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## Finite Type Link Concordance Invariants

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## FINITE TYPE LINK CONCORDANCE INVARIANTS

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## ABSTRACT

This paper is a follow-up to [10], in which the author showed that the only real-valued finite type invariants of link homotopy are the linking numbers of the components. In this paper, we extend the methods used to show that the only real-valued finite type invariants of link concordance are, again, the linking numbers of the components. *Keywords:* Finite type invariants; link concordance.

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## 1. INTRODUCTION

This paper extends the arguments in the author's previous work on link homotopy [10] to link concordance. Many of the arguments are nearly identical; the main differences lie in the base cases to the main theorem, in Section 3. We will begin with a brief overview of finite type invariants. In 1990, V.A. Vassiliev introduced the idea of *Vassiliev* or *finite type* knot invariants, by looking at certain groups associated with the cohomology of the space of knots. Shortly thereafter, Birman and Lin [4] gave a combinatorial description of finite type invariants. We will give a summary of this combinatorial theory. For more details, see Bar-Natan [1].



two such embeddings will differ by crossing changes, but these changes will not effect the value of a type  $d$  Vassiliev invariant on the singular knot. Then, for any type  $d$  invariant  $\gamma$ , we define  $w_d(\gamma)(D) = \gamma(K_D)$ . Bar-Natan shows that this is, in fact, a weight system. The 1-term relation is satisfied because of the first Reidemeister move, and the 4-term relation is essentially the result of rotating a third strand a full turn around a double point.  $v_d$  is much more complicated to define, using the Kontsevich integral. For a full treatment of the Kontsevich integral, see Bar-Natan [1] and Le and Murakami [9]. Using a Morse function, any knot (or link or string link) can be decomposed into elementary *tangles*:

Le and Murakami define a map  $Z$  from an elementary tangle with  $k$  strands to the space of chord diagrams on  $k$  strands. This map respects composition of tangles: if  $T_1 \cdot T_2$  is the tangle obtained by placing  $T_1$  on top of  $T_2$ , then  $Z(T_1 \cdot T_2) = Z(T_1)Z(T_2)$ . Le and Murakami prove that this map gives an isotopy invariant of knots and links. Given a degree  $d$  weight system  $W$ , and a knot  $K$ , we now define  $v_d(W)(K) = W(Z(K))$ . Bar Natan shows that  $w_d$  and  $v_d$  are “almost” inverses. More precisely,  $w_d(v_d(W)) = W$  and  $v_d(w_d(\gamma)) - \gamma$  is a knot invariant of type  $d - 1$ . As a result, (see [4, 1, 12]) the space  $W_d$  of weight systems of degree  $d$  is isomorphic to  $V_d/V_{d-1}$ . For convenience, we will usually take the dual approach, and simply study the space of chord diagrams of degree  $d$  modulo the 1-term and 4-term relations. The dimensions of these spaces have been computed for  $d \leq 12$  (see Bar-Natan [1] and Kneissler [8]). It is useful to combine all of these spaces into a graded module via direct sum. We can give this module a Hopf algebra structure by defining an appropriate product and co-product:

- We define the product  $D_1 \cdot D_2$  of two chord diagrams  $D_1$  and  $D_2$  as their connect sum. This is well-defined modulo the 4-term relation (see [1]).

- We define the co-product  $\Delta(D)$  of a chord diagram  $D$  as follows:

$$\Delta(D) = \sum_J D'_J \otimes D''_J$$

where  $J$  is a subset of the set of chords of  $D$ ,  $D'_J$  is  $D$  with all the chords in  $J$  removed, and  $D''_J$  is  $D$  with all the chords not in  $J$  removed.

It is easy to check the compatibility condition  $\Delta(D_1 \cdot D_2) = \Delta(D_1) \cdot \Delta(D_2)$ .

FIGURE 1. STU relation

FIGURE 2. AS and IHX relations

**1.2. Univalent Diagrams.** It is often useful to consider the Hopf algebra of bounded univalent diagrams, rather than chord diagrams. These diagrams, introduced by Bar-Natan [1] (Bar-Natan calls them *Chinese Character Diagrams*), can be thought of as a shorthand for writing certain linear combinations of chord diagrams. We define a *bounded univalent graph* to be a univalent graph, with oriented vertices, together with a bounding circle to which all the univalent vertices are attached. We also require that each component of the graph have at least one univalent vertex (so every component is connected to the boundary circle). We define the space  $A$  of bounded univalent diagrams as the quotient of the space of all bounded univalent graphs by the  $STU$  relation, shown in Figure 1. As consequences of  $STU$  relation, the anti-symmetry ( $AS$ ) and  $IHX$  relations, see Figure 2, also hold in  $A$ . Bar-Natan shows that  $A$  is isomorphic to the algebra of chord diagrams. We can get an algebra  $B$  of *univalent diagrams* by simply removing the bounding circle from the diagrams in  $A$ , leaving graphs with trivalent and univalent vertices, modulo the  $AS$  and  $IHX$  relations. Bar-Natan shows that the spaces  $A$  and  $B$  are isomorphic. The map  $\chi$  from  $B$  to  $A$  takes a diagram to the linear combination of all ways of attaching the univalent vertices to a bounding circle, divided by total number of such ways (T. Le noticed that this factor, missing in [1], is necessary to preserve the comultiplicative structure of the algebras). The inverse map  $\sigma$  turns a diagram into a linear combination of diagrams by performing sequences of “basic operations,” and then removes the bounding circle. The two basic operations are:

## 2. STRING LINKS, LINKS AND CONCORDANCE

**2.1. String Links.** Bar-Natan [2] extends the theory of finite type invariants to string links.

FIGURE 3. The link relation for chord diagrams

**Definition 3.** (see [6]) Let  $D$  be the unit disk in the plane and let  $I = [0,1]$  be the unit interval. Choose  $k$  points  $p_1, \dots, p_k$  in the interior of  $D$ , aligned in order along the  $x$ -axis. A **string link**  $\sigma$  of  $k$  components is a smooth proper imbedding of  $k$  disjoint copies of  $I$  into  $D \times I$ :

$$\sigma : \bigsqcup_{i=1}^k I_i \rightarrow D \times I$$

such that  $\sigma|_{I_i}(0) = p_i \times 0$  and  $\sigma|_{I_i}(1) = p_i \times 1$ . The image of  $I_i$  is called the  $i$ th string of the string link  $\sigma$ .

