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## On the existence of finite type link homotopy invariants

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# On the Existence of Finite Type Link Homotopy Invariants

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## ABSTRACT

We show that for links with at most 5 components, the only finite type homotopy invariants are products of the linking numbers. In contrast, we show that for links with at least 9 components, there must exist finite type homotopy invariants which are *not* products of the linking numbers. This corrects the errors of the first author in [10, 11].  
*Keywords:* Finite type invariants; link homotopy.

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## 1. INTRODUCTION

In [10, 11] the first author claimed, erroneously, that there are no finite type link homotopy or concordance invariants other than the pairwise linking numbers (and their products). However, the proofs of this result in both of these paper contained a serious algebraic error. The purpose of this paper is to show the opposite - in fact, there *do* exist finite type link homotopy (and, hence, concordance) invariants other than the linking

numbers. However, the proof is not constructive; it is still an open problem to actually construct such an invariant (see Section 4).

There have been many excellent introductions to the theory of finite type invariants, such as [4, 1, 5]; we will not try to replicate them here. We will provide a few basic definitions in order to clarify our notation and terminology. It should be mentioned that our approach and proofs are combinatorial in nature.

**1.1. Singular Links.** Recall that, in the most general sense, a *link invariant* is a map from the set of equivalence classes of links under isotopy to another set  $G$ . We will need to have some additional structure on  $G$ . For our purposes,  $G$  will be the field of complex numbers  $\mathbb{C}$ . In this theory, it is also convenient to look at invariants of *regular* isotopy (i.e. links with framing), rather than just isotopy. So we will not allow the first Reidemeister move. We first note that we can extend any link invariant to an invariant of *singular* links, where a singular link is an immersion of several copies of  $S^1$  into 3-space which is an embedding except for a finite number of isolated double points. Given a link invariant  $v$ , we extend it via the relation:

An invariant  $v$  of singular links is then said to be of *finite type*, if there is an integer  $d$  such that  $v$  is zero on any link with more than  $d$  double points.  $v$  is then said to be of *type  $d$* . We denote by  $V_d$  the vector space over  $\mathbb{C}$  generated by  $\mathbb{C}$ -valued finite type invariants of type  $d$ . We can completely understand the space of  $\mathbb{C}$ -valued finite type invariants by understanding all of the quotient spaces  $V_d/V_{d-1}$ .

**1.2. Link homotopy and link concordance.** The idea of *link homotopy* (or just *homotopy*) was introduced by Milnor [13]. Two links are homotopic if one can be transformed into the other through a sequence of ambient isotopies of  $S^3$  and crossing changes of a component with itself (but *not* crossing changes of different components). Habegger and Lin [6] succeeded in classifying links up to homotopy. We construct a theory of finite type invariants in exactly the same way as before; the difference is that the invariants are trivial when evaluated on a link with a singularity in which a component intersects itself. In this case, the two "resolutions" of the singular point are homotopically equivalent, so the value of a homotopy invariant on their difference is zero. We will denote the vector space of type  $d$  link homotopy invariants by  $V_d^h$ .

**Definition 1.** Consider two  $k$ -component links  $L_0$  and  $L_1$ . These can be thought of as embeddings:

$$L_i : \bigsqcup_{i=1}^k S^1 \hookrightarrow \mathbf{R}^3$$

A **(link) concordance** between  $L_0$  and  $L_1$  is an embedding:

$$H : \left( \bigsqcup_{i=1}^k S^1 \right) \times I \hookrightarrow \mathbf{R}^3 \times I$$

such that  $H(x, 0) = (L_0(x), 0)$  and  $H(x, 1) = (L_1(x), 1)$ . A concordance is an isotopy if and only if  $H$  is level preserving; i.e. if the image of  $H_t$  is a link at level  $t$  for each  $t \in I$ .

We will denote the vector space of type  $d$  link concordance invariants by  $V_d^c$ .

**1.3. Univalent diagrams.** It is a marvelous fact that the vector spaces  $V_d^*/V_{d-1}^*$  can be given relatively simple combinatorial descriptions in terms of *univalent diagrams*. These are spaces of univalent graphs (with colored endpoints and oriented vertices) with various relations imposed upon them. That these descriptions are isomorphic to the original vector spaces is largely due to Kontsevich and his integral (see [5] for an excellent exposition of the Kontsevich integral). The description of the space for link homotopy was

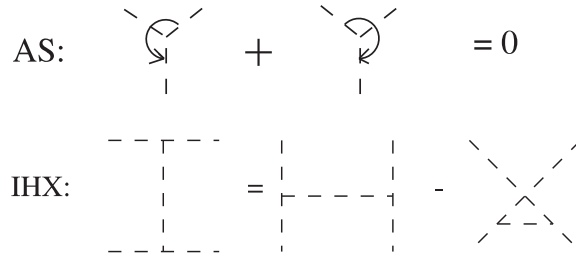


FIGURE 1. AS and IHX relations

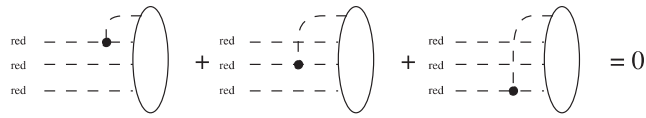


FIGURE 2. The link relation for univalent diagrams

developed by Bar-Natan and others [1, 3]. The modification for concordance was found by Habegger and Masbaum [7]. For a more detailed development, see [10, 11].

