does not hold, as we show next.

**Example 2.4.** In $C([0, 1])$, denote by $i$ the function $i(x) = x$, and by $e$ the function $e(x) = 1$. Let $A = \{f \in C([0, 1]) : f = ae + g \text{ where } a \in \mathbf{R},\ g \in \langle i \rangle\}$ with coordinate operations. Then $A$ is a commutative semiprime $f$-ring.

We will show that the sum of two semiprime $l$-ideals of $A$ is semiprime. So suppose $I$, $J$ are semiprime $l$-ideals. If $I$ or $J$ contains an element $f = ae+g$ such that $a \neq 0$, then it can be shown that $I$ or $J$ is square dominated. Then by Theorem 2.1, $I+J$ is semiprime. So we may now suppose that both $I, J \subseteq \langle i \rangle$. If $f^2 \in I+J$, then there is $i_1 \in I^+$, $j_1 \in J^+$ such that $f^2 = i_1 + j_1$. Also, $f \in \langle i \rangle$ which implies $|f| \leq ni$ and $f^2 \leq n^2 i^2$ for some $n \in \mathbf{N}$. So $i_1 \leq n^2 i^2$ and $j_1 \leq n^2 i^2$. Therefore $\sqrt{i_1} \leq ni$ and $\sqrt{i_1} \in A$. Since $I$ is semiprime, $\sqrt{i_1} \in I$. Similarly, $\sqrt{j_1} \in J$. So $f \leq \sqrt{i_1} + \sqrt{j_1}$ implies $f \in I+J$. Thus $I+J$ is semiprime.

Next we show that not every minimal prime $l$-ideal of $A$ is square dominated. Let $f$ be a function such that $0 \leq f \leq i$, $f(x) = 0$ for all $x \in [1/4, 1]$, $f(x) = 0$ for all $x \in [1/(4n+2), 1/4n]$, and $f(1/(4n+3)) = 1/(4n+3)$ for all $n \in \mathbf{N}$. Also, let $g$ be a function such that $0 \leq g \leq i$, $g(x) = 0$ for all $x \in [1/4, 1]$, $g(1/(4n+1)) = 1/(4n+1)$, and $g(x) = 0$ for all $x \in [1/(4n+4), 1/(4n+2)]$ for all $n \in \mathbf{N}$. Then $g \in \{f\}^d$, and there is no element $h \in A$ which satisfies $g \leq h^2$ and $h^2 \in \{f\}^d$. So $\{f\}^d$ is not square dominated, and Lemma 1.1 implies that not every minimal prime $l$-ideal of $A$ is square dominated.

Next we turn our attention to the sum of two $z$-ideals which are $l$-ideals and to the sum of two $d$-ideals. Note that in a commutative $f$-ring with identity element, a $z$-ideal is not always an $l$-ideal. However it can easily be seen that in a commutative $f$-ring with identity element, if every maximal ideal is an $l$-ideal (or equivalently if for all $x \geq 1$, $x^{-1}$ exists), then a $z$-ideal is always an $l$-ideal. In a commutative $f$-ring with identity element, every $d$-ideal is an $l$-ideal. G. Mason has established three results concerning $z$-ideals which we will use in the proof of the next theorem. The first is as follows.

($\alpha$) In a commutative ring with identity element, every $z$-ideal is semiprime [9, 1.0].

The second was proven for a commutative ring with identity element, and the third was proven for a commutative ring with identity element in which the prime ideals containing a given prime form a chain. With only very slight modifications to the proofs, these results can be given in the context of $f$-rings.

($\beta$) If, in a commutative $f$-ring with identity element, $P$ is minimal in the class of prime $l$-ideals containing a $z$-ideal $I$ which is an $l$-ideal, then $P$ is also a $z$-ideal. In particular, minimal prime $l$-ideals are $z$-ideals [9, 1.1].

($\gamma$) If, in a commutative $f$-ring with identity element, the sum of any two minimal prime $l$-ideals is a prime $z$-ideal, then the sum of any two prime $l$-ideals not in a chain is a $z$-ideal [10, 3.2].

One can easily mimic the proofs to ($\beta$) and ($\gamma$) to show analogous results about $d$-ideals.

($\beta'$) If, in a commutative $f$-ring with identity element, $P$ is minimal in the class of prime $l$-ideals containing a $d$-ideal $I$, then $P$ is also a $d$-ideal. In particular, minimal prime $l$-ideals are $d$-ideals.

($\gamma'$) If, in a commutative $f$-ring with identity element, the sum of any two minimal prime $l$-ideals is a prime $d$-ideal, then the sum of any two prime $l$-ideals which are not in a chain is a $d$-ideal.

