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A comparison of two types of rough sets induced by coverings

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ABSTRACT

Rough set theory is an important technique in knowledge discovery in databases. In covering-based rough sets, many types of rough set models were established in recent years. In this paper, we compare the covering-based rough sets defined by Zhu with ones defined by Xu and Zhang. We further explore the properties and structures of these types of rough set models. We also consider the reduction of coverings. Finally, the axiomatic systems for the lower and upper approximations defined by Xu and Zhang are constructed.

1. Introduction

The basic notion of rough sets and approximation spaces were proposed by Pawlak [16–20] during the early 1980s. The rough set theory may serve as a new mathematical approach to vagueness, and has attracted the interest of researchers and practitioners in various fields of science and technology.

In Pawlak original rough set theory, partition or equivalence (indiscernibility) relation is a primitive concept. However, equivalence relations are too restrictive for many applications. To address this issue, many proposals have been put forward for generalizing and interpreting rough sets. For example, rough set model is extended to arbitrary binary relations [4,9–11,13,14,24,25,29,31–34,40] and coverings [3,6,37–39,41]. Some researchers even extended classical rough sets to fuzzy sets [5,15,22,26–28,36], fuzzy lattices [12], Boolean algebras [21], completely distributive lattices [2] and residuated lattices [23].

Various kinds of upper approximations in covering rough sets were studied [39]. For instance, in [38], Zhu defined a new type of covering-based rough sets from the topological view and explored the topological properties of this type of rough sets. In [30], based on coverings, Xu and Zhang proposed another new lower and upper approximations and defined a measure of roughness with this lower and upper approximations. What connections among these approximations? In this paper, we further explore the structures and properties of these two new rough sets. We establish the close relationship between these two new covering-based rough sets. A further exploration may serve the purpose of bringing more insights into covering-based rough sets.

The paper is structured as follows: in Section 2, we present some definitions and properties of generalized rough sets induced by binary relations and coverings. In Section 3, we study the relationship among covering-based rough sets defined by Zhu [38], Xu and Zhang [30] and binary relation-based rough sets. In Section 4, we investigate the transformation of covering

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and the exact sets. The transformation of coverings can be considered to be a kind reduction of covering. We also obtain the algebraic structure of the exact sets. In Section 5, axiomatic systems for the covering-based rough sets defined by Xu and Zhang [30] are constructed. Finally, we conclude the paper in Section 6.

2. Some relevant concepts and results

In this section, we consider fundamental properties of generalized rough sets induced by arbitrary binary relations and coverings.

2.1. Generalized rough sets induced by arbitrary binary relations

Let U be a non-empty set of objects called the universe. U can be an infinite set, i.e., we do not restrict the universe to the finite. Let R be an equivalence relation on U. We use U/R to denote the family of all equivalence classes of R (or classifications of U), and $[x]_R$ to denote an equivalence class in R containing an element $x \in U$. The pair (U, R) is called an approximation space. For any $X \subseteq U$, we can define the lower and upper approximation of X [16,17,34] by

$$\underline{R}X = \{x|[x]_R \subseteq X\} \quad \text{and} \quad \overline{R}X = \{x|[x]_R \cap X \neq \emptyset\},$$

respectively. The pair $(\underline{R}X, \overline{R}X)$ is referred to as the rough set of X. The rough set $(\underline{R}X, \overline{R}X)$ gives rise to a description of X under the present knowledge, i.e., the classification of U.

Much research [8,35,37] has pointed out the necessity to introduce a more general approach by considering an arbitrary binary relation (or even an arbitrary fuzzy relation in two universes) $R \subseteq U \times U$ in the set U of objects instead of an equivalence relation.

Suppose R is an arbitrary binary relation on U, the pair (U, R) is called an approximation space. With respect to R, we can define the R-left and R-right neighborhoods of an element x in U as follows:

$$l_R(x) = \{y|y \in U, yRx\} \quad \text{and} \quad r_R(x) = \{y|y \in U, xRy\},$$

respectively. The binary relation R can be determined by its left neighborhoods and vice versa, this is also true for right neighborhoods. The neighborhood $r_R(x)$ (or $l_R(x)$) becomes an equivalence class containing x if R is an equivalence relation. For an arbitrary relation R, by substituting equivalence class $[x]_R$ with right neighborhood $r_R(x)$, Yao [32–34] defined the operators $\underline{R}$ and $\overline{R}$ from $P(U)$ to itself by

$$\underline{R}X = \{x|r_R(x) \subseteq X\} \quad \text{and} \quad \overline{R}X = \{x|r_R(x) \cap X \neq \emptyset\}.$$

