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Finite Type Link Homotopy Invariants

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FINITE TYPE LINK HOMOTOPY INVARIANTS

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ABSTRACT

In [2], Bar-Natan used unitrivalent diagrams to show that finite type invariants classify string links up to homotopy. In this paper, I will construct the correct spaces of chord diagrams and unitrivalent diagrams for links up to homotopy. I will use these spaces to show that, far from classifying links up to homotopy, the only rational finite type invariants of link homotopy are the linking numbers of the components. *Keywords:* Finite type invariants; link homotopy.

CONTENTS

1. Introduction	1
1.1. Singular Knots and Chord Diagrams	1
1.2. Unitrivalent Diagrams	4
2. String Links, Links and Homotopy	5
2.1. String Links	5
2.2. Links	5
2.3. Link Homotopy	6
3. The Size of B^{hl}	7
3.1. Base cases	7
3.2. Main Theorem	10
4. Acknowledgements	15
References	15

1. INTRODUCTION

We will begin with a brief overview of finite type invariants. In 1990, V.A. Vassiliev introduced the idea of *Vassiliev* or *finite type* knot invariants, by looking at certain groups associated with the cohomology of the space of knots. Shortly thereafter, Birman and Lin [4] gave a combinatorial description of finite type invariants. We will give a brief overview of this combinatorial theory. For more details, see Bar-Natan [1].

1.1. Singular Knots and Chord Diagrams. Recall that, in the most general sense, a *knot invariant* is a map from the set of equivalence classes of knots under isotopy to another set G . We will need to have some additional structure on G .

For our purposes, G will always be at least an associative, commutative ring with an identity. We first note that we can extend any knot invariant to an invariant of *singular* knots, where a singular knot is an immersion of S^1 in 3-space which is an embedding except for a finite number of isolated double points. Given a knot invariant v , we extend it via the relation:

An invariant v of singular knots is then said to be of *finite type*, if there is an integer d such that v is zero on any knot with more than d double points. v is then said to be of *type d* . We denote by $V_d(G)$ the G -module generated by G -valued finite type invariants of type d . We can completely understand the space of G -valued finite type invariants by understanding all of the G -modules $V_d(G)/V_{d-1}(G)$. An element of this module is completely determined by its behavior on knots with exactly d singular points. Since such an element is zero on knots with more than d singular points, any other (non-singular) crossing of the knot can be changed without affecting the value of the invariant. This means that elements of V_d/V_{d-1} can be viewed as functions on the space of *chord diagrams*:

Definition 1. A **chord diagram of degree d** is an oriented circle, together with d chords of the circles, such that all of the $2d$ endpoints of the chords are distinct. The circle represents a knot, the endpoints of a chord represent 2 points identified by the immersion of this knot into 3-space.

Functions on the space of chord diagrams which are derived from knot invariants will satisfy certain relations. This leads us to the definition of a *weight system*:

Definition 2. A **G -valued weight system of degree d** is a G -valued function W on the space of chord diagrams of degree d which satisfies 2 relations:

- (1-term relation)

- (4-term relation)

Outside of the solid arcs on the circle, the diagrams can be anything, as long as it is the same for all four diagrams.

We let $W_d(G)$ denote the space of G -valued weight systems of degree d .

From now on, we will be considering the case when $G = \mathbb{R}$. We will simplify our notation by letting $V_d = V_d(\mathbb{R})$ and $W_d = W_d(\mathbb{R})$. Bar-Natan [1] defines maps $w_d : V_d \rightarrow W_d$ and $v_d : W_d \rightarrow V_d$. w_d is defined by *embedding* a chord diagram D in \mathbf{R}^3 as a singular knot K_D , with the chords corresponding to singularities of the embedding