**Definition 2.**  $B^h$  is defined as the vector space of (disjoint unions of) univalent diagrams modulo the following relations:

- The antisymmetry (AS) relation (see Figure 1).
- The IHX relation (see Figure 1).
- The link relation (see Figure 2), where the sum is over all univalent vertices of the diagram with the same color. Another example of diagrams appearing in a link relation can be found in Figure 3.
- Any diagram with a loop is trivial.
- Any diagram with a connected component which has two univalent vertices of the same color is trivial.

The **degree**  $d$  of a diagram in  $B^h$  is defined to be one half of the number of vertices of the diagram. Let  $B_d^h$  be the vector space of univalent diagrams of degree  $d$  (notice that all of the relations involve diagrams of the same degree, so they apply equally well to  $B_d^h$ ). So  $B^h$  is just the graded vector space  $\bigoplus_{d=1}^{\infty} B_d^h$ . We define  $B^c$  to be the space of univalent diagrams modulo only the first four relations (so components can have multiple endpoints with the same color), and  $B^c = \bigoplus_{d=1}^{\infty} B_d^c$  in the same way.

**Theorem 1.** [1, 3, 6]  $B_d^h \cong V_d^h/V_{d-1}^h$ , and  $B_d^c \cong V_d^c/V_{d-1}^c$ .

## 2. NON-EXISTENCE RESULTS FOR $B^h$ AND $B^c$

Now that we have properly defined the spaces  $B^h$  and  $B^c$  of univalent diagrams for link homotopy, we want to analyze them more closely. Let  $B^h(k)$  (respectively  $B^c(k)$ ) denote the space of univalent diagrams for link homotopy (resp. concordance) with  $k$  possible colors for the univalent vertices (i.e. we are looking at links with  $k$  components).

**2.1. Previous results for  $B^h(k)$ .** Consider a diagram  $D \in B^h(k)$ . Each component of  $D$  is a tree diagram with at most one endpoint of each color. Since a univalent tree with  $n$  endpoints has  $2n - 2$  vertices, and hence degree  $n - 1$ ,  $D$  cannot have any components of degree greater than  $k - 1$ . **Notation:** Before we continue, we will introduce a bit of notation which will be useful in this section. Given a univalent

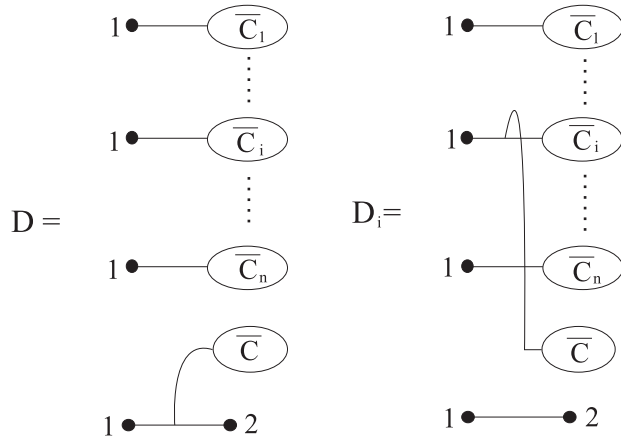


FIGURE 3. Diagrams arising from the link relation

diagram  $D$ , we define  $m(D; i, j)$  to be the number of components of  $D$  which are simply line segments with ends colored  $i$  and  $j$ , as shown below:

$$i \text{ --- } j$$

We call these components *struts*. Recall the following (correct) results from [10]. We include the proofs for completeness, and as a warm up for the more complicated proof in Section 2.2:

**Theorem 2.** *If  $D$  has a component  $C$  of degree  $k-1$  (with  $k \geq 3$ ), then  $D$  is trivial in  $B^h(k)$ .*

PROOF:  $C$  has one endpoint of each color  $1, 2, \dots, k$ . Without loss of generality, we may assume that  $C$  has a branch as shown, where  $\bar{C}$  denotes the remainder of  $C$ :

$$C: \begin{array}{c} \bar{C} \\ | \\ 1 \text{ --- } 2 \end{array}$$

We are going to apply the link relation with the color 1. Let  $\{C_1, \dots, C_n\}$  be the components of  $D$  with an endpoint colored 1. So, ignoring the other components of  $D$ , we have the diagrams of Figure 3 (where  $\bar{C}_i$  denotes all of  $C_i$  except for the endpoint colored 1). The link relation then implies that  $D + \sum D_i = 0$ . If  $C_i$  is just a line segment with endpoints colored 1 and 2, then  $D_i = D$ . Otherwise,  $\bar{C}_i$  will have an endpoint of some color  $j \in 3, \dots, k$ . In this case, since  $\bar{C}$  has an endpoint of each color  $3, \dots, k$ , including  $j$ ,  $D_i$  will have a component with two endpoints colored  $j$ , and hence be trivial in  $B^h$ . Therefore, we find that  $D + m(D; 1, 2)D = 0$  where  $m(D; 1, 2) \geq 0$ . We can divide both sides by  $1 + m(D; 1, 2)$  (since we are working over a field of characteristic 0) to conclude that  $D = 0$ .  $\square$