$\underline{R}X$ is called a lower approximation of X and $\overline{R}X$ an upper approximation of X. The pair $(\underline{R}X, \overline{R}X)$ is referred to as a generalized rough set induced by a binary relation R. Note that the definition of the lower and upper approximations is not unique. For example, we can use the R-left neighborhood $l_R(x) = \{y \in U|yRx\}$ to define the lower and upper approximations. With generalized rough sets induced by an arbitrary binary relation, the following properties hold [11,12,34].

Proposition 2.1. *Let U be an arbitrary universal set, $P(U)$ the power set of U, and R an arbitrary binary relation on U. Then the lower and upper approximation operators satisfy the following properties:*

(1) $\overline{R}\{x\} = l_R(x), x \in U$ *and* $\overline{R}X = \cup_{x \in X} l_R(x)$ *for all* $X \in P(U)$;
(2) $\overline{R}\emptyset = \emptyset$ *and* $\underline{R}U = U$;
(3) *For any given index set I and* $X_i \in P(U), i \in I, \overline{R}(\cup_{i \in I} X_i) = \cup_{i \in I} \overline{R}X_i$, *and* $\underline{R}(\cap_{i \in I} X_i) = \cap_{i \in I} \underline{R}X_i$;
(4) *If* $X, Y \in P(U)$ *and* $X \subseteq Y$, *then* $\underline{R}X \subseteq \underline{R}Y$ *and* $\overline{R}X \subseteq \overline{R}Y$;
(5) $\underline{R}X \cup \underline{R}Y \subseteq \underline{R}(X \cup Y)$, *and* $\overline{R}(X \cap Y) \subseteq \overline{R}X \cap \overline{R}Y$ *for all* $X, Y \in P(U)$;
(6) $(\underline{R}X)^C = \overline{R}X^C$, *and* $(\overline{R}X)^C = \underline{R}X^C$ *for all* $X \in P(U)$, *where* X^C *denotes the complement of* X;
(7) *R is reflexive* $\iff X \subseteq \overline{R}X \iff \underline{R}X \subseteq X$ *for all* $X \in P(U)$;
(8) *R is transitive* $\iff \overline{R}\,\overline{R}X \subseteq \overline{R}X \iff \underline{R}X \subseteq \underline{R}\,\underline{R}X$ *for all* $X \in P(U)$;
(9) *R is symmetric* $\iff (X, \overline{R}Y) = (Y, \overline{R}X) \iff [X, \underline{R}Y] = [Y, \underline{R}X]$, *where* (X, Y) *and* $[X, Y]$ *denote the inner and outer products* [13] *of* $X, Y \in P(U)$, *respectively*;
(10) *Let S be another binary relation on U, then* $\overline{R}X \subseteq \overline{S}X$ *for all* $X \in P(U)$ *if and only if* $R \subseteq S$;
(11) *Let S be another binary relation on U, then* $\underline{R}X \subseteq \underline{S}X$ for all $X \in P(U)$ *if and only if* $R \supseteq S$;
(12) *Let S be another binary relation on U, then* $\overline{R}X = \overline{S}X$ *for all* $X \in P(U)$ *if and only if* $R = S$;
(13) *Let S be another binary relation on U, then* $\underline{R}X = \underline{S}X$ *for all* $X \in P(U)$ *if and only if* $R = S$.

Now we consider the following two interesting subsets of $P(U)$.

$$G = \{X|X \in P(U), \overline{R}X = \emptyset\}$$

and

$$H = \{X|X = \overline{R}Y \text{ for some } Y \in P(U)\}.$$

Proposition 2.2. *Let U be an arbitrary universal set, $P(U)$ the power set of U, and R an arbitrary binary relation on U. Then*

(1) $G = \{X | X \in P(U), \forall x \in X, l_R(x) = \emptyset\}$;
(2) $\langle G, \cap, \cup\rangle$ *is a completely distribute lattice. Its least element is* $\emptyset$ *and its greatest element is* $\{x | x \in U, l_R(x) = \emptyset\}$;
(3) $\emptyset \in H$;
(4) *For any given index I*, if $X_i \in H, i \in I$, *then* $\cup_{i\in I} X_i \in H$;
(5) *If R is idempotent, i.e.,* $R^2 = R$, *then* $G \cap H = \emptyset$.