**Theorem 2.5.** *Let $A$ be a commutative $f$-ring with identity element in which minimal prime $l$-ideals are square dominated.*

(1) *If the sum of any two minimal prime $l$-ideals of $A$ is a $z$-ideal, then the sum of any two $z$-ideals which are $l$-ideals of $A$ is a $z$-ideal.*

(2) *If the sum of any two minimal prime $l$-ideals of $A$ is a $d$-ideal, then the sum of any two $d$-ideals of $A$ is a $d$-ideal.*

*Proof.* We first show part (1). Suppose $I$, $J$ are $z$-ideals which are $l$-ideals. Then $I$, $J$ are semiprime $l$-ideals by ($\alpha$), and by Theorem 2.3, $I + J$ is a semiprime $l$-ideal. We will show that $I + J$ is the intersection of $z$-ideals. To do so, we let $z \in A$ such that $z \notin I + J$, and we will show there is a $z$-ideal containing $I + J$ but not $z$. Since $I + J$ is a semiprime $l$-ideal, it is the intersection of prime $l$-ideals. So there is a prime $l$-ideal $P$ containing $I + J$ but not $z$. Let $P_1, P_2 \subseteq P$ be prime $l$-ideals minimal with respect to containing $I$, $J$ respectively. By ($\beta$), $P_1$, $P_2$ are prime $z$-ideals. It follows from ($\gamma$) that $P_1 + P_2$ is a $z$-ideal. Also, $I + J \subseteq P_1 + P_2$ and $z \notin (P_1 + P_2)$ since $P_1 + P_2 \subseteq P$.

The proof of part (2) is analogous. □

Recall that for any element $a$ of an $f$-ring, $\{a\}^d$ is a $z$-ideal and a $d$-ideal. Recall also that a prime $l$-ideal $P$ of a commutative semiprime $f$-ring is minimal if and only if $a \in P$ implies there is a $b \notin P$ such that $ab = 0$.

**Corollary 2.6.** *Let $A$ be a commutative semiprime $f$-ring with identity element in which minimal prime $l$-ideals are square dominated.*

(1) *If for every $a, b \in A^+$, $\{a\}^d + \{b\}^d$ is a $z$-ideal, then the sum of any two $z$-ideals which are $l$-ideals of $A$ is a $z$-ideal.*
(2) *If for every $a, b \in A^+$, $\{a\}^d + \{b\}^d$ is a $d$-ideal, then the sum of any two $d$-ideals of $A$ is a $d$-ideal.*

*Proof.* To show part (1), we need only show that the sum of any two minimal prime $l$-ideals is a $z$-ideal. Let $P$, $Q$ be minimal prime $l$-ideals. Suppose $a$, $b$ are in the same set of maximal ideals and $b \in P+Q$. Then $b = p+q$ for some $p \in P$, $q \in Q$. Also, there is $p_1, q_1 \in A^+$ such that $p_1 \notin P$, $q_1 \notin Q$, and $pp_1 = 0$, $qq_1 = 0$. So $b = p+q \in \{p_1\}^d + \{q_1\}^d$. By hypothesis, $\{p_1\}^d + \{q_1\}^d$ is a $z$-ideal. So $a \in \{p_1\}^d + \{q_1\}^d \subseteq P+Q$.

The proof of part (2) is analogous. □

An $f$-ring (and more generally a Riesz space) $A$ is called *normal* if $A = \{a^+\}^d + \{a^-\}^d$ for all $a \in A$, or equivalently if $a \wedge b = 0$ implies $A = \{a\}^d + \{b\}^d$. In [8, 2.5] it is shown that in a commutative, semiprime normal $f$-ring with identity element, every minimal prime $l$-ideal is square dominated.

**Corollary 2.7.** *Let $A$ be a commutative semiprime normal $f$-ring with identity element. In $A$, the sum of any two $z$-ideals which are $l$-ideals is a $z$-ideal and the sum of any two $d$-ideals is a $d$-ideal.*

*Proof.* In view of the fact that minimal prime $l$-ideals of $A$ are square dominated and in light of Theorem 2.5, we need only show that the sum of any two minimal prime $l$-ideals is a $z$-ideal and a $d$-ideal. So let $P$, $Q$ be minimal prime $l$-ideals. We will show that if $P \neq Q$, then $P+Q = A$. If $P \neq Q$, then there is an element $p \in P \backslash Q$. Since $P$ is a minimal prime, there is an element $q \notin P$ such that $pq = 0$. Then $p \wedge q = 0$, and $\{p\}^d + \{q\}^d = A$. But $\{p\}^d \subseteq Q$, $\{q\}^d \subseteq P$. So $A = P+Q$. □

Huijsmans and de Pagter show in [6, 4.4] that in a normal Riesz space the sum of two $d$-ideals is a $d$-ideal, so the $d$-ideal portion of the previous corollary is known.