**Theorem 3.** *If  $D$  has a component  $C$  of degree  $k-2$  (with  $k \geq 4$ ), then  $D$  is trivial in  $B^h(k)$ .*

PROOF: Without loss of generality,  $C$  has endpoints colored  $1, 2, \dots, k-1$ . We will prove the lemma by inducting on  $m(D; 1, k)$ ; inducting among the set of diagrams having a component with endpoints colored  $1, 2, \dots, k-1$ . As in the previous theorem, we may assume that  $C$  has a branch as shown:

$$C: \begin{array}{c} \bar{C} \\ | \\ 1 \text{ --- } 2 \end{array}$$

And conclude that  $D + \sum D_i = 0$ , where the  $D_i$  are defined as before. Since  $\bar{C}$  contains endpoints of all colors except 1, 2, and  $k$ ,  $D_i$  has two endpoints of the same color (and hence is trivial) unless  $C_i$  has one of the following 3 forms (as in Theorem 2):

$$\begin{aligned} (1) \quad C_i &= 1 - - - - - 2 \\ (2) \quad C_i &= 1 - - - - - k \\ (3) \quad C_i &= \begin{array}{c} k \\ | \\ 1 - - - - - 2 \end{array} \end{aligned}$$

In the first case,  $D_i = D$ ; and in the second case,  $D_i = D'$ , where  $D'$  is the same as  $D$  except that:

- $C$  is replaced by a component  $C'$  identical to it except that the endpoint colored 2 in  $C$  is colored  $k$  in  $C'$  (so  $\bar{C}' = \bar{C}$ ).
- A line segment with endpoints colored 1 and  $k$  has been replaced by a line segment with endpoints colored 1 and 2. In other words,  $m(D'; 1, 2) = m(D; 1, 2) + 1$  and  $m(D'; 1, k) = m(D; 1, k) - 1$ .

In the third case,  $D_i$  has a component of degree  $k - 1$ , and so is trivial by the previous theorem. Therefore, we find that  $D + m(D; 1, 2)D + m(D; 1, k)D' = 0$ . If  $m(D; 1, k) = 0$  we conclude, as before, that  $D$  is trivial modulo the link relation, which proves the base case of our induction.

For the inductive step, we use the IHX relation on  $C'$  to decompose  $D' = \sum_{i \neq 1, 2, k} \pm D'_i$ , where  $D'_i$  is the same as  $D'$  except that  $C'$  has been replaced by a component  $C'_i$  with endpoints of the same colors (although arranged differently), and a branch as shown:

$$C'_i : \begin{array}{c} \bar{C}'_i \\ | \\ i - - - - - k \end{array}$$

(The decomposition is simply a matter of letting the endpoint colored  $k$  “travel” the tree - see Figure 4 for an example.) In particular,  $m(D'_i; a, b) = m(D'; a, b)$  for all colors  $a$  and  $b$ . We now apply the link relation to  $D'_i$  using color  $i$  (and component  $C'_i$ ). In this case, the only other components which matter (modulo trivial diagrams) are ones which look like one of the following:

$$\begin{aligned} (1) \quad & i - - - - - k \\ (2) \quad & i - - - - - 2 \\ (3) \quad & \begin{array}{c} 2 \\ | \\ i - - - - - k \end{array} \end{aligned}$$

As before, the first case gives  $D'_i$  again, the third case is trivial by Theorem 2, and the second case gives a diagram  $D''_i$  such that:

- $C'_i$  is replaced by a component  $C''_i$  identical to it except that the endpoint colored  $k$  in  $C'_i$  is colored 2 in  $C''_i$  (so  $\bar{C}''_i = \bar{C}'_i$ ).
- A line segment with endpoints colored  $i$  and 2 has been replaced by a line segment with endpoints colored  $i$  and  $k$ . In other words,  $m(D''_i; i, k) = m(D'_i; i, k) + 1$  and  $m(D''_i; i, 2) = m(D'_i; i, 2) - 1$ .

Otherwise,  $D''_i$  is the same as  $D'_i$ ; in particular,  $m(D''_i; 1, k) = m(D'_i; 1, k) = m(D'; 1, k) = m(D; 1, k) - 1$ . Then the link relation tells us that  $D'_i + m(D'; i, k)D'_i + m(D'; 2, i)D''_i = 0$ . Since  $D''_i$  has a component of degree  $k - 2$  with endpoints colored 1, ...,  $k - 1$  (namely,  $C''_i$ ), the inductive hypothesis implies that  $D''_i$  is trivial. Therefore,  $(1 + m(D'; i, k))D'_i = 0$ , so  $D'_i$  is trivial in  $B^h(k)$ . This is true for every  $i$ , so it immediately follows that  $D'$ , and hence  $D$ , are also trivial in  $B^h(k)$ .  $\square$