Proof.

Part (1) is a restatement of Proposition 2.1(1).
(2) By Proposition 2.1(2), $\overline{R}\emptyset = \emptyset$, we have $\emptyset \in G$, thus $G \neq \emptyset$. For any given index I and $X_i \in G, i \in I$, since $\overline{R}(\cup_{i\in I} X_i) = \cup_{i\in I}\overline{R}X_i = \emptyset$, we have $\cup_{i\in I} X_i \in G$. Similarly, $\overline{R}(\cap_{i\in I} X_i) \subseteq \overline{R}X_i = \emptyset$, this means that $\cap_{i\in I} X_i \in G$. Thus G is a completely distribute lattice and it has a least element $\emptyset$ and a greatest element $\{x | x \in U, l_R(x) = \emptyset\}$.
(3) $\emptyset = \overline{R}\emptyset \in H$.
(4) If $X_i \in H$, then there exists some $Y_i \in P(U)$ such that $X_i = \overline{R}Y_i$, and $\cup_{i\in I} X_i = \cup_{i\in I}\overline{R}Y_i = \overline{R}(\cup_{i\in I} Y_i) \in H$.
(5) Suppose that $X \in G \cap H$, then there exists some $Y \in P(U)$ such that $X = \overline{R}Y$ and $\overline{R}X = \emptyset$. Thus $X = \overline{R}Y = \overline{R^2}Y = \overline{R}(\overline{R}Y) = \overline{R}X = \emptyset$. □

2.2. Generalized rough sets induced by coverings

This subsection will recall some definitions and results about generalized rough sets induced by coverings, which can be found in [1,30,37–39].

Definition 2.1. Let U be a universal set and C be a family of subsets of U. C is called a covering of U if none subsets in C is empty, and $\cup_{K\in C} K = U$. The order pair $\langle U, C\rangle$ is called a covering approximation space if C is a covering of U.

Definition 2.2. Let $\langle U, C\rangle$ be a covering approximation space, $x \in U$, the minimal description of x is defined as

$$Md(x) = \{K | x \in K \in C \wedge (\forall S \in C \wedge x \in S \subseteq K \Rightarrow K = S)\}.$$

Zhu [38] has given the following definition of the lower and upper approximation in a covering approximation space $\langle U, C\rangle$.

Definition 2.3 (*See [38]*). Let $\langle U, C\rangle$ be a covering approximation space. $N(x) = \cap\{K \in C | x \in K\}$ is called the neighborhood of an element $x \in U$. For a subset $X \subseteq U$, the lower approximation of X is defined as $X_+ = \cup\{K \in C | K \subseteq X\}$ and the upper approximation of X is defined as $X^+ = X_+ \cup \{N(x) | x \in X - X^+\}$.

For the lower and upper approximations, Zhu [38] gave a counter example to show that the dual properties do not hold generally. That is, $(X^+)^C \neq (X^C)_+$ and $(X_+)^C \neq (X^C)^+$. Zhu [38] also obtained another representation of the upper approximation. That is,

$$X^+ = \cup_{x\in X} N(x).$$

For a covering approximation space $\langle U, C\rangle$, Xu and Zhang [30] introduced new covering lower and upper approximations as follows.

Definition 2.4 *[30]*. Let $\langle U, C\rangle$ be a covering approximation space. For any $X \subseteq U$, the lower and upper approximation of X are defined as follows:

$$C_*X = \{x \in U | (\cap Md(x)) \subseteq X\} \quad \text{and} \quad C^*X = \{x \in U | (\cap Md(x)) \cap X \neq \emptyset\},$$

we can verify that $\cap Md(x) = \cap\{K \in C | x \in K\} = N(x)$, thus C_*X and C^*X can be rewritten as

$$C_*X = \{x | N(x) \subseteq X\} \quad \text{and} \quad C^*X = \{x | N(x) \cap X \neq \emptyset\}.$$

Unlike the lower and upper approximations defined by Zhu [38], the operators C_*, C^* are dual to each other. An interesting question is what connections between these two approximations? Next we will establish the connections and answer the question.

3. Relationships between X^+ and C^*X

This section studies the relationship between the upper approximation X^+ defined by Zhu [38] and C^*X defined by Xu and Zhang [30]. By using the neighborhood $N(x)$ of an element $x \in U$, we construct a binary relation R on U as follows:

$$xRy \text{ if and only if } y \in N(x). \tag{1}$$

That is, for $x \in U$, R-right neighborhood $r_R(x) = N(x)$.