Essentially, everything works the same way for string links as for knots. The bounding circle of the bounded univalent diagrams now becomes a set of bounding line segments, each labeled with a color, to give an algebra  $A^{sl}$  (the multiplication is given by placing one diagram on top of another). The univalent diagrams are unchanged, except that each univalent vertex is also labeled with a color to give the space  $B^{sl}$ . The isomorphisms  $\chi$  and  $\sigma$  between  $A$  and  $B$  easily extend to isomorphisms  $\chi^{sl}$  and  $\sigma^{sl}$  between  $A^{sl}$  and  $B^{sl}$ , just working with each color separately. In addition, there are obvious maps  $w_d^{sl}$  and  $v_d^{sl}$  analogous to  $w_d$  and  $v_d$  (we just need to keep track of colors).

**2.2. Links.** The obvious definition of chord diagrams for links is simply to replace the bounding line segments with bounding circles. However, these diagrams are difficult to work with, and it is in particular unclear how to define the univalent diagrams. Unlike for a knot, closing up the components of a string link of several components is not a trivial operation, so we need to place some relations on the space of univalent diagrams. Since we understand the spaces of chord diagrams and univalent diagrams for string links, it would be useful to be able to express these spaces for links as quotients of the spaces for string links. The question is then, what relations do we need? One relation is fairly obvious. When we construct the space  $A^l$  of bounded univalent diagrams for links, we replace the bounding line segments of  $A^{sl}$  with directed circles. Bar-Natan et. al. observed (see Theorem 3, [3]) that this is exactly equivalent to saying that the “top” edge incident to one of the line segments can be brought around the circle to be on the “bottom.” So we can write  $A^l$  as the quotient of  $A^{sl}$  by relation (1), shown in Figure 3 (where the figure shows *all* the chords with endpoints on the red component). Then the Kontsevich integral for links,  $Z^l$ , is defined by cutting the link to make a string link, applying the Kontsevich integral for string links, and then taking the quotient by relation (1). Now  $w_d^l$  and  $v_d^l$  are defined similarly to  $w_d$  and  $v_d$ . Given a link invariant  $\gamma$  and a diagram  $D$  in  $A^l$ ,  $w_d^l(\gamma)(D) = \gamma(L_{\hat{D}})$ , where  $\hat{D}$  is the closure

FIGURE 4. The link relation for Bounded Univalent Diagrams

FIGURE 5. The link relation for univalent diagrams

of the diagram  $D$  (i.e. the bounding line segments are closed to circles).  $L_{\hat{D}}$  is well-defined by Theorem 3 of [3]. Defining  $v_d^l$  is even easier, now that we have  $Z^l$ . Given a weight system (element of the graded dual of  $A^l$ )  $W$  and a link  $L$ , we define  $v_d^l(W)(L) = W(Z^l(L))$ . One advantage of this formulation of  $A^l$  is that it enables us to define the space  $B^l$  of univalent diagrams as a quotient of the already known (see [2]) space  $B^{sl}$ . This was done by Bar-Natan et. al. [3]. Using the  $STU$  relation, we can rewrite relation (1) as in Figure 4. This suggests how we should define the space  $B^l$ . We will take the quotient of  $B^{sl}$  by the relations (\*) shown in Figure 5, where the univalent vertices shown are *all* the univalent vertices of a given color. With these definitions, Bar-Natan et. al. proved that  $A^l$  and  $B^l$  are isomorphic:

**Theorem 1.** (Theorem 3, [3]) *The isomorphism between  $A^{sl}$  and  $B^{sl}$  descends to an isomorphism between  $A^l$  and  $B^l$ .*

### 2.3. Link Concordance.

**Definition 4.** *Consider two  $k$ -component links  $L_0$  and  $L_1$ , and two  $k$ -component string links  $SL_0$   $SL_1$ . These can be thought of as embeddings (proper embeddings, in the case of the string links):*

$$L_i : \bigsqcup_{i=1}^k S^1 \hookrightarrow \mathbf{R}^3$$

$$SL_i : \bigsqcup_{i=1}^k I \hookrightarrow \mathbf{R}^2 \times I$$

A **(link) concordance** between  $L_0$  and  $L_1$  is an embedding:

$$H : \left( \bigsqcup_{i=1}^k S^1 \right) \times I \hookrightarrow \mathbf{R}^3 \times I$$

such that  $H(x, 0) = (L_0(x), 0)$  and  $H(x, 1) = (L_1(x), 1)$ . Similarly, a **(string link) concordance** between  $SL_0$  and  $SL_1$  is an embedding:

$$H : \left( \bigsqcup_{i=1}^k I \right) \times I \hookrightarrow (\mathbf{R}^2 \times I) \times I$$

such that  $H(x, 0) = (SL_0(x), 0)$ ,  $H(x, 1) = (SL_1(x), 1)$ , and  $H(i, t) = (i, t)$  for  $i = 0, 1$ . A concordance is an isotopy if and only if  $H$  is level preserving; i.e. if the image of  $H_t$  is a (string) link at level  $t$  for each  $t \in I$ .

We want to extend the results of the last section to string links and links considered up to concordance. For string links, this has already been done by Habegger and Masbaum [7]. They describe the algebras  $A^{csl}$  and  $B^{csl}$  of bounded and unbounded unitrivalent diagrams for string links up to concordance (which they denote  $A^t$  and  $B^t$ ), and observe that they are isomorphic. In brief, we take the quotient of  $A^{sl}$  (resp.  $B^{sl}$ ) by the space of diagrams with non-trivial first homology. In other words, we are left with tree diagrams. It is then straightforward to define  $w_d^{csl}$  and  $v_d^{csl}$  in the usual way, and show that they are “almost” inverses in the same sense that  $w_d$  and  $v_d$  are. All of this extends to links just as it did for isotopy. We define  $A^{cl}$  as the quotient of  $A^{csl}$  by relation (1), and  $B^{cl}$  as the quotient of  $B^{csl}$  by relation (\*). We then define  $Z^{cl}$ ,  $w_d^{cl}$ , and  $v_d^{cl}$  just as we did for links up to homotopy. Finally, the arguments of Bar-Natan et. al. carry through to show:

**Theorem 2.** (Theorem 3, [3]) *The isomorphism between  $A^{csl}$  and  $B^{csl}$  descends to an isomorphism between  $A^{cl}$  and  $B^{cl}$ .*

**Remark:** By results of Habegger and Masbaum (see Theorem 5.5 of [7]),  $Z^{cl}$  is the *universal* finite type invariant of link concordance. By this we mean that it dominates all other such invariants.