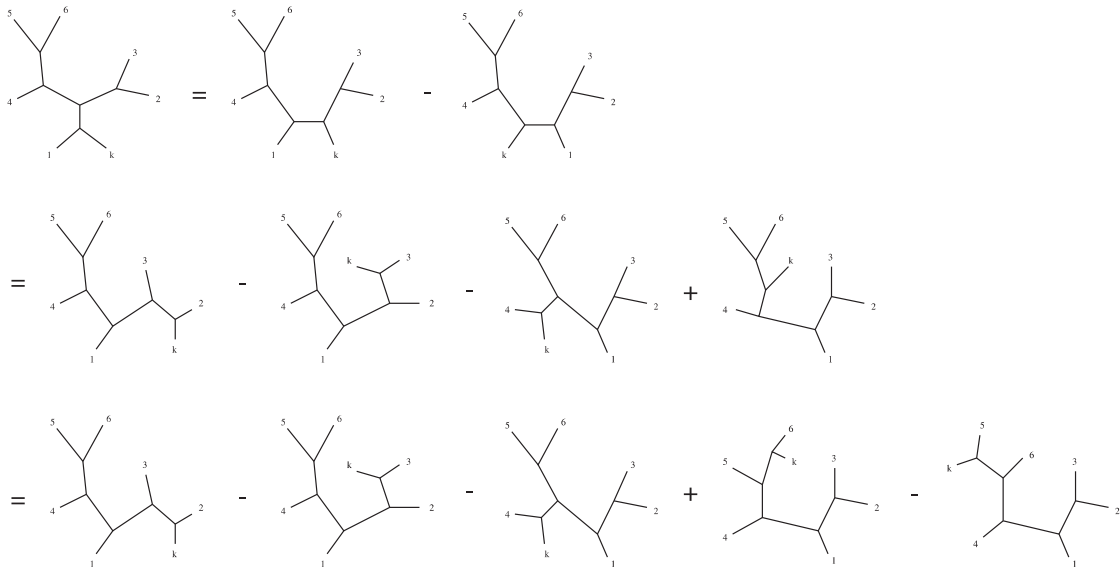


FIGURE 4. Using the IHX relation to decompose a diagram

So the largest possible degree of a component of a diagram in  $B^h(k)$  is  $k - 3$  (if  $k \geq 4$ ). In particular, this means that if  $k$  is 3 or 4, then the largest possible degree of a component of a diagram in  $B^h(k)$  is 1. It is well-known that the pairwise linking numbers are the only type 1 link homotopy invariants, and are dual to struts via the isomorphism of Theorem 1. Their products are dual to disjoint unions of struts. So we have as a corollary:

**Corollary 1.** *On links with at most 4 components, the only finite type homotopy invariants are the pairwise linking numbers and their products.*

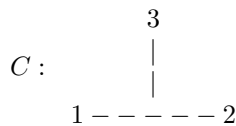
The obvious question is whether this result will generalize to links with more components. In the next section we will show, by a rather involved combinatorial argument, that it *does* extend to links with five components. However, in Section 3 we will show that it fails for links with more than 8 components.

REMARK: The error in [10] (replicated in [11]) was in the attempt to generalize the result to all  $k$ . On page 785 of [10], line 15, the possibility that  $c = a$  was neglected. This adds another term to the sum, which ends up cancelling everything out. This was pointed out by Alexander Merkov [12].

**2.2. The case of  $B^h(5)$ .** In this section we consider the  $B^h(5)$ . We know that no diagram in this space has a component of degree 3 or more. So the question is whether a diagram can have a component of degree 2. Any such component will be a "Y-component" - i.e. a graph with three (colored) univalent vertices connected to a single trivalent vertex.

**Theorem 4.** *If  $D \in B^h(5)$  has a component  $C$  of degree 2, then  $D$  is trivial.*

PROOF: This proof is significantly more delicate than that for Theorem 3, involving an extra level of induction. Without loss of generality,  $C$  has endpoints colored 1, 2, 3.



Our first induction is on  $m(D; 1, 4) + m(D; 1, 5)$ ; inducting among the set of diagrams having a component with endpoints colored 1, 2, 3. We will begin by proving the base case of this induction.

Let  $\{C_1, \dots, C_n\}$  be the other components of  $D$  with an endpoint colored 1. Then we apply the link relation as in Figure 3. We will apply the relation along the color 1, fixing the color 3. This means that  $\bar{C}$  (in Figure 3) is just a single univalent vertex, colored 3. This will be successively attached to the components  $C_i$  to form the diagrams  $D_i$  (in the figure  $\bar{C}_i$  denotes all of  $C_i$  except for the endpoint colored 1). We will refer to this operation as "expanding along 1, fixing 3."

So then  $D + \sum D_i = 0$ .  $D_i$  has two endpoints of the same color unless  $C_i$  has one of the following 4 forms:

$$\begin{aligned} (1) \quad C_i &= 1 \text{ --- } 2 \\ (2) \quad C_i &= 1 \text{ --- } 4 \\ (3) \quad C_i &= 1 \text{ --- } 5 \\ (4) \quad C_i &= \begin{array}{c} a \\ | \\ 1 \text{ --- } b \end{array} \quad a, b \in \{2, 4, 5\} \end{aligned}$$

In the first case,  $D_i = D$ . In the fourth case,  $D_i$  has a component of degree 3, and so is trivial by Theorem 3.