Theorem 3.1. *If R is the binary relation on U as defined in above formula (1), then*

(1) *R is reflexive and transitive*;
(2) $C^*X = \overline{R}X$ *and* $C_*X = \underline{R}X$ for all $X \subseteq U$;
(3) *Conversely, for each reflexive and transitive binary relation R on U, there exists a covering approximation space $\langle U, C\rangle$ such that $\overline{R}X = C^*X$ and $\underline{R}X = C_*X$ for all $X \subseteq U$.*

Proof.

(1) By definition, $x \in N(x) = \cap\{K \in C | x \in K\}$, thus R is reflexive. We now prove that R is transitive. Suppose that xRy and yRz. Then $y \in N(x)$ and $z \in N(y)$. This, in turn, implies that $N(y) \subseteq N(x)$ and $N(z) \subseteq N(y)$. Thus $z \in N(x)$ and xRz, we are done.
(2) The proof of follows from the definitions of R, C^*X and C_*X.
(3) By using the reflexive and transitive binary relation R on U, we define $C = \{r_R(x) | x \in U\}$. Since R is reflexive, we have $x \in r_R(x)$ and $\cup_{x \in U} r_R(x) = U$, thus C is a covering of U. Now we prove that, for this covering, $\forall x \in U, N(x) = r_R(x)$.

Since R is transitive, we obtain that if $x \in r_R(y)$, then $r_R(x) \subseteq r_R(y)$. Since R is reflexive, we have

$$N(x) = \cap\{K \in C | x \in K\} = \cap\{r_R(y) | x \in r_R(y)\} = r_R(x),$$

thus $C^*X = \{x | N(x) \cap X \neq \emptyset\} = \{x | r_R(x) \cap X \neq \emptyset\} = \overline{R}X$. Similarly, $\underline{R}X = C_*X$. □

Theorem 3.1 shows that if $\langle U, C\rangle$ is a covering approximation space, then $\langle U, C\rangle$ can be used to construct a reflexive and transitive relation R on U such that $C^*X = \overline{R}X$ and $C_*X = \underline{R}X$. Conversely, a reflexive and transitive relation R on U can also induce a covering approximation space $\langle U, C\rangle$ such that $\overline{R}X = C^*X$ and $\underline{R}X = C_*X$. However, we do not guarantee that R is symmetric, this can be seen from the following counter example.

Example 3.1. Let $U = \{1, 2, 3, 4\}$ be the universal set. $K_1 = \{1\}, K_2 = \{1, 2\}$, $K_3 = \{1, 2, 3\}$ and $K_4 = U$. $C = \{K_1, K_2, K_3, K_4\}$ is a covering. $r_R(1) = \{1\}$, $r_R(2) = \{1, 2\}, r_R(3) = \{1, 2, 3\}$ and $r_R(4) = U$. Note that $(3, 2) \in R$, but $(2, 3) \notin R$. This shows that R is not a symmetric relation on U.

Zhu has considered the properties of the unary coverings. Recall that a covering C of U is called unary if $|Md(x)| = 1$ for all $x \in U$. The following corollary establishes a one to one correspondence between unary coverings and reflexive and transitive binary relations.

Corollary 3.1. *Suppose that C is a unary covering of U, then there exists some reflexive and transitive binary relation R on U such that $Md(x) = \{r_R(x)\}$, where $r_R(x)$ is the R-right neighborhood of an element x in U. Conversely, if R is a reflexive and transitive binary relation on U, then $C = \{r_R(x)\}$ is a unary covering of U and $Md(x) = \{r_R(x)\}$ for all $x \in U$.*

Proof. If C is unary, by Theorem 3.1(1), we have a reflexive and transitive binary relation R on U such that $Md(x) = \{N(x)\} = \{r_R(x)\}$. Conversely, if R is a reflexive and transitive binary relation on U. By the reflexive property of R, C is a covering of U. Since R is transitive, $x \in r_R(y)$ implies $r_R(x) \subseteq r(y)$. Therefore, $Md(x) = \{r_R(x)\}$ and $|Md(x)| = 1$ for all $x \in U$. □

From the covering approximation spaces, we can find that when C is a partition, C_* and C^* will be the Pawlak lower and upper approximations. However, we can also find that when C is not a partition, C_* and C^* may be the Pawlak lower and upper approximations. We give such an example as follows.