### 3. THE SIZE OF $B^{cl}$

Now that we have properly defined the space  $B^{cl}$  of unitrivalent diagrams for link homotopy, we want to analyze it more closely. We will consider the case when  $B^{cl}$  is a vector space over the reals (or, more generally, a module over a ring of characteristic 0). In particular, we would like to know exactly which diagrams of  $B^{csl}$  are in the kernel of the relation (\*) (i.e. are 0 modulo (\*)). We will find that the answer is “almost everything” - to be precise, any unitrivalent diagram with a component of degree 2 or more. We will start by proving a couple of base cases, and then prove the rest of the theorem by induction. Let  $B^{csl}(k)$  denote the space of unitrivalent diagrams for string link concordance with  $k$  possible colors for the univalent vertices (i.e. we are looking at links with  $k$  components). Consider a diagram  $D \in B^{csl}(k)$ . Recall from the previous sections that each component of  $D$  is a tree diagram. **Notation:** Before we continue, we will introduce two bits of notation which will be useful in this section.

- Given a unitrivalent diagrams  $D$ , we define  $m(D; i, j)$  to be the number of components of  $D$  which are simply line segments with ends colored  $i$  and  $j$ , as shown below:

$$i - - - - - j$$

- Components of a diagram with degree greater than one will be called *large* components. Components of degree one will be called *small* components.



FIGURE 6. Using IHX to get “standard” tree diagram

**3.1. Knots and Two Component Links.** We begin by considering the case of knot concordance, when  $k = 1$ . Ng [11] has already shown that the only finite type invariant of knot concordance is the  $\mathbf{Z}_2$ -valued Arf invariant, so there are no real-valued finite type invariants of knot concordance. We will begin, as a warm-up, by showing this result using unitrivalent diagrams. Bar-Natan has shown that the spaces of unitrivalent diagrams for knots and string links of one component are isomorphic, so the relation (\*) has no effect.

**Lemma 1.**  $B^{csl}(1) = B^{cl}(1) = 0$ .

PROOF: First, we consider the case when  $D \in B^{csl}(1)$  has a large component  $C$ . Since  $C$  is a tree, we can use the *IHX* relation as in Figure 6 to rewrite  $C$  as a sum of diagrams:

Clearly, twisting the branch at the right or left end of  $T$  gives us the same diagram, but by the *AS* relation, this flips the sign. Therefore,  $T = -T = 0$ , so  $D = 0$ . The case when all the components of  $D$  have degree one is somewhat more subtle, though it is essentially an application of the 1-term relation. If  $D$  has only small components, then  $\chi(D)$  (see Section 1.2) is a linear combination of bounded unitrivalent diagrams with no interior vertices (i.e. chord diagrams). By repeated applications of the *STU* relation, we can isolate a chord in each of these diagrams, at the expense of adding a linear combination of diagrams which *do* have internal vertices. The diagrams with isolated chords disappear by the 1-term relation, so we are left with a linear combination of bounded unitrivalent diagrams with at least one internal vertex. Now we apply  $\sigma$  to this linear combination to get a linear combination of (unbounded) unitrivalent diagrams. Since the basic operations  $U$  and  $S$  of  $\sigma$  can never decrease the number of internal vertices, every unitrivalent diagram in the image of  $\sigma$  will have at least one internal vertex; i.e. at least one large component. Hence, by the first case, all of these diagrams are 0. So we have shown that  $\sigma(\chi(D)) = 0$ . But  $\sigma$  and  $\chi$  are inverse isomorphisms, so this means  $D = 0$ , as desired.  $\square$  Next we consider links of two components, i.e.  $k = 2$ . In this case we need to consider the effect of the relation (\*).

**Lemma 2.** *Let  $D \in B^{csl}(2)$ . If  $D$  has a component  $C$  of degree  $d \geq 2$ , then  $D$  is trivial modulo (\*).*

PROOF:  $D$  is a diagram with all endpoints colored 1 or 2. Note that  $C$  must have endpoints of both colors, or  $D$  will be trivial by the same argument as in Lemma 1. In fact, any terminal branch of  $C$  must have the form (where  $\bar{C}$  denotes

FIGURE 7. Diagrams arising from relation (\*)

the remainder of  $C$ ):

$$C : \begin{array}{c} \bar{C} \\ | \\ 1 - - - - - 2 \end{array}$$

Otherwise, if the two endpoints have the same color,  $C$  (and hence  $D$ ) is trivial by the  $AS$  relation. The proof is by induction on the number of large components of  $D$ . In the base case, there is only one such component,  $C$ . So all the other components  $C_i$  are simply line segments labeled  $a$  and  $b$ , where  $a, b \in \{1, 2\}$ , as shown:

$$C_i : a - - - - - b$$

Now we apply the relation (\*) to the branch of  $C$  shown above using the color 1, as in Figure 7 (where  $\bar{C}_i$  represents the remainder of the component  $C_i$ ). We denote the merger of  $C_i$  and  $\bar{C}$  as  $C'_i$ . We do not need to consider other vertices of  $C$  colored 1, since these terms result in a diagram with a loop, which are trivial in concordance. Since we are expanding using the color 1, we only need to consider  $C_i$  where  $a$  or  $b$  is 1. If they are both 1, then by the  $AS$  relation:

$$C'_i : \begin{array}{c} \bar{C} \\ | \\ 1 - - - - - 1 \end{array} = 0$$

Therefore, we need only consider  $C_i$  with one endpoint colored 1 and the other colored 2. But in this case,  $C'_i = C$ , so  $D_i = D$ , and we get an equation  $D + \sum D_i = (1 + m)D = 0$  for some  $m \geq 0$ . Hence  $D = 0$  modulo (\*). This concludes the base case. For the inductive step, we assume that the theorem is true for diagrams with  $n$  large components, and consider a diagram  $D$  with  $n + 1$  large components. Let