In the second case,  $D_i = D_4$ , where  $D_4$  is the same as  $D$  except that:

- $C$  is replaced by a component  $C'$  identical to it except that the endpoint colored 2 in  $C$  is colored 4 in  $C'$ .

$$C' : \begin{array}{c} 3 \\ | \\ 1 \text{ --- } 4 \end{array}$$

- A strut with endpoints colored 1 and 4 has been replaced by a strut with endpoints colored 1 and 2. In other words,  $m(D'; 1, 2) = m(D; 1, 2) + 1$  and  $m(D'; 1, 4) = m(D; 1, 4) - 1$ .

NOTATION: For the remainder of this proof, we will represent diagrams by giving the changes made from  $D$ . We will draw the new component of degree 2 which has replaced  $C$  (we will see that for all of our diagrams, any other components of degree 2 remain unchanged). Although the total number of struts is always preserved, some struts have been replaced by others. We represent a strut by the (unordered) pair of the colors of its endpoints, and use an arrow to show how the struts have been traded. For example, we will represent  $D_4$  as follows:

$$D_4 = \begin{array}{c} 3 \\ | \\ 1 \text{ --- } 4 \end{array} \quad (1, 4) \rightarrow (1, 2)$$

Finally, in the third case,  $D_i = D_5$ , which is defined similarly to  $D_4$ .

$$D_5 = \begin{array}{c} 3 \\ | \\ 1 \text{ --- } 5 \end{array} \quad (1, 5) \rightarrow (1, 2)$$

Therefore, we find that  $D + m(D; 1, 2)D + m(D; 1, 4)D_4 + m(D; 1, 5)D_5 = 0$ . If  $m(D; 1, 4) + m(D; 1, 5) = 0$ , then we conclude  $D + m(D; 1, 2)D = 0$  (since  $m(D; i, j) \geq 0$ ). Since  $m(D; 1, 2) \geq 0$ , we can divide by  $1 + m(D; 1, 2)$  to conclude  $D = 0$ , which proves the base case of our first induction.

Henceforth, for convenience, we will let  $m(i, j) = m(D; i, j)$ .

We will now assume the inductive hypothesis that any diagram  $E \in B^h(5)$  with a component of degree 2 with endpoints colored 1, 2 and 3, and such that  $m(E; 1, 4) + m(E; 1, 5) < m(1, 4) + m(1, 5)$ , is trivial. Our



inductive step consists of using this hypothesis to prove that  $D_4$  and  $D_5$  are trivial. This will immediately imply that  $D + m(1, 2)D = 0$ , and hence that  $D = 0$ .

We will prove that  $D_4$  is trivial. The proof that  $D_5$  is trivial is very similar. This proof involves a second induction. We will be looking at diagrams which do not have a component with endpoints colored 1, 2 and 3, so are not directly trivial by the (first) inductive hypothesis. However, we will find that (modulo the inductive hypothesis), we can effectively "swap" struts in these diagrams so that the number of struts with certain colors on their endpoints always decreases. Since there are only a finite number of such struts, the supply eventually disappears, and we are able to conclude that the diagrams are trivial.

We begin with  $D_4$ . We expand along 3, fixing 1.

$$D_4 + m(3, 4)D_4 + m(2, 3)D_{42} + m(3, 5)D_{45} = 0$$

$$D_{42} = \begin{array}{c} 3 \\ | \\ 1 \text{ --- } 2 \end{array} \begin{array}{l} (1, 4) \rightarrow (1, 2) \\ (2, 3) \rightarrow (3, 4) \end{array}$$

$$D_{45} = \begin{array}{c} 3 \\ | \\ 1 \text{ --- } 5 \end{array} \begin{array}{l} (1, 4) \rightarrow (1, 2) \\ (3, 5) \rightarrow (3, 4) \end{array}$$

$D_{42}$  has a component of degree 2 with endpoints colored 1, 2, and 3. Also,  $m(D_{42}; 1, 4) + m(D_{42}; 1, 5) = (m(1, 4) - 1) + m(1, 5) = m(1, 4) + m(1, 5) - 1$ . So by the inductive hypothesis,  $D_{42} = 0$ . Therefore:

$$D_4 = \frac{-m(3, 5)}{1 + m(3, 4)} D_{45}$$

Consider  $D_{45}$ . We expand along 3, fixing 5.

$$D_{45} + m(1, 3)D_{45} + m(2, 3)D_{452} + m(3, 4)D_{454} = 0$$

$$D_{452} = \begin{array}{c} 3 \\ | \\ 2 \text{ --- } 5 \end{array} \begin{array}{l} (1, 4) \rightarrow (1, 2) \\ (3, 5) \rightarrow (3, 4) \\ (2, 3) \rightarrow (1, 3) \end{array}$$

$$D_{454} = \begin{array}{c} 3 \\ | \\ 4 \text{ --- } 5 \end{array} \begin{array}{l} (1, 4) \rightarrow (1, 2) \\ (3, 5) \rightarrow (3, 4) \\ (3, 4) \rightarrow (1, 3) \end{array} \Rightarrow \begin{array}{l} (1, 4) \rightarrow (1, 2) \\ (3, 5) \rightarrow (1, 3) \end{array}$$

Therefore:

$$D_4 = \frac{-m(3, 5)}{1 + m(3, 4)} \frac{1}{1 + m(1, 3)} (-m(2, 3)D_{452} - m(3, 4)D_{454})$$