Example 3.2. Let $U = \{a, b, c, d, e\}$ be a universal set, $K_1 = \{a, b\}, K_2 = \{c, d\}, K_3 = \{d, e\}$ and $K_4 = \{c, e\}$. $C = \{K_1, K_2, K_3, K_4\}$ is a covering of U. By direct computation, $N(a) = N(b) = \{a, b\}, N(c) = \{c\}, N(d) = \{d\}$, and $N(e) = \{e\}$. $\{N(a), N(c), N(d), N(e)\}$ is a partition of U. For covering C, C_* and C^* are the Pawlak lower and upper approximations.

More generally, for covering C, we ask whether C_* and C^* are Pawlak lower and upper approximations. We will consider the problem.

Theorem 3.2. *Let $\langle U, C\rangle$ be a covering approximation space. then C_* and C^* are the Pawlak lower and upper approximations if and only if $\{N(x) | x \in U\}$ is a partition of U.*

Proof. By Theorem 3.1(2), $C_*X = \underline{R}X$ and $C^*X = \overline{R}X$. $\underline{R}X$ and $\overline{R}X$ are the Pawlak lower and upper approximations if and only if R is an equivalence relation on U. This means that $\{N(x) | x \in U\}$ is a partition of U. □

Recall that an operation on a relation R on U is the formation of the inverse, usually written R^{-1}. The relation R^{-1} is a relation on U defined by

$$yR^{-1}x \text{ if and only if } xRy.$$

It is easy to verify that $l_{R^{-1}}(x) = r_R(x)$ and $r_{R^{-1}}(x) = l_R(x)$ for all $x \in U$. The following theorem presents the relationship between the upper approximations X^+ and C^*X.

Theorem 3.3. *Let $\langle U, C\rangle$ be a covering approximation space and R be a binary relation on U as defined in above formula (1). Then*

(1) $C^*X = \overline{R}X$ *for all* $X \subseteq U$;
(2) $X^+ = \overline{R^{-1}}X$ *for all* $X \subseteq U$;
(3) $C^*X = \cup_{x\in X} l_R(x)$ *and* $X^+ = \cup_{x\in X} r_R(x)$ *for all* $X \subseteq U$;
(4) *If* $C^*X = X^+$ *for all* $X \subseteq U$, *then R is an equivalence relation on U, therefore,* C^*X *and* X^+ *are Pawlak rough set upper approximations.*

Proof.

Part (1) is the restatement of Theorem 3.1(1).
(2) We note that the R^{-1}-right neighborhood $r_R(x)$ coincides with the R-left neighborhood $l_R(x)$ for all $x \in U$ and vice versa. By substituting R with R^{-1} in Proposition 2.1(1), we obtain $\overline{R^{-1}}X = \cup_{x\in X} r_R(x)$. Comparing $\overline{R^{-1}}X = \cup_{x\in X} r_R(x)$ with $X^+ = \cup_{x\in X} N(x)$, we have $X^+ = \overline{R^{-1}}X$.
(3) $C^*X = \overline{R}X = \cup_{x\in X} l_R(x)$, and $X^+ = \overline{R^{-1}}X = \cup_{x\in X} l_{R^{-1}}(x) = \cup_{x\in X} r_R(x)$.
(4) If $C^*X = X^+$ for all $X \subseteq U$, then $\overline{R}X = \overline{R^{-1}}X$, by Proposition 2.1(12), $R = R^{-1}$ and R is symmetric. By Theorem 3.1, R is also reflexive and transitive. Thus R is an equivalence relation on U. □

4. Transformation of coverings and the exact sets

The concept of reduction of coverings is introduced by Zhu and Wang [37]. In this section, we consider two problems: One is transformation of coverings and the other is the algebraic structure of all exact sets.

Here we propose the concept of transformation of coverings. We will show that the transformation of coverings makes the upper approximations X^+, C^*X and the lower approximation C_*X to be invariant. For these aims, we first give an example of transformation of coverings.

Example 4.1. Let $U = \{a,b,c,d\}, K_1 = \{a,b\}, K_2 = \{a,c\}, K_3 = \{a,d\}, K_4 = \{b,c\}$, and $C = \{K_1,K_2,K_3,K_4\}$ is a covering of U. It is easily to verify that $N(a) = \{a\}, N(b) = \{b\}, N(c) = \{c\}$ and $N(d) = \{a,d\}$. $C' = \{N(a),N(b),N(c),N(d)\}$ is also a covering of U. Moreover, C and C' generate the same upper approximations X^+, C^*X and the lower approximation C_*X for all $X \subseteq U$.