We conclude with an example showing that in a commutative $f$-ring with identity element in which minimal prime $l$-ideals are square dominated, the sum of two minimal prime $l$-ideals is not necessarily a $d$-ideal or $z$-ideal. So the hypothesis that the sum of any two minimal prime $l$-ideals is a $z$-ideal or a $d$-ideal cannot be omitted from Theorem 2.5. The example makes use of a construction of Henriksen and Smith which appears in [5].

**Example 2.8.** In $C([0, 1])$, let $i$ denote the function defined by $i(x) = x$ and let $I = \{f \in C([0, 1]) : |f^n| \leq mi \text{ for some } n, m \in \mathbf{N}\}$. Then $I$ is a semiprime $l$-ideal of $C([0, 1])$. Let $A = \{(f, g) \in C([0, 1]) \times C([0, 1]) : f-g \in I\}$. Then as shown in [5], $A$ is a commutative semiprime $f$-ring with identity element.

As shown in [5, §3], the minimal prime $l$-ideals of $A$ have the form $\{(f+g, f): f \in P, g \in I\}$ or $\{(f, f+g): f \in P, g \in I\}$ for some minimal prime $l$-ideal $P$ of $C([0, 1])$. Using this fact, it is not hard to show that minimal prime $l$-ideals of $A$ are square dominated.

Define a function $h$ by $h(x) = \sum_{n=1}^{\infty} 1/2^n x^{1/n}$. Then $h \in C([0, 1])$ and $h \notin I$. Let $P$ be a prime $l$-ideal such that $I \subseteq P$ and $h \notin P$, and let $P_1$ be a minimal prime $l$-ideal contained in $P$. In $A$, let $Q_1 = \{(f+g, f): f \in P_1, g \in I\}$ and $Q_2 = \{(f, f+g): f \in P_1, g \in I\}$. Then $Q_1$ and $Q_2$ are minimal prime $l$-ideals of $A$ and so are $z$-ideals and $d$-ideals. But $Q_1 + Q_2$ is not a $z$-ideal, since the only maximal ideal $(h, h)$ or $(i, i)$ is contained in is $M = \{(f, g): f(0) = g(0) = 0\}$ and yet $(i, i) \in Q_1 + Q_2$ but $(h, h) \notin Q_1 + Q_2$. To see directly that $Q_1 + Q_2$ is not a $d$-ideal, note that $\{(i, i)\}^{dd} = A$ and $A \nsubseteq Q_1 + Q_2$.

## REFERENCES

1. A. Bigard, K. Keimel, and S. Wolfenstein, *Groupes et anneaux reticules*, Lecture Notes in Math., vol. 608, Springer-Verlag, New York, 1977.
2. L. Gillman and M. Jerison, *Rings of continuous functions*, Springer-Verlag, New York, 1960.
3. L. Gillman and C. Kohls, *Convex and pseudoprime ideals in rings of continuous functions*, Math. Z. **72** (1960), 399–409.
4. M. Henriksen, *Semiprime ideals of f-rings*, Sympos. Math. **21** (1977), 401–409.
5. M. Henriksen and F. A. Smith, *Sums of z-ideals and semiprime ideals*, Gen. Topology and its Relations to Modern Analysis and Algebra **5** (1982), 272–278.
6. C. B. Huijsmans and B. de Pagter, *On z-ideals and d-ideals in Riesz spaces* I, Nederl. Akad. Wetensch. Indag. Math. **42** (1980), 183–195.
7. ——, *Ideal theory in f-algebras*, Trans. Amer. Math. Soc. **269** (1982), 225–245.
8. S. Larson, *Pseudoprime l-ideals in a class of f-rings*, Proc. Amer. Math. Soc. **104** (1988), 685–692.
9. G. Mason, *z-ideals and prime ideals*, J. Algebra **26** (1973), 280–297.
10. ——, *Prime z-ideals of $C(X)$ and related rings*, Canad. Math. Bull. **23** (1980), 437–443.
11. D. Rudd, *On two sum theorems for ideals of $C(X)$*, Michigan Math. J. **17** (1970), 139–141.
12. H. Subramanian, *l-prime ideals in f-rings*, Bull. Soc. Math. France **95** (1967), 193–203.

DEPARTMENT OF MATHEMATICS, LOYOLA MARYMOUNT UNIVERSITY, LOS ANGELES, CALIFORNIA 90045