Neither of the new diagrams are trivial inductively, so we will need to analyze both of them. First, we consider  $D_{452}$ . We will show that, modulo the inductive hypothesis, we can swap a strut (3,5) (i.e. a strut with endpoints colored 3 and 5) for a strut (3,4), while simultaneously swapping a strut (2,4) for a strut (2,5). We begin by expanding along 2, fixing 3.

$$D_{452} + m(2, 5)D_{452} + (m(1, 2) + 1)D_{4521} + m(2, 4)D_{4524} = 0$$

$$D_{4521} = \begin{array}{c} 3 \\ | \\ 2 \text{ --- } 1 \end{array} \begin{array}{l} (1, 4) \rightarrow (1, 2) \\ (3, 5) \rightarrow (3, 4) \\ (2, 3) \rightarrow (1, 3) \\ (1, 2) \rightarrow (2, 5) \end{array} \Rightarrow \begin{array}{l} (1, 4) \rightarrow (2, 5) \\ (3, 5) \rightarrow (3, 4) \\ (2, 3) \rightarrow (1, 3) \end{array}$$

$$D_{4524} = \begin{array}{c} 3 \\ | \\ 2 \text{ --- } 4 \end{array} \begin{array}{l} (1, 4) \rightarrow (1, 2) \\ (3, 5) \rightarrow (3, 4) \\ (2, 3) \rightarrow (1, 3) \\ (2, 4) \rightarrow (2, 5) \end{array}$$

By the inductive hypothesis,  $D_{4521} = 0$ . Therefore:

$$D_{452} = \frac{-m(2, 4)}{1 + m(2, 5)} D_{4524}$$

Now consider  $D_{4524}$ . We expand along 3, fixing 2.

$$D_{4524} + (m(3, 4) + 1)D_{4524} + (m(1, 3) + 1)D_{45241} + (m(3, 5) - 1)D_{45245} = 0$$

$$D_{45241} = \begin{array}{c} 3 \\ | \\ 2 \text{ --- } 1 \end{array} \begin{array}{l} (1, 4) \rightarrow (1, 2) \\ (3, 5) \rightarrow (3, 4) \\ (2, 3) \rightarrow (1, 3) \\ (2, 4) \rightarrow (2, 5) \\ (3, 4) \rightarrow (1, 3) \end{array} \Rightarrow \begin{array}{l} (1, 4) \rightarrow (1, 2) \\ (3, 5) \rightarrow (1, 3) \\ (2, 3) \rightarrow (1, 3) \\ (2, 4) \rightarrow (2, 5) \end{array}$$

$$D_{45245} = \begin{array}{c} 3 \\ | \\ 2 \text{ --- } 5 \end{array} \begin{array}{l} (1, 4) \rightarrow (1, 2) \\ (3, 5) \rightarrow (3, 4) \\ (2, 3) \rightarrow (1, 3) \\ (2, 4) \rightarrow (2, 5) \\ (3, 5) \rightarrow (3, 4) \end{array}$$

By the inductive hypothesis,  $D_{45241} = 0$ . Therefore:

$$D_{4524} = \frac{-(m(3, 5) - 1)}{2 + m(3, 4)} D_{45245}$$

Combining this with the result of the previous step, we have:

$$D_{452} = \frac{-m(2, 4)}{1 + m(2, 5)} \frac{-(m(3, 5) - 1)}{2 + m(3, 4)} D_{45245} = \frac{m(2, 4)}{1 + m(2, 5)} \frac{(m(3, 5) - 1)}{2 + m(3, 4)} D_{45245}$$

Notice that the component of degree 2 in  $D_{45245}$  is the same as that in  $D_{452}$ , so we could start the whole procedure again. Also notice that we have swapped the struts as we wanted. Inductively, we can see that:

$$D_{452} = \prod_{k=1}^n \frac{(m(2, 4) - k + 1)}{k + m(2, 5)} \frac{(m(3, 5) - k)}{1 + k + m(3, 4)} D_{452(45)^n}$$

Eventually,  $n > m(3, 5)$ , so  $D_{452} = 0$ . This means that:

$$D_4 = \frac{-m(3, 5)}{1 + m(3, 4)} \frac{-m(3, 4)}{1 + m(1, 3)} D_{454}$$

Now we examine  $D_{454}$ . We expand along 4, fixing 3.

$$D_{454} + m(4, 5)D_{454} + (m(1, 4) - 1)D_{4541} + m(2, 4)D_{4542} = 0$$

$$D_{4541} = \begin{array}{c} 3 \\ | \\ 4 \text{ --- } 1 \end{array} \begin{array}{l} (1, 4) \rightarrow (1, 2) \\ (3, 5) \rightarrow (1, 3) \\ (1, 4) \rightarrow (4, 5) \end{array}$$

$$D_{4542} = \begin{array}{c} 3 \\ | \\ 4 \text{ --- } 2 \end{array} \begin{array}{l} (1, 4) \rightarrow (1, 2) \\ (3, 5) \rightarrow (1, 3) \\ (2, 4) \rightarrow (4, 5) \end{array}$$