(so there are d singularities). Any two such embeddings will differ by crossing changes, but these changes will not effect the value of a type d Vassiliev invariant on the singular knot. Then, for any $\gamma \in V_d$, we define $w_d(\gamma)(D) = \gamma(K_D)$. Bar-Natan shows that this is, in fact, a weight system. The 1-term relation is satisfied because of the first Reidemeister move, and the 4-term relation is essentially the result of rotating a third strand a full turn around a double point. Note that this argument will work for any ring G , not just $G = \mathbb{R}$. v_d is much more complicated to define, using the Kontsevich integral. This does require that we consider real-valued invariants; it is still an open question whether an analogous procedure can be found for other rings (or even finite fields). For a full treatment of the Kontsevich integral, see Bar-Natan [1] and Le and Murakami [8]. We will simply mention the few facts and properties we need, primarily following Le and Murakami. Using a Morse function, any knot (or link or string link) can be decomposed into elementary *tangles*:

Le and Murakami define a map Z from an elementary tangle with k strands to the space of chord diagrams on k strands. This map respects composition of tangles: if $T_1 \cdot T_2$ is the tangle obtained by placing T_1 on top of T_2 , then $Z(T_1 \cdot T_2) = Z(T_1)Z(T_2)$. Le and Murakami prove that this map gives a real-valued isotopy invariant of knots and links. Given a degree d weight system W , and a knot K , we now define $v_d(W)(K) = W(Z(K))$. Bar Natan [1] shows that w_d and v_d are “almost” inverses. More precisely, $w_d(v_d(W)) = W$ and $v_d(w_d(\gamma)) - \gamma \in V_{d-1}$. As a result, (see [4, 1, 10]) the space W_d of weight systems of degree d is isomorphic to V_d/V_{d-1} . For convenience, we will usually take the dual approach, and simply study the real vector space of chord diagrams of degree d modulo the 1-term and 4-term relations. The dimensions of these spaces have been computed for $d \leq 12$ (see Bar-Natan [1] and Kneissler [7]). It is useful to combine all of these spaces into a graded module via direct sum. We can give this module a Hopf algebra structure by defining an appropriate product and co-product:

- We define the product $D_1 \cdot D_2$ of two chord diagrams D_1 and D_2 as their connect sum. This is well-defined modulo the 4-term relation (see [1]).

- We define the co-product $\Delta(D)$ of a chord diagram D as follows:

$$\Delta(D) = \sum_J D'_J \otimes D''_J$$

FIGURE 1. STU relation

FIGURE 2. AS and IHX relations

where J is a subset of the set of chords of D , D'_J is D with all the chords in J removed, and D''_J is D with all the chords not in J removed.

It is easy to check the compatibility condition $\Delta(D_1 \cdot D_2) = \Delta(D_1) \cdot \Delta(D_2)$.

1.2. Univalent Diagrams. It is often useful to consider the Hopf algebra of bounded univalent diagrams, rather than chord diagrams. These diagrams, introduced by Bar-Natan [1] (there called *Chinese Character Diagrams*), can be thought of as a shorthand for writing certain linear combinations of chord diagrams. We define a *bounded univalent diagram* to be a univalent graph, with oriented vertices, together with a bounding circle to which all the univalent vertices are attached. We also require that each component of the graph have at least one univalent vertex (so every component is connected to the boundary circle). We define the space A of bounded univalent diagrams as the quotient of the space of all bounded univalent graphs by the *STU* relation, shown in Figure 1. As consequences of *STU* relation, the anti-symmetry (*AS*) and *IHX* relations, see Figure 2, also hold in A . Bar-Natan shows that A is isomorphic to the algebra of chord diagrams. We can get an algebra B of *univalent diagrams* by simply removing the bounding circle from the diagrams in A , leaving graphs with trivalent and univalent vertices, modulo the *AS* and *IHX* relations. Bar-Natan shows that the spaces A and B are isomorphic. The map χ from B to A takes a diagram to the linear combination of all ways of attaching the univalent vertices to a bounding circle, divided by total number of such ways (T. Le noticed that this factor, missing in [1], is necessary to preserve the comultiplicative structure of the algebras). The inverse map σ turns a diagram into a linear combination of diagrams by performing sequences of “basic operations,” and then removes the bounding circle. The two

basic operations are:

2. STRING LINKS, LINKS AND HOMOTOPY

2.1. String Links. Bar-Natan [2] extended the theory of finite type invariants to string links.

Definition 3. (see Habegger and Lin [5]) Let D be the unit disk in the plane and let $I = [0,1]$ be the unit interval. Choose k points p_1, \dots, p_k in the interior of D , aligned in order along the x -axis. A **string link** σ of k components is a smooth proper imbedding of k disjoint copies of I into $D \times I$:

$$\sigma : \bigsqcup_{i=1}^k I_i \rightarrow D \times I$$

such that $\sigma|_{I_i}(0) = p_i \times 0$ and $\sigma|_{I_i}(1) = p_i \times 1$. The image of I_i is called the i th string of the string link σ .

Essentially, everything works the same way for string links as for knots. The bounding circle of the bounded univalent diagrams now becomes a set of bounding line segments, each labeled with a color, to give an algebra A^{sl} (the multiplication is given by placing one diagram on top of another). The univalent diagrams are unchanged, except that each univalent vertex is also labeled with a color to give the space B^{sl} . The isomorphisms χ and σ between A and B easily extend to isomorphisms χ^{sl} and σ^{sl} between A^{sl} and B^{sl} , just working with each color separately. In addition, there are obvious maps w_d^{sl} and v_d^{sl} analogous to w_d and v_d (we just need to keep track of colors).

2.2. Links. The obvious definition of chord diagrams for links is simply to replace the bounding line segments with bounding circles. However, these diagrams are difficult to work with, and it is unclear how to define the univalent diagrams. Unlike for a knot, closing up the components of a string link of several components is not a trivial operation, so we need to impose some relations on the space of univalent diagrams. Since we understand the spaces of chord diagrams and univalent diagrams for string links, it would be useful to be able to express these spaces for links as quotients of the spaces for string links. The question is then, what relations do we need? One relation is fairly obvious. When we construct the space A^l of bounded univalent diagrams for links, we replace the bounding line segments of A^{sl} with directed circles. Bar-Natan et. al. observed (see Theorem 3, [3]) that this is exactly equivalent to saying that the “top” edge incident to one of the line segments can be brought around the circle to be on the “bottom.” So we can write A^l as the quotient of A^{sl} by relation (1), shown in Figure 3 (where the figure shows *all* the chords with endpoints on the red component). Then the Kontsevich integral for links, Z^l , is defined by cutting the link to make a string link, applying the Kontsevich integral for string links, and then taking the quotient by relation (1). Now w_d^l and v_d^l are defined similarly to w_d and v_d . Given a link invariant γ of type d and a diagram D of degree d in A^l , $w_d^l(\gamma)(D) = \gamma(L_{\hat{D}})$, where \hat{D} is the closure of the diagram D (i.e. the bounding line segments are closed to

FIGURE 3. The link relation for chord diagrams

FIGURE 4. The link relation for Bounded Unitrivalent Diagrams

FIGURE 5. The link relation for unitrivalent diagrams

circles). $L_{\hat{D}}$ is well-defined by Theorem 3 of [3]. Defining v_d^l is even easier, now that we have Z^l . Given a weight system (element of the graded dual of A^l) W and a link L , we define $v_d^l(W)(L) = W(Z^l(L))$. One advantage of this formulation of A^l is that it enables us to define the space B^l of unitrivalent diagrams as a quotient of the already known space B^{sl} . This was done by Bar-Natan et. al. Using the *STU* relation, we can rewrite relation (1) as in Figure 4. This suggests how we should define the space B^l . We will take the quotient of B^{sl} by the relations (*) shown in Figure 5, where the univalent vertices shown are *all* the univalent vertices of a given color. With these definitions, Bar-Natan et. al. proved that A^l and B^l are isomorphic:

Theorem 1. (Theorem 3, [3]) *The isomorphism between A^{sl} and B^{sl} descends to an isomorphism between A^l and B^l .*