Let $\mathbb{C}(U)$ denote the set of all coverings of U, we define the transformation F from $\mathbb{C}(U)$ to $\mathbb{C}(U)$ as follows.

$$F : \mathbb{C}(U) \to \mathbb{C}(U), F(C) = C' = \{N(x)|x \in U\}. \tag{2}$$

Lemma 4.1. *If* $F : \mathbb{C}(U) \to \mathbb{C}(U)$ *is the transformation as defined in above formula (2), then* $F(F(C)) = F(C)$.

Proof. For any covering C of U, since $x \in N(x)$, C' is a covering of U and F, indeed, is a transformation from $\mathbb{C}(U)$ to $\mathbb{C}(U)$. Recall that a binary relation R on U has defined in Section 3:

$$xRy \text{ if and only if } y \in N(x).$$

Denote $N(x) = \cap\{K \in C|x \in K\}$ and $N'(x) = \cap\{K \in C'|x \in K\}$. We now prove $N(x) = N'(x)$. Since R is reflexive and transitive, we have

$$N'(x) = \cap\{K \in C'|x \in K\} = \cap\{r(y)|x \in r_R(y)\} = r_R(x) = N(x).$$

Thus $F(F(C)) = F(C') = \{N'(x)|x \in U\} = \{N(x)|x \in U\} = F(C)$. □

Corollary 4.1. *If a covering C is unary, then* $F(C) = \{Md(x)|x \in U\}$.

The $F(C) = C'$ can be seen as a kind of reduction of C. More generally, we have the following result.

Theorem 4.1. *Let* $\langle U, C\rangle$ *be a covering approximation space. Then C and* $F(C) = C' = \{N(x)|x \in U\}$ *generate the same upper approximations* X^+, C^*X *and the lower approximation* C_*X *for all* $X \subseteq U$. *That is,* C_*, C^* *and* X^+ *are invariant under the transformation F.*

Proof. By Lemma 4.1, the upper approximations X^+, C^*X and the lower approximation C_*X are defined by $N(x)$. Since $N(x)$ is invariable under the transformation F. Thus C and $F(C)$ generate the same upper approximations X^+, C^*X and the lower approximation C_*X for all $X \subseteq U$. □

However, in the process of transformation of coverings, since X_+ is not defined by $N(x)$, we cannot guarantee that X_+ is invariant. The counter example as follows.

Example 4.2. Let $U = \{a,b,c\}$ be a universal set, $K_1 = \{a,b\}, K_2 = \{b,c\}$ and $K_3 = \{a,c\}$. Then $C = \{K_1,K_2,K_3\}$ is a covering of U. By direct computation, $N(a) = \{a\}, N(b) = \{b\}, N(c) = \{c\}$ and $C' = \{N(a),N(b),N(c)\}$. Suppose $X = \{a\}$, for covering $C, X_+ = \emptyset$, but for covering $C', X_+ = \{a\}$.

In covering rough sets, when $C^*X = C_*X$, we say that X is an exact set. This section investigates the exact sets for a covering approximation space $\langle U, C\rangle$. Consider the set of all exact sets

$$T = \{X | X \in P(U), C_*X = C^*X\}.$$

We prove that T is a Boolean algebra.

Theorem 4.2. *Let $\langle U, C\rangle$ be a covering approximation space. $T = \{X|X \in P(U), C_*X = C^*X\}$, then $\langle T, \cap, \cup, ^C, U, \emptyset\rangle$ is a Boolean algebra (the Boolean subalgebra of $P(U)$).*

Proof. Since $\emptyset = C_*\emptyset = C^*\emptyset$, we have $\emptyset \in T$. Similarly, $U \in T$. If $X \in T$, then $C^*X = C_*X$, by Theorem 3.1, $\underline{R}X = \overline{R}X$. Thus $(\underline{R}X)^C = (\overline{R}X)^C$, that is, $\underline{R}X^C = \overline{R}X^C$. This implies $C_*X^C = C^*X^C$ and $X^C \in T$.