Notice that  $D_{4542}$  has a degree 2 component which is the same, up to sign, as  $D_{4524}$ . So, by using the same kind of argument used for  $D_{452}$ , we can show that  $D_{4542} = 0$  (we will look at the diagrams  $D_{4542(54)^n}$ ). So:

$$D_{454} = \frac{m(1,4) - 1}{1 + m(4,5)} (-D_{4541})$$

Therefore, we can express  $D_4$  in terms of  $D_{4541}$ :

$$D_4 = \frac{m(3,5)}{1 + m(3,4)} \frac{m(3,4)}{1 + m(1,3)} \frac{m(1,4) - 1}{1 + m(4,5)} (-D_{4541})$$

Notice that we have swapped a strut (1,4) for a strut (4,5), and a strut (3,5) for a strut (3,4). Since  $D_{4541}$  has the same degree 2 component (up to sign) as  $D_4$ , we can repeat the same sequence of operations. Inductively, we will find that:

$$D_4 = \prod_{k=1}^n \frac{(m(3,5) - k + 1)}{1 + m(3,4)} \frac{m(3,4)}{k + m(1,3)} \frac{(m(1,4) - k)}{k + m(4,5)} (-1)^n D_{4(541)^n}$$

Eventually,  $n > m(1,4)$ , so  $D_4 = 0$ .

By a similar argument,  $D_5$  will equal 0, so we can conclude that  $D + m(1,2)D = 0$ . Therefore,  $D = 0$ , which completes the proof.  $\square$

**Corollary 2.** *On links with at most 5 components the only finite type invariants are the pairwise linking numbers and their products.*

It appears to be difficult to extend the proof of Theorem 4 to  $B^h(6)$ . In this case, the IHX relation comes into play, and it is not clear that one can decrease  $m(1,4) + m(1,5)$  monotonically.

**2.3. Previous results for  $B^c(k)$ .** Unlike for homotopy, there is no *a priori* limit on the size of the components of diagrams in  $B^c(k)$ . However, we are able to prove some non-existence results for small values of  $k$ . The proofs of the following results can be found in [11]:

**Theorem 5.** *The only nontrivial diagrams in  $B^c(k)$  for  $k = 1, 2, 3$  are disjoint unions of struts. In other words, any diagram with a component of degree greater than 1 is trivial.*

As with homotopy, it seems difficult to extend the methods used to prove Theorem 5 to higher values of  $k$ . The attempt in [11] fell prey to the same error as in [10] (see the Remark in Section 2.1).

REMARK: If we allow the first Reidemeister move, we can prove that  $B^c(1)$  (with the new relation) is trivial, confirming a result of [14] that the Arf invariant (which is  $\mathbb{Z}_2$ -valued) is the only finite type knot concordance invariant.

### 3. EXISTENCE RESULTS FOR $B^h$

In this section we demonstrate the existence of non-trivial diagrams in  $B^h(k)$  which are *not* just the products of small components, for  $k \geq 9$ . Since  $B^h$  is just a quotient of  $B^c$ , this implies the existence of non-trivial diagrams in  $B^c$ . This can also be proved directly using the same methods, but we will leave that as an exercise for the reader. The arguments used are not constructive; we simply use a counting argument to show that (within a certain subspace) there are more diagrams than relations. Since all the relations are just linear combinations, we have a homogeneous system of linear equations with more equations than unknowns, and we conclude that there must be non-trivial solutions. Each such solution corresponds to some finite type invariant which is not just a product of linking numbers.

We consider the subspace  $Y^h(k)$  of  $B^h(k)$  which is spanned by diagrams which have a single Y-component (degree 2 component) and all other components are struts (degree 1). This is the space of diagrams which have exactly one trivalent vertex. Since all the relations of  $B^h$  preserve the number of trivalent vertices (i.e. any two diagrams in a given relation have the same number of trivalent vertices),  $Y^h(k)$  is closed under the relations of  $B^h$ . We will show that  $Y^h(k)$  contains non-trivial diagrams for  $k \geq 9$ .

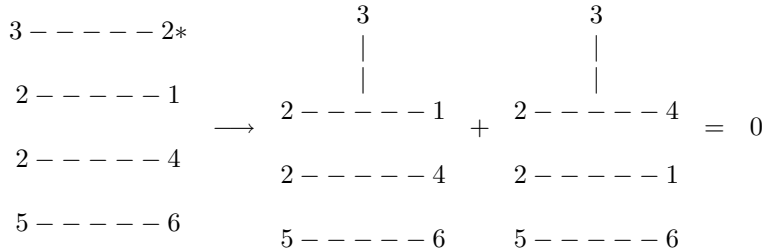
We count the number of diagrams in  $Y^h(k)$  which have exactly  $n$  struts. We count these diagrams by counting the number of ways of coloring the endpoints of the Y-component (i.e. of choosing 3 distinct colors), and then counting the number of ways of choosing the  $n$  struts (i.e. of choosing  $n$  pairs of distinct colors).

Notice that this count does not distinguish the orientation of the trivalent vertex. This would double the number of diagrams, except that the new ones are just the negatives of the previous ones by the antisymmetry relation. So we will leave them out, and simply not count the antisymmetry relations among our relations. Since our diagrams only have one trivalent vertex, there are no IHX relations. Also, since our diagrams have no loops, and the endpoints of any component are given distinct colors, we can ignore the fourth and fifth relations of Definition 2. This means that the only relations we need to count are the link relations.