2.3. Link Homotopy. The idea of *link homotopy* (or just *homotopy*) was introduced by Milnor [9]. Two links are homotopic if one can be transformed into the other through a sequence of ambient isotopies of S^3 and crossing changes of a component with itself (but *not* crossing changes of different components). The definition for string links is similar. Habegger and Lin [5] succeeded in classifying string links and links up to homotopy. We want to extend the results of the last section to string links and links considered up to homotopy. For string links, this has already been done by Bar-Natan [2]. Bar-Natan describes the algebras A^{hsl} and B^{hsl} of bounded and unbounded unitrivalent diagrams for string links up to homotopy. In

brief, we take the quotient of A^{sl} (resp. B^{sl}) by the space of *boring* diagrams. A diagram is *boring* if it has (1) two univalent vertices on the same component (resp. assigned the same color), or (2) non-trivial first homology. In other words, we are left with tree diagrams with no more than one univalent vertex on each component (resp. of each color). Bar-Natan then defines w_d^{hsl} and v_d^{hsl} in the usual way, and shows that they are “almost” inverses in the same sense that w_d and v_d are. All of this extends to links just as it did for isotopy. We define A^{hl} as the quotient of A^{hsl} by relation (1), and B^{hl} as the quotient of B^{hsl} by relation (*). We then define Z^{hl} , w_d^{hl} , and v_d^{hl} just as we did for links up to homotopy. Finally, the arguments of Bar-Natan et. al. carry through to show:

Theorem 2. (*Theorem 3, [3]*) *The isomorphism between A^{hsl} and B^{hsl} descends to an isomorphism between A^{hl} and B^{hl} .*

Remark: By results of Habegger and Masbaum (see Remark 2.1 of [6]), Z^{hl} is the *universal* finite type invariant of link homotopy. By this we mean that it dominates all other such invariants.

3. THE SIZE OF B^{hl}

Now that we have properly defined the space B^{hl} of unitrivalent diagrams for link homotopy, we want to analyze it more closely. We will consider the case when B^{hl} is a vector space over the reals (or, more generally, a module over a ring of characteristic 0). In particular, we would like to know exactly which diagrams of B^{hsl} are in the kernel of the relation (*) (i.e. are 0 modulo (*)). We will find that the answer is “almost everything” - to be precise, any unitrivalent diagram with a component of degree 2 or more. We will start by proving a couple of base cases, and then prove the rest of the theorem by induction. Let $B^{hsl}(k)$ denote the space of unitrivalent diagrams for string link homotopy with k possible colors for the univalent vertices (i.e. we are looking at links with k components). Consider a diagram $D \in B^{hsl}(k)$. Recall from the previous sections that each component of D is a tree diagram with at most one endpoint of each color. Since a unitrivalent tree with n endpoints has $2n - 2$ vertices, and hence degree $n - 1$, D cannot have any components of degree greater than $k - 1$. **Notation:** Before we continue, we will introduce a bit of notation which will be useful in this section. Given a unitrivalent diagram D , we define $m(D; i, j)$ to be the number of components of D which are simply line segments with ends colored i and j , as shown below:

$$i - - - - - j$$

3.1. Base cases.

Lemma 1. *If D has a component C of degree $k-1$ (with $k \geq 3$), then D is trivial modulo (*).*

PROOF: C has one endpoint of each color $1, 2, \dots, k$. Without loss of generality, we may assume that C has a branch as shown, where \bar{C} denotes the remainder of C :

$$C : \begin{array}{c} \bar{C} \\ | \\ 1 - - - - - 2 \end{array}$$

FIGURE 6. Diagrams arising from relation (*)