$C$  be one of these components. As before,  $C$  has a branch:

$$C : \begin{array}{c} \bar{C} \\ | \\ 1 - - - - 2 \end{array}$$

Also, every small component  $C_i$  looks like (where  $a, b \in \{1, 2\}$ ):

$$C_i : a - - - - b$$

Once again, we apply the relation (\*) to the color 1, using this branch of  $C$ , and find that  $D + \sum D_i = 0$ . Whenever  $C_i$  is small,  $D_i = D$  (as in the base case, we need only consider the case when  $a = 1$  and  $b = 2$ ). If  $C_i$  is large, then  $D_i$  has one fewer large component than  $D$  does, since the bulk of  $C$  has joined with  $C_i$ , leaving behind a line segment of degree one. So  $D_i$  is trivial modulo (\*) by the inductive hypothesis. Hence we are left with a sum of copies of  $D$ , and conclude that  $(1 + m)D = 0$  for some  $m \geq 0$ , so  $D = 0$  modulo (\*). This concludes the induction and the proof.  $\square$

**3.2. Three Component Links.** The case when  $k = 3$  is our final special case before the proof of the general theorem, and it is significantly more complicated than the previous two lemmas. The main step is to show that no component of  $D$  can have two endpoints of the same color (essentially, this reduces the problem to the case of link homotopy, treated in [10]). Once again, we will be inducting on the number of large components of the diagram. The base case contains most of the work of the proof, so we present it as a separate lemma.

**Lemma 3.** *If  $k=3$ ,  $D$  has exactly one large component  $C$ , and  $C$  has two endpoints of the same color, then  $D$  is trivial modulo (\*).*

PROOF: Without loss of generality, we will say that  $C$  has two endpoints colored 1. If these endpoints are on the same final branch, as shown below, then  $D$  will be trivial by the AS relation (since we will have  $D = -D$ ).  $\bar{C}$  denotes the remainder of  $C$ :

$$C : \begin{array}{c} \bar{C} \\ | \\ 1 - - - - 1 \end{array}$$

Otherwise, we can use the IHX relation to move one of the endpoints colored 1 out to the ends of the component  $C$ , as shown in Figure 8 (where we move the endpoint colored  $k$ ). So it suffices, without loss of generality, to consider  $C$  with a branch as shown below:

$$C : \begin{array}{c} \bar{C} \\ | \\ 1 - - - - 2 \end{array}$$

Our proof will be by induction on  $m(D; 2, 3)$ , inducting among diagrams with exactly one large component, which have a branch with colors 1 and 2 as shown above. For the base case,  $m(D; 2, 3) = 0$ . This means that there are no small components of  $D$  colored 2 and 3. So the only degree 1 components with an endpoint colored 2 have their other endpoint colored 1 or 2:

$$C_i : a - - - - 2, a \in \{1, 2\}$$

FIGURE 8. Using the IHX relation to decompose a diagram

Now we apply the relation (\*) to  $D$  as in Figure 7, only now we are applying it using the color 2. So we only need to consider the components of  $D$  with endpoints colored 2. Because of the loop relation for concordance, we can ignore other endpoints of  $C$  which might be colored 2, and just consider the remaining components. These are all small components with endpoints colored  $a$  and 2 as described above. As in Lemma 2, the case when  $a = 2$  can be ignored by the  $AS$  relation, so we are reduced to the case of small components with endpoints colored 1 and 2. Therefore, we obtain the equation  $D + \sum D_i = 0$ , where every  $D_i = D$ . So  $D + m(D; 1, 2)D = (1 + m(D; 1, 2))D = 0$ . Since  $m(D; 1, 2) \geq 0$ , we conclude that  $D = 0$  modulo (\*). This concludes the base case of the induction. For the inductive step, we will have small components with endpoints colored 2 and 3. Our goal is to reduce the number of such components. Now, when we apply the relation (\*), we can again ignore small components with both endpoints colored 2, by the  $AS$  relation. We get an equation  $D + m(D; 1, 2)D + m(D; 2, 3)D' = 0$ , where  $D'$  is identical to  $D$  except that a small component with endpoints colored 2 and 3 has been replaced by one with endpoints colored 1 and 2, and the endpoint of  $C$  colored 1 above has been colored 3. We denote this analogue of  $C$  in  $D'$  by  $C'$ . Diagrammatically, we can represent  $D'$  by showing the changes that have been made to  $D$ :

$$D' : \begin{array}{ccc} & \bar{C} & \\ & | & \\ & \text{---} & (2, 3) \rightarrow (1, 2) \\ 3 & \text{---} & \text{---} & \text{---} & 2 \end{array}$$

Note that  $m(D'; 2, 3) = m(D; 2, 3) - 1$ . As we did in Lemma 1, we can apply the  $IHX$  relation to  $C'$ , fixing the branch shown above. As a result, it suffices to

consider the case when  $C'$  has the form shown below (where  $n$  is the degree of  $C'$ ):

We can assume that one of the endpoints  $\alpha_3, \dots, \alpha_n$  is colored 1. If not, then  $\alpha_{n+1} = 1$  (since we know there is a second endpoint of  $C$  colored 1), and we can switch  $\alpha_n$  and  $\alpha_{n+1}$  using the *AS* relation (at the cost of reversing the sign of  $D'$ ). We can apply the *IHX* relation as in Figure 6 to move the endpoint colored 2 along the “spine” of  $C'$  (i.e. the path from the endpoint colored 3 to the endpoint colored  $\alpha_{n+1}$ ) and obtain the decomposition  $D' = \sum D^i$ , where we transform  $C$  into  $C^i$  as shown below:

Notice that the spine of  $C^i$  is shorter than the spine of  $C$  by one edge; and that one of the branches along the spine has grown correspondingly. We observe that if  $\alpha_i = 1$ , then  $D^i = 0$  by the inductive hypothesis, since it now has a branch colored 1 and 2, and  $m(D^i; 2, 3) = m(D'; 2, 3) = m(D; 2, 3) - 1$ . So the endpoint colored 1 will not be incorporated into the larger branch, and has moved one position closer to the endpoint colored 3 at the far left. In the case when  $\alpha_3 = 1$ ,  $D^3 = 0$ , and  $C^i$  has the form below for  $i > 3$ :