Let  $u(n, k)$  be the number of elements of  $Y^h(k)$  which have  $n$  struts (i.e. our number of "unknowns"). There are  $\binom{k}{3}$  ways of choosing the labels for the Y-component. There are  $\binom{k}{2}$  possible struts. The number of ways of selecting  $n$  of them, with repetition allowed, is simply  $\binom{\binom{k}{2}+n-1}{n}$ . Putting these together, we find:

$$u(n, k) = \binom{k}{3} \binom{\binom{k}{2} + n - 1}{n}$$

Now we want to count the link relations among these diagrams. Notice that the diagrams in  $Y^h(k)$  with  $n$  struts are exactly the diagrams in  $Y^h(k)$  with  $2n + 3$  endpoints. Since the link relation preserves the number of endpoints, as well as the number of univalent vertices, if one diagram in a relation is an element of  $Y^h(k)$  with  $n$  struts, so is every other diagram in the relation. Let  $r(n, k)$  be the number of link relations among elements in  $Y^h(k)$  which have  $n$  struts. We can think of one of these relations as consisting of  $n + 1$  struts, together with one "special" strut. The special strut will have a distinguished endpoint. The link relation is created by attaching this endpoint in turn to all the other struts which have an endpoint of the same color, forming a series of diagrams with a single Y-component. An example is shown below:



There are  $k(k-1)$  ways of coloring the "special" strut (not  $\binom{k}{2}$ , since the endpoints are distinguished). Then, as before, there are  $\binom{\binom{k}{2}+n}{n+1}$  ways of choosing the other  $n + 1$  struts. We conclude that:

$$r(n, k) = k(k-1) \binom{\binom{k}{2} + n}{n+1}$$

To compare these two counts, we look at the quotient of the number of relations by the number of diagrams:

$$\frac{r(n, k)}{u(n, k)} = \frac{k(k-1) \binom{\binom{k}{2}+n}{n+1}}{\binom{k}{3} \binom{\binom{k}{2}+n-1}{n}} = \frac{6}{k-2} \frac{\binom{k}{2} + n}{n+1}$$

For a fixed value of  $k$ , we can look at the limit of this ratio as  $n \rightarrow \infty$ :

$$\lim_{n \rightarrow \infty} \frac{r(n, k)}{u(n, k)} = \frac{6}{k-2}$$

If  $k \geq 9$ , then this limit is less than 1, which means there are more relations than diagrams, so there must be non-trivial diagrams. If we plug in  $k = 9$  and solve for the ratio to be 1, we obtain  $n = 209$ . So if we have 210 struts (i.e. diagrams of degree 212) there will definitely be nontrivial diagrams.

**Theorem 6.** *There is a non-trivial homotopy invariant on links with 9 components, of type 212, which is not a product of linking numbers.*

REMARK: In fact, we have slightly overcounted the relations. We have counted diagrams where the distinguished endpoint of the special strut has a color which does not appear elsewhere in the diagram, so it cannot be attached to any other strut to form a Y-component. However, if we take the limit of the number of these extra relations divided by  $u(n, k)$  as  $n$  tends to  $\infty$ , we get 0. So removing these relations from the count does not significantly improve our result.

In general, of course, many of the relations are dependent. So we would expect that there are also non-trivial diagrams when  $k = 8$ , and the ratio tends to 1, and possibly for even lower values of  $k$ .

#### 4. QUESTIONS

**Question 1.** *What is an explicit example of a non-trivial finite type link homotopy invariant which is not a product of linking numbers?*

Any such invariant would immediately give a finite type invariant for string links. Bar-Natan [2] has shown that the finite type invariants for string links are exactly the Milnor invariants, which classify string links up to homotopy. However, Milnor's invariants, other than the linking numbers, have indeterminacies which prevent them from being lifted to links as integer- (or  $\mathbb{C}$ -) valued invariants. Apparently, it is possible to find some product in which the indeterminacies "cancel" and the product can be lifted, which is unexpected.

**Question 2.** *What is the first value of  $k$  for which  $B^h(k)$  has non-trivial diagrams which are not disjoint unions of struts?*

We know that such diagrams exist for  $k \geq 9$ , and that they do *not* exist for  $k \leq 5$ , but the situation for  $k = 6, 7, 8$  is still unknown. It seems likely there are non-trivial diagrams in  $B^h(8)$ , but as yet we do not have a proof.

**Question 3.** *What is the first value of  $k$  for which  $B^c(k)$  has non-trivial diagrams which are not disjoint unions of struts?*

As for  $B^h(k)$ , we know that such diagrams exist for  $k \geq 9$ , and do not exist for  $k \leq 3$ , but the situation for  $4 \leq k \leq 8$  is unknown.

**Question 4.** *Can we refine the methods of Section 3 to prove that there are nontrivial diagrams for lower values of  $k$ ?*

We mentioned in Section 3 that the link relations preserve the number of trivalent vertices of the diagram. They also preserve the number of univalent vertices of each color. Perhaps this could be used to find smaller "closed" subspaces with fewer dependent relations, so that the ratio of relations to diagrams is smaller in the limit.

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