We are going to apply (*) with the color 1. Let $\{C_1, \dots, C_n\}$ be the components of D with an endpoint colored 1. So, ignoring the other components of D , we have the diagrams of Figure 6 (where \bar{C}_i denotes all of C_i except for the endpoint colored 1). (*) then implies that $D + \sum D_i = 0$. If C_i is just a line segment with endpoints colored 1 and 2, then $D_i = D$. Otherwise, D_i will have a *boring* component (since \bar{C}_i will have an endpoint of some color $j \in 3, \dots, k$, and \bar{C} has an endpoint of each color $3, \dots, k$, including j , D_i will have a component with two endpoints colored j), and hence be trivial in B^{hsl} . Therefore, we find that $D + m(D; 1, 2)D = 0$ where $m(D; 1, 2) \geq 0$. We can divide both sides by $1 + m(D; 1, 2)$ (since we are working over the reals, which have characteristic 0) to conclude that $D = 0$. \square

Lemma 2. *If D has a component C of degree $k-2$ (with $k \geq 4$), then D is trivial modulo (*).*

PROOF: Without loss of generality, C has endpoints colored $1, 2, \dots, k-1$. We will prove the lemma by inducting on $m(D; 1, k)$; inducting among the set of diagrams having a component with endpoints colored $1, 2, \dots, k-1$. As in the previous lemma, we may assume that C has a branch as shown:

$$C : \begin{array}{c} \bar{C} \\ | \\ 1 \text{ --- } 2 \end{array}$$

And conclude that $D + \sum D_i = 0$, where the D_i are defined as before. Since \bar{C} contains endpoints of all colors except 1, 2, and k , D_i is boring unless C_i has one of the following 3 forms (as in Lemma 1):

$$(1) C_i = 1 \text{ --- } 2$$

$$(2) C_i = 1 \text{ --- } k$$

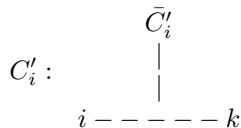
$$(3) C_i = \begin{array}{c} k \\ | \\ 1 \text{ --- } 2 \end{array}$$

FIGURE 7. Using the IHX relation to decompose a diagram

In the first case, $D_i = D$; and in the second case, $D_i = D'$, where D' is the same as D except that:

- C is replaced by a component C' identical to it except that the endpoint colored 2 in C is colored k in C' (so $\bar{C}' = \bar{C}$).
- A line segment with endpoints colored 1 and k has been replaced by a line segment with endpoints colored 1 and 2. In other words, $m(D'; 1, 2) = m(D; 1, 2) + 1$ and $m(D'; 1, k) = m(D; 1, k) - 1$.

In the third case, D_i has a component of degree $k - 1$, and so is 0 modulo (*) by the previous lemma. Therefore, as in the previous lemma, we find that $D + m(D; 1, 2)D + m(D; 1, k)D' = 0$. If $m(D; 1, k) = 0$ we conclude, as before, that D is trivial modulo (*), which proves the base case of our induction. For the inductive step, we use the IHX relation on C' to decompose $D' = \sum_{i \neq 1, 2, k} \pm D'_i$, where D'_i is the same as D' except that C' has been replaced by a component C'_i with endpoints of the same colors (although arranged differently), and a branch as shown:



(The decomposition is simply a matter of letting the endpoint colored k “travel” the tree - see Figure 7 for an example.) In particular, $m(D'_i; a, b) = m(D'; a, b)$ for all colors a and b . We now apply (*) to D'_i using color i (and component C'_i), similarly to what we’ve done before. In this case, the only other components which matter (modulo boring diagrams) are ones which look like one of the following:

- (1) $i \text{ --- } k$
- (2) $i \text{ --- } 2$

$$(3) \quad \begin{array}{c} 2 \\ | \\ i - - - - k \end{array}$$

As before, the first case gives D'_i again, the third case is trivial by Lemma 1, and the second case gives a diagram D''_i such that:

- C'_i is replaced by a component C''_i identical to it except that the endpoint colored k in C'_i is colored 2 in C''_i (so $\bar{C}''_i = \bar{C}'_i$).
- A line segment with endpoints colored i and 2 has been replaced by a line segment with endpoints colored i and k . In other words, $m(D''_i; i, k) = m(D'_i; i, k) + 1$ and $m(D''_i; i, 2) = m(D'_i; i, 2) - 1$.