Now if we use the relation (\*) to expand  $D^i$  along the color 1, we find that  $D^i + m(D^i; 1, 3)D^i + m(D^i; 1, 2)D_2^i = 0$ , where  $D_2^i$  is the result of replacing a small component colored 1 and 2 with a small component colored 1 and 3, and changing the endpoint of  $C^i$  colored 3 to one colored 2. Then  $D_2^i$  has a large component with a branch colored 1 and 2, and  $m(D_2^i; 2, 3) = m(D^i; 2, 3) = m(D; 2, 3) - 1$ , so  $D_2^i$  is trivial modulo (\*) by the inductive hypothesis. Therefore, we find that  $(1 + m(D^i; 1, 3))D^i = 0$ , and hence  $D^i = 0$  modulo (\*). If  $\alpha_3 \neq 1$ , the first branch on the spine of  $C^i$  (adjacent to the endpoint colored 3) will only have endpoints colored 2 and 3 (not 1). Now we repeat the process by expanding the (non-trivial)  $C^i$ 's, using the first branch on the spine of  $C^i$ . In order to continue the process, we need to show the following fact: CLAIM: If  $K$  is a unitrivalent diagram with all endpoints colored 2 or 3, then the following diagram is trivial modulo (\*):

Here  $K$  is assumed to be a subdiagram of the only component of degree greater than 1 in a diagram  $E$  such that  $m(E; 2, 3) = m(D; 2, 3) - 1$ . PROOF OF CLAIM: Using the *IHX* relation as in Figure 8, we can decompose this diagram into a linear combination of diagrams where the endpoint colored 1 has migrated out to one of the ends of  $K$ , leaving a diagram with a branch with endpoints colored 1 and 2 or 1 and 3. In the first case, the diagram is trivial by the inductive hypothesis.

In the second case, the diagram is trivial by the argument used above for  $\alpha_3 = 1$ . So we conclude that  $E$  is trivial modulo (\*).  $\square$  (Claim) Using the claim, we can continue the process, moving the endpoint colored 1 to the left at each stage, until we are left with diagrams where the first branch, adjacent to the endpoint colored 3, consists of a single endpoint colored 1. These diagrams are trivial by the argument above (when  $\alpha_3 = 1$ ). We can conclude that all of the  $D^i$ 's are trivial modulo (\*). Therefore,  $D'$  is trivial modulo (\*), and we obtain the equation  $D + m(D; 1, 2)D = (1 + m(D; 1, 2))D = 0$ . We conclude that  $D = 0$  modulo (\*), which finishes the induction and the proof.  $\square$

**Lemma 4.** *If  $k=3$  and  $D$  has a component  $C$  with two endpoints of the same color, then  $D$  is trivial modulo (\*).*

PROOF: First, we assume that  $C$  is large. We will induct on the number of large components of  $D$ . The base case of the induction has already been proved, in Lemma 3. The general case follows exactly the same argument. The only modification is to notice that, whenever (\*) is applied, the diagrams  $D_i$  (as in Figure 7) which arise from large components  $C_i$  will have fewer large components than  $D$  (since  $C$  and  $C_i$  have been joined, leaving behind a component of degree one). So by the inductive hypothesis, these diagrams can be ignored at every stage. Therefore, exactly the same proof shows that  $D$  is trivial modulo (\*). If  $C$  is small, with both endpoints the same color, we can use the previous case together with the argument from Lemma 1 to show  $D$  is trivial.  $\square$

**Lemma 5.** *If  $k=3$  and  $D$  has a large component  $C$ , then  $D$  is trivial modulo (\*).*

PROOF: By Lemma 4, there is (up to sign) only one possible diagram for  $C$ :

$$C = \begin{array}{c} 3 \\ | \\ 1 - - - - 2 \end{array}$$

Now we apply the relation (\*) to  $D$  using  $C$  and the color 1. By Lemma 4, we need only consider components with endpoints colored 1 or 2, and no component can have two endpoints of the same color. Therefore, we need only consider components  $C_i$  as shown:

$$C_i = 1 - - - - 2$$

This gives us the equation  $D + m(D; 1, 2)D = (1 + m(D; 1, 2))D = 0$ . We conclude that  $D = 0$  modulo (\*), which completes the proof for the case  $k = 3$ .  $\square$

**3.3. The General Case.** We are now ready to begin our proof of the general case. As we did for the case when  $k = 3$ , we will induct on the number of large components of  $D$ . Once again, for clarity, we will prove the base case (which contains most of the work) as a separate lemma.

**Lemma 6.** *If  $D$  has exactly one large component  $C$ , then  $D$  is trivial modulo (\*).*

PROOF: The method of proof for this lemma is very similar to the earlier lemmas. We will successively apply (\*) (and do a single expansion via IHX) until we obtain a set of diagrams which are all either trivial or repetitions of earlier diagrams. We can then backtrack to show that everything disappears. However, we will need to apply (\*) four times. This unfortunately makes keeping track of the diagrams

somewhat confusing - we have done our best. Without loss of generality, as before, we can assume that  $C$  has a branch as shown:

$$C : \begin{array}{c} \bar{C} \\ | \\ 1 - - - - 2 \end{array}$$

We apply (\*) using the color 1 and find that  $D+m(D; 1, 2)D+\sum_{a \neq 1, 2} m(D; 1, a)D_a = 0$ , where  $D_a$  is the same as  $D$  except that:

- $C$  has been replaced by a component  $C_a$  identical to it except that the endpoint colored 2 is now colored  $a$  (so  $\bar{C}_a = \bar{C}$ )
- A line segment with endpoints colored 1 and  $a$  has been replaced by a line segment with endpoints colored 1 and 2. In other words,  $m(D_a; 1, a) = m(D; 1, a) - 1$  and  $m(D_a; 1, 2) = m(D; 1, 2) + 1$ .

We will denote this as shown below:

$$D_a : \begin{array}{c} \bar{C} \\ | \\ 1 - - - - a \end{array} \quad (1, a) \rightarrow (1, 2)$$

As in Lemma 3, we use the IHX relation to decompose  $D_a = \sum_{\alpha_i \neq a} \pm D_a^i$  (summing over the endpoints of  $C_a$ , with colors  $\alpha_i$ ), where the analogue  $C_a^i$  of  $C_a$  in  $D_a^i$  has a branch as shown, and the other components of the diagram are the same as  $D_a$ :

$$D_a^i : \begin{array}{c} \bar{C}_a^i \\ | \\ i - - - - a \end{array} \quad (1, a) \rightarrow (1, 2)$$