We prove that if $X, Y \in T$, then $X \cup Y \in T$. Suppose that $X, Y \in T$, then $C^*(X \cup Y) = C^*(X) \cup C^*(Y) = X \cup Y$. Since $X \subseteq X \cup Y$ implies $C_*(X) \subseteq C_*(X \cup Y)$, we have $X = C_*(X) \subseteq C_*(X \cup Y)$. Similarly, $Y \subseteq C_*(X \cup Y)$. Thus $X \cup Y \subseteq C_*(X \cup Y) \subseteq X \cup Y$ and $X \cup Y \in T$.

Similarly, we can prove that if $X, Y \in T$, then $X \cap Y \in T$. Thus T is a Boolean algebra. This completes the proof. □

Lemma 4.2. *Let $\langle U, C\rangle$ be a covering approximation space. Then for any given index set I,*

(1) $C_*(\cup_{i\in I} C_*X_i) = \cup_{i\in I} C_*X_i$;
(2) $C^*(\cap_{i\in I} C^*X_i) = \cap_{i\in I} C^*X_i$.

Proof.

(1) It is clear that $C_*(\cup_{i\in I} C_*X_i) \subseteq \cup_{i\in I} C_*X_i$. We only need to prove $\cup_{i\in I} C_*X_i \subseteq C_*(\cup_{i\in I} C_*X_i)$. For $x \in \cup_{i\in I} C_*X_i$, then $\exists i \in I$ such that $x \in C_*X_i$. Since $C_*(C_*X_i) = C_*X_i$, this implies $x \in C_*(C_*X_i)$. By the definition of operator C_*, $N(x) \subseteq C_*X_i \subseteq \cup_{i\in I} C_*X_i$. Thus $x \in C_*(\cup_{i\in I} C_*X_i)$. This means $\cup_{i\in I} C_*X_i \subseteq C_*(\cup_{i\in I} C_*X_i)$ and $C_*(\cup_{i\in I} C_*X_i) = \cup_{i\in I} C_*X_i$.
(2) By duality, the proof of part (2) is analogous to that of part (1). □

Theorem 4.3. *Let $\langle U, C\rangle$ be a covering approximation space. $S = \{X|X \in P(U), C_*X = X\}$, then $\langle S, \cap, \cup, U, \emptyset\rangle$ is a completely distribute lattice.*

Proof. It is clear that $\emptyset, U \in S$. For any given index I and $X_i \in S$, we first prove $\cap_{i\in I} X_i \in S$. Since $C_*(\cap_{i\in I} X_i) = \cap_{i\in I} C_*X_i = \cap_{i\in I} X_i$, we have $\cap_{i\in I} X_i \in S$. By Lemma 4.1, $\cup_{i\in I} X_i \in S$. Thus $\langle S, \cap, \cup, ^C, U, \emptyset\rangle$ is a completely distribute lattices. □

Theorem 4.4. *Let $\langle U, C\rangle$ be a covering approximation space. $W = \{X|X \in P(U), C^*X = X\}$, then $\langle W, \cap, \cup, U, \emptyset\rangle$ is a completely distribute lattice.*

Proof. The proof is analogous to that of Theorem 4.3. □

In Pawlak rough sets, for any subset $X \subseteq U$, $\overline{R}X = X$ can imply $\underline{R}X = X$ and vice versa. However, $C^*X = X$ cannot imply $C_*X = X$. This can be seen from the following example.

Example 4.3. Let $U = \{a, b, c, d\}$ be a universal set. $K_1 = \{a, b, d\}, K_2 = \{b\}$, $K_3 = \{c\}$ and $K_4 = \{d\}$. Then $C = \{K_1, K_2, K_3, K_4\}$ is a covering of U. Suppose that $X = \{a, b, c\}$, by direct computation, $\overline{R}X = X$. But $C_*X = \{b, c\} \neq X$.

It is clear that sets S, T and W have the following properties:

(1) If $X \in W$, then $X^C \in S$ and vice versa;
(2) $T = S \cap W$.

Example 4.4. Let $U = \{a, b, c, d\}$ be a universal set. $K_1 = \{a, b, c\}, K_2 = \{b, c\}$, $K_3 = \{c\}$ and $K_4 = \{d\}$. Then $C = \{K_1, K_2, K_3, K_4\}$ is a covering of U. Moreover, $W = \{\emptyset, U, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}\}$, $S = \{\emptyset, U, \{b, c, d\}, \{a, b, c\}, \{c, d\}, \{b, c\}, \{d\}, \{c\}\}$ and $T = \{\emptyset, U, \{d\}, \{a, b, c\}\}$.