By abuse of notation, we write the color  $\alpha_i$  as simply  $i$ . Since we will never have to compare different  $D_a^i$ 's, this will not cause any confusion. Note that, aside from having endpoints of the same colors,  $C_a^i$  looks nothing like  $C_a$ . Now we apply (\*) to  $D_a^i$ , using the color  $i$ . In the pictures we use to describe the various diagrams that we produce in what follows, we will just be showing how the diagrams differ from  $D_a^i$ . This will involve showing how  $C_a^i$  has been altered, and which line segments have been added or removed. At each stage, we will eliminate loop diagrams without comment. We obtain the relation:

$$D_a^i + m(D_a^i; i, a)D_a^i + m(D_a^i; 2, i)D_{a2}^i + \sum_{b \neq i, 2, a} m(D_a^i; i, b)D_{ab}^i = 0$$

where:

$$D_{a2}^i : \begin{array}{c} \bar{C}_a^i \\ | \\ i - - - - 2 \end{array} \quad (2, i) \rightarrow (i, a)$$

$$D_{ab}^i : \begin{array}{c} \bar{C}_a^i \\ | \\ i - - - - b \end{array} \quad (i, b) \rightarrow (i, a)$$

Next we apply (\*) to  $D_{ab}^i$ , using the color  $b$ , and find that:

$$D_{ab}^i + m(D_{ab}^i; i, b)D_{ab}^i + m(D_{ab}^i; 2, b)D_{ab2}^i + \sum_{c \neq b, i, 2} m(D_{ab}^i; b, c)D_{abc}^i = 0$$

where:

$$D_{ab2}^i : \begin{array}{c} \bar{C}_a^i \\ | \\ 2 \text{ --- } \text{---} \text{---} \text{---} b \end{array} \quad \begin{array}{l} (i, b) \rightarrow (i, a) \\ (2, b) \rightarrow (i, b) \end{array} \Rightarrow (2, b) \rightarrow (i, a)$$

$$D_{abc}^i : \begin{array}{c} \bar{C}_a^i \\ | \\ c \text{ --- } \text{---} \text{---} \text{---} b \end{array} \quad \begin{array}{l} (i, b) \rightarrow (i, a) \\ (c, b) \rightarrow (i, b) \end{array} \Rightarrow (c, b) \rightarrow (i, a)$$

Now we apply (\*) to  $D_{ab2}^i$  using the color 2, and to  $D_{abc}^i$ , using the color  $c$ . We get two relations:

$$D_{ab2}^i + m(D_{ab2}^i; 2, b)D_{ab2}^i + m(D_{ab2}^i; 2, i)D_{ab2i}^i + \sum_{c \neq b, i, 2} m(D_{ab2}^i; 2, c)D_{ab2c}^i = 0$$

$$D_{abc}^i + m(D_{abc}^i; b, c)D_{abc}^i + m(D_{abc}^i; 2, c)D_{abc2}^i + m(D_{abc}^i; i, c)D_{abci}^i + \sum_{d \neq b, c, i, 2} m(D_{abc}^i; c, d)D_{abcd}^i = 0$$

where:

$$D_{ab2i}^i : \begin{array}{c} \bar{C}_a^i \\ | \\ 2 \text{ --- } \text{---} \text{---} \text{---} i \end{array} \quad \begin{array}{l} (2, b) \rightarrow (i, a) \\ (2, i) \rightarrow (2, b) \end{array} \Rightarrow (2, i) \rightarrow (i, a)$$

$$D_{ab2c}^i : \begin{array}{c} \bar{C}_a^i \\ | \\ 2 \text{ --- } \text{---} \text{---} \text{---} c \end{array} \quad \begin{array}{l} (2, b) \rightarrow (i, a) \\ (2, c) \rightarrow (2, b) \end{array} \Rightarrow (2, c) \rightarrow (i, a)$$

$$D_{abc2}^i : \begin{array}{c} \bar{C}_a^i \\ | \\ c \text{ --- } \text{---} \text{---} \text{---} 2 \end{array} \quad \begin{array}{l} (c, b) \rightarrow (i, a) \\ (2, c) \rightarrow (c, b) \end{array} \Rightarrow (2, c) \rightarrow (i, a)$$

$$D_{abci}^i : \begin{array}{c} \bar{C}_a^i \\ | \\ c \text{ --- } \text{---} \text{---} \text{---} i \end{array} \quad \begin{array}{l} (c, b) \rightarrow (i, a) \\ (i, c) \rightarrow (c, b) \end{array} \Rightarrow (i, c) \rightarrow (i, a)$$

$$D_{abcd}^i : \begin{array}{c} \bar{C}_a^i \\ | \\ c \text{ --- } \text{---} \text{---} \text{---} d \end{array} \quad \begin{array}{l} (c, b) \rightarrow (i, a) \\ (c, d) \rightarrow (c, b) \end{array} \Rightarrow (c, d) \rightarrow (i, a)$$

We make several observations:

- $D_{ab2i}^i = -D_{a2}^i$ .
- $D_{ab2c}^i = -D_{abc2}^i$ .
- $D_{abci}^i = -D_{ac}^i$ .
- $D_{abcd}^i = D_{adc}^i = -D_{acd}^i$ .



Now that we have these recursive relations, we can plug them into our various equations. We will use the following equalities:

$$\begin{aligned} m(D_{ab}^i; i, b) + 1 &= m(D_a^i; i, b) \\ m(D_{ab2}^i; 2, b) + 1 &= m(D_{ab}^i; 2, b) = m(D_a^i; 2, b) \\ m(D_{abc}^i; b, c) + 1 &= m(D_a^i; b, c) \end{aligned}$$

And for all the other coefficients we have:

$$m(D_*^i; x, y) = m(D_a^i; x, y)$$

For convenience, we will write  $m(x, y) = m(D_a^i; x, y)$  in what follows:

$$\begin{aligned} (m(i, a) + 1)D_a^i + m(2, i)D_{a2}^i &= \sum_{b \neq a, i, 2} -m(i, b)D_{ab}^i \\ &= \sum_{b \neq a, i, 2} -(m(D_{ab}^i; i, b) + 1)D_{ab}^i \\ &= \sum_{b \neq a, i, 2} \left( m(2, b)D_{ab2}^i + \sum_{c \neq b, i, 2} m(b, c)D_{abc}^i \right) \end{aligned}$$