5. Axiomatization of the operators C^* and C_*

Axiomatic approach is significant in rough set theory, the axiomatic approach aims to study the logical characters of rough sets, which may help to develop methods for application. Zhu [38] gave the axiomatic systems for lower approximation X_+ and upper approximation X^+. This section presents the axiomatic systems for the operators C^* and C_*.

Theorem 5.1. *Let U be a universal set and $P(U)$ be the power set of U. If an operator $H : P(U) \to P(U)$ satisfies the following properties:*

(1) $H(\emptyset) = \emptyset$;
(2) *For any index set I and $X_i \in P(U), i \in I, H(\cup_{i\in I}X_i) = \cup_{i\in I}H(X_i)$;*
(3) $X \cup H(H(X)) = H(X)$.

Then there exists some covering C of U such that $C^* = H$.

Proof. By Theorem 1 in Kondo [8], axioms (1) and (2) guarantee that there exists some binary relation R on U such that $H(X) = \overline{R}X$ for all $X \subseteq U$. Since axiom $X \cup H(H(X)) = H(X)$ is equivalent to $X \subseteq H(X)$ and $H(H(X)) \subseteq H(X)$. This, in turn, implies that R is reflexive and transitive. Consider

$$C = \{r_R(x) | x \in U\},$$

where $r_R(x) = \{y | xRy\}$. It is obvious that C is a covering of U. Now we prove $C^* = H$. Since R is reflexive and transitive, $Md(x) = \{r_R(x)\}$. According to the definition of C^*, $C^*X = \overline{R}X = H(X)$. Thus $C^* = H$. □

For the covering lower approximation, we have the following dual result.

Theorem 5.2. *Let U be a universal set and $P(U)$ be the power set of U. If an operator $L : P(U) \to P(U)$ satisfies the following properties:*

(1) $L(U) = U$;
(2) *For any index set I and $X_i \in P(U), i \in I, L(\cap_{i\in I}X_i) = \cap_{i\in I}L(X_i)$;*
(3) $X \cap L(L(X)) = L(X)$.

Then there exists some covering C of U such that $C_* = L$.

Recall that in a topological space U [7], a closure operator on U is an operator which assigns to each $X \subseteq U$ a subset $c(X)$ of U such that the following four statements are true. (1) $c(\emptyset) = \emptyset$, (2) For each $X \subseteq U, X \subseteq c(X)$, (3) For each $X \subseteq U, c(c(X)) = c(X)$ and (4) For each $X \subseteq U$ and $Y \subseteq U$, $c(X \cup Y) = c(X) \cup c(Y)$. Duality, an interior operator on U is an operator which assigns to each $X \subseteq U$ a subset $i(X)$ of U such that the following four statements are true. (1) $i(U) = U$, (2) For each $X \subseteq U, i(X) \subseteq X$, (3) For each $X \subseteq U, i(i(X)) = i(X)$ and (4) For each $X \subseteq U$ and $Y \subseteq U$, $i(X \cap Y) = i(X) \cap i(Y)$. From Theorems 4.1 and 4.2 the following holds. □

Corollary 5.1. *Let $\langle U, C\rangle$ be a covering approximation space. Then C_* is an interior operator and C^* a closure one.*

Note that if U is a finite universal set, by Theorems 4.1 and 4.2, a closure operator (or an interior one) $f : P(U) \to P(U)$ can be generated by C^* (or C_*) for some covering of U. However, if U is an infinite universal set, by Kondo [8], there exists some closure operator $f : P(U) \to P(U)$ such that f cannot generated by C^* for any covering C of U. That is, $f \neq C^*$ for any covering C of U. The reason is that, in a topological space, the union of the infinite closed sets may be not a closed set.

6. Conclusions

In [38], Zhu defined a new type of covering rough sets. In [30], Xu and Zhang defined another new type of covering rough sets. Naturally, an important and interesting problem is to establish the connection between these two new type of covering rough sets. This paper considered the problem. Thus this paper can be seen as the further exploration for these two new types of covering rough sets.

By using rough sets induced by binary relations, we established the relationship between covering rough sets defined by Zhu [38] and covering rough sets defined by Xu and Zhang [30]. We proposed a new concept of reduction of coverings. We also proved that the set of all exact sets for covering rough sets defined by Xu and Zhang is a Boolean algebra. Finally, the axiomatic systems for the lower and upper approximations defined by Xu and Zhang are constructed.

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