Note that:

$$\begin{aligned} m(2, b)D_{ab2}^i &= (m(D_{ab2}^i; 2, b) + 1)D_{ab2}^i \\ &= -m(2, i)D_{ab2i}^i + \sum_{c \neq b, i, 2} -m(2, c)D_{ab2c}^i \\ m(b, c)D_{abc}^i &= (m(D_{abc}^i; b, c) + 1)D_{abc}^i \\ &= -m(2, c)D_{abc2}^i - m(i, c)D_{abci}^i - \sum_{d \neq c, b, i, 2} m(c, d)D_{abcd}^i \end{aligned}$$

Therefore:

$$\begin{aligned} &m(2, b)D_{ab2}^i + \sum_{c \neq b, i, 2} m(b, c)D_{abc}^i = \\ &-m(2, i)D_{ab2i}^i + \sum_{c \neq b, i, 2} \left( -m(2, c)(D_{ab2c}^i + D_{abc2}^i) - m(i, c)D_{abci}^i - \sum_{d \neq c, b, i, 2} m(c, d)D_{abcd}^i \right) = \\ &m(2, i)D_{a2}^i + \sum_{c \neq b, i, 2} \left( m(i, c)D_{ac}^i + \sum_{d \neq c, b, i, 2} m(c, d)D_{acd}^i \right) \end{aligned}$$

We plug this back in above to find:

$$(m(i, a) + 1)D_a^i + m(2, i)D_{a2}^i = \sum_{b \neq a, i, 2} \left( m(2, i)D_{a2}^i + \sum_{c \neq b, i, 2} \left( m(i, c)D_{ac}^i + \sum_{d \neq c, b, i, 2} m(c, d)D_{acd}^i \right) \right)$$

We notice that:

$$\begin{aligned}
\sum_{c \neq b, i, 2} \sum_{d \neq c, b, i, 2} m(c, d) D_{acd}^i &= \frac{1}{2} \sum_{\substack{c \neq d \\ c, d \neq b, i, 2}} m(c, d) (D_{acd}^i + D_{adc}^i) \\
&= \frac{1}{2} \sum_{\substack{c \neq d \\ c, d \neq b, i, 2}} m(c, d) (D_{acd}^i - D_{adc}^i) \\
&= 0
\end{aligned}$$

Therefore:

$$(m(i, a) + 1)D_a^i + m(2, i)D_{a2}^i = \sum_{b \neq a, i, 2} \left( m(2, i)D_{a2}^i + \sum_{c \neq b, i, 2} m(i, c)D_{ac}^i \right)$$

Returning to our first equation, we obtain (simply replacing  $b$  by  $c$  in the second equality):

$$\begin{aligned}
(m(i, a) + 1)D_a^i + m(2, i)D_{a2}^i &= \sum_{b \neq a, i, 2} -m(i, b)D_{ab}^i \\
&= \sum_{c \neq a, i, 2} -m(i, c)D_{ac}^i \\
&= \left( \sum_{c \neq b, i, 2} -m(i, c)D_{ac}^i \right) + m(i, a)D_{aa}^i - m(i, b)D_{ab}^i
\end{aligned}$$

Since  $D_{aa}^i = D_a^i$ , we can cancel and rearrange terms to write:

$$\sum_{c \neq b, i, 2} m(i, c)D_{ac}^i = -D_a^i - m(i, b)D_{ab}^i - m(2, i)D_{a2}^i$$

Therefore:

$$\begin{aligned}
(m(i, a) + 1)D_a^i + m(2, i)D_{a2}^i &= \sum_{b \neq a, i, 2} (m(2, i)D_{a2}^i - D_a^i - m(i, b)D_{ab}^i - m(2, i)D_{a2}^i) \\
&= \sum_{b \neq a, i, 2} (-D_a^i - m(i, b)D_{ab}^i) \\
&= - \left( \sum_{b \neq a, i, 2} D_a^i \right) + (m(i, a) + 1)D_a^i + m(2, i)D_{a2}^i
\end{aligned}$$

Finally, cancelling gives us:

$$\sum_{b \neq a, i, 2} D_a^i = 0$$

We now need to consider the possibilities for the color  $i$ . If  $i = 2$ , then:

$$\sum_{b \neq a, i, 2} D_a^i = \sum_{b \neq a, 2} D_a^i = (k - 2)D_a^i = 0$$

However, if  $i \neq 2$ , then:

$$\sum_{b \neq a, i, 2} D_a^i = (k - 3)D_a^i = 0$$

Since we have already dealt with the cases when  $k \leq 3$  in Lemma 1, Lemma 2 and Lemma 5, we can conclude that  $D_a^i$  is trivial modulo (\*) for every  $i$ . Hence,  $D_a$  is trivial for every  $a$ . Finally, this means that  $(1 + m(D; 1, 2))D = 0$ , so  $D$  will also be trivial modulo (\*). This completes the proof.  $\square$

**Theorem 3.** *If  $D$  has any large components  $C$ , then  $D$  is trivial modulo (\*).*

PROOF: As in Lemma 4, we induct on the number of large components of  $D$ , and the argument for the general case is almost identical to the argument for the base case. We just have to observe that whenever we apply the relation (\*), we can ignore large components of  $D$  (other than the one we're working with), since the diagrams they give rise to have fewer large components than  $D$  does. This completes the induction and the proof.  $\square$  This theorem tells us that the only elements of  $B^{csl}$  which are *not* in the kernel of the relation (\*) are univalent diagrams all of whose components are of degree 1 (i.e. line segments). By the arguments of Lemma 1, we also know that the degree 1 components with both endpoints of the same color are also trivial, so we only need to consider line segments with different colors on the two endpoints. Restricted to the space generated by these elements, (\*) is clearly trivial, so  $B^{cl}$  is in fact simply the polynomial algebra over the reals generated by these univalent diagrams (since (\*) is trivial on this space,  $B^{cl}$  inherits a multiplication from  $B^{csl}$ ). We formalize this as a corollary:

**Corollary 1.**  *$B^{cl}(k)$  (and hence  $A^{cl}(k)$ ) is isomorphic to the algebra  $\mathbf{R}[x_{ij}]$ , where each  $x_{ij}$  is of degree 1, and  $1 \leq i < j \leq k$ .*

It is well-known that these diagrams correspond to the pairwise linking numbers of the components, so we conclude:

**Corollary 2.** *The pairwise linking numbers of the components of a link are the only finite type link concordance invariants of the link.*

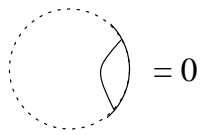
#### 4. ACKNOWLEDGEMENTS

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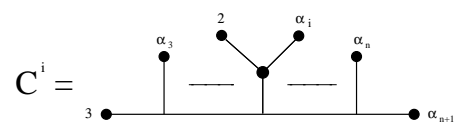
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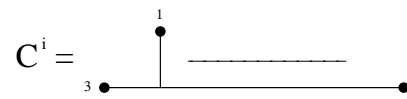
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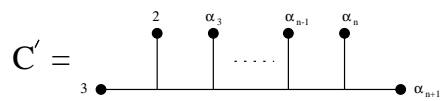
$$\text{Diagram 1} - \text{Diagram 2} + \text{Diagram 3} - \text{Diagram 4} = 0$$

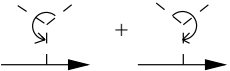

The image shows a mathematical equation involving four diagrams of a sphere with a dashed boundary and three internal lines meeting at a central point. The diagrams are arranged horizontally and separated by minus and plus signs. The first diagram has the lines meeting at the bottom. The second diagram has the lines meeting at the top. The third diagram has the lines meeting at the top, with the left and right lines curved outwards. The fourth diagram has the lines meeting at the bottom, with the left and right lines curved outwards. The equation ends with an equals sign followed by a zero.

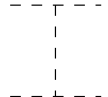
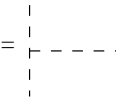





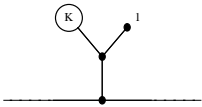


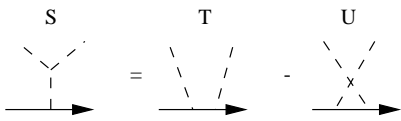


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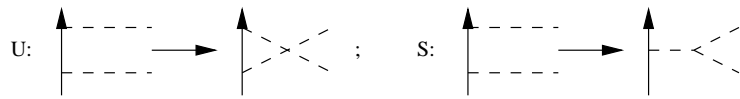
IHX:  =  - 





$$\text{S} = \text{T} - \text{U}$$


The image shows three Feynman diagrams labeled S, T, and U, arranged in a subtraction equation. Diagram S is a tree-level diagram with a horizontal incoming arrow from the left, a vertical line connecting it to a vertex, and two dashed lines branching from that vertex. Diagram T is a tree-level diagram with a horizontal incoming arrow from the left, a dashed line connecting it to a vertex, and two dashed lines branching from that vertex. Diagram U is a tree-level diagram with a horizontal incoming arrow from the left, a dashed line connecting it to a vertex, and two dashed lines branching from that vertex, with an 'x' mark at the vertex. The diagrams are separated by an equals sign and a minus sign.



$$\Delta(\textcircled{\diagup}) = \textcircled{\diagup} \otimes \textcircled{\phantom{\diagup}} + \textcircled{\diagdown} \otimes \textcircled{\diagup} + \textcircled{\diagup} \otimes \textcircled{\diagdown} + \textcircled{\phantom{\diagup}} \otimes \textcircled{\diagup}$$

The image shows a mathematical identity between three Feynman diagrams. The first diagram on the left is a square with two diagonal lines crossing in the center, forming an 'X'. The second diagram in the middle is a square with a diagonal line from the bottom-left to the top-right, and a separate diagonal line from the top-left to the bottom-right. The third diagram on the right is a square with a diagonal line from the top-left to the bottom-right, and a separate diagonal line from the bottom-left to the top-right. An equals sign is placed between the first and second diagrams, and a minus sign is placed between the second and third diagrams.

$$\text{Diagram 1} = \text{Diagram 2} - \text{Diagram 3}$$

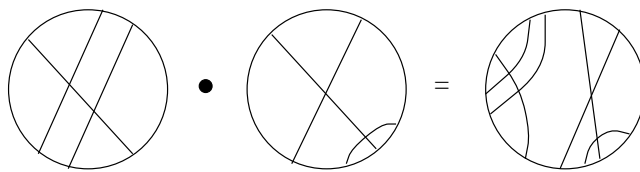


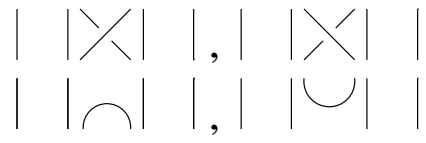
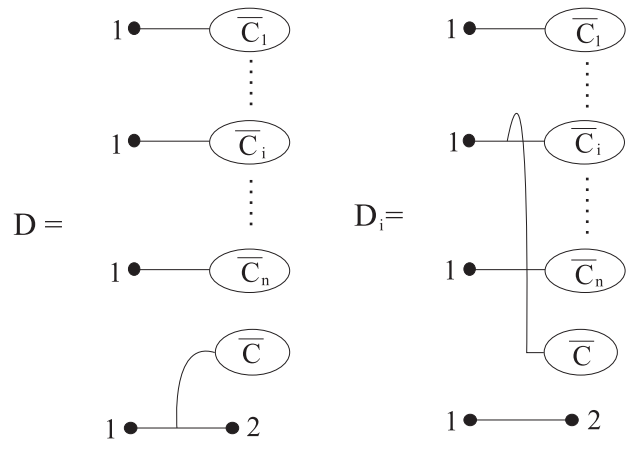


The diagram shows an equation with three terms on the left, each followed by a plus sign, and an equals sign followed by a zero on the right. Each term consists of a vertical line with an upward-pointing arrowhead at the top. The word "red" is written below the line. To the right of the line are four horizontal dashed lines. A solid black dot is placed on the second dashed line from the top. A dashed line extends from the dot to the right, where it meets a vertical line that forms the left side of an oval loop. The top of the loop is also dashed, and the right side is solid. The three terms are identical in structure, but the dot is positioned at different vertical levels relative to the dashed lines in each term: the first term has the dot on the second dashed line, the second term has it on the third, and the third term has it on the fourth. The entire expression is set against a white background.

$$\text{red} + \text{red} + \text{red} = 0$$

$$\begin{array}{c} \text{red} \\ \text{red} \\ \text{red} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \bullet \\ | \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \left( \right) + \begin{array}{c} \text{red} \\ \text{red} \\ \text{red} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array} \left( \right) + \begin{array}{c} \text{red} \\ \text{red} \\ \text{red} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \bullet \end{array} \left( \right) = 0 \quad (*)$$





$$\Gamma = \bullet \text{---} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \text{---} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \text{---} \dots \text{---} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \text{---} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \text{---} \bullet$$