Otherwise, D''_i is the same as D'_i ; in particular, $m(D''_i; 1, k) = m(D'_i; 1, k) = m(D'; 1, k) = m(D; 1, k) - 1$. Then (*) tells us that $D'_i + m(D'; i, k)D''_i + m(D'; 2, i)D''_i = 0$. Since D''_i has a component of degree $k - 2$ with endpoints colored $1, \dots, k - 1$ (namely, C''_i), the inductive hypothesis implies that D''_i is trivial modulo (*). Therefore, $(1 + m(D'; i, k))D'_i = 0$ modulo (*), so D'_i is trivial modulo (*). This is true for every i , so it immediately follows that D' , and hence D , are also trivial modulo (*). \square

3.2. Main Theorem. The lemmas of Section 3.1 will act as base cases for the main theorem of this section:

Theorem 3. *If D has a component C of degree 2 or higher, then D is trivial modulo (*).*

PROOF: The method of proof for this theorem is essentially the same as that for Lemma 2. We will successively apply (*) (and do a single expansion via IHX) until we obtain a set of diagrams which are all either trivial or repetitions of earlier diagrams. We can then backtrack to show that everything disappears. However, rather than applying (*) twice, as in Lemma 2, we will need to apply it four times. This unfortunately makes keeping track of the diagrams somewhat confusing - we have done our best. Again, the proof is by induction; in this case, it is a nested double induction. The outer induction is backwards, on the degree of the largest component of D . The base cases of this induction are given by Lemma 1 and Lemma 2. So we assume that a diagram is trivial modulo (*) if it has a component of degree $\geq n + 1$, and let D be a diagram whose largest component C has degree n . (Of course, $n \geq 2$, and by Lemmas 1 and 2 we can assume $k > n + 2$.) Without loss of generality, C has endpoints colored $1, 2, \dots, n + 1$. The inner induction, which is the rest of the proof, is on $\sum_{a=n+2}^k m(D; 1, a)$, inducting among diagrams with a component with endpoints colored $1, 2, \dots, n + 1$. Without loss of generality, as before, we can assume that C has a branch as shown:

$$C : \quad \begin{array}{c} \bar{C} \\ | \\ 1 - - - - 2 \end{array}$$

We apply (*) using the color 1 and find, after removing boring diagrams and those which are trivial by the first inductive hypothesis, that $D + m(D; 1, 2)D + \sum_{a=n+2}^k m(D; 1, a)D_a = 0$, where D_a is the same as D except that:

- C has been replaced by a component C_a identical to it except that the endpoint colored 2 is now colored a (so $\bar{C}_a = \bar{C}$)
- A line segment with endpoints colored 1 and a has been replaced by a line segment with endpoints colored 1 and 2. In other words, $m(D_a; 1, a) = m(D; 1, a) - 1$ and $m(D_a; 1, 2) = m(D; 1, 2) + 1$.

We will denote this as shown below:

$$D_a : \begin{array}{c} \bar{C} \\ | \\ 1 \text{ --- } a \end{array} \quad (1, a) \rightarrow (1, 2)$$

Notice that if $\sum_{a=n+2}^k m(D; 1, a) = 0$, then $m(D; 1, a) = 0$ for each a , since these are all non-negative integers. In this case, $D + m(D; 1, 2)D = 0$, and hence $D = 0$. This proves the base case of the second induction. As in Lemma 2, we use the IHX relation to decompose $D_a = \sum_{i=3}^{n+1} \pm D_a^i$, where the analogue C_a^i of C_a in D_a^i has a branch as shown, and the other components of the diagram are the same as D_a :

$$D_a^i : \begin{array}{c} \bar{C}_a^i \\ | \\ i \text{ --- } a \end{array} \quad (1, a) \rightarrow (1, 2)$$

Note that, aside from having endpoints of the same colors, C_a^i looks nothing like C_a . \bar{C}_a^i has endpoints colored $1, 3, 4, \dots, i-1, i+1, \dots, n+1$. We will keep this in mind. Now we apply (*) to D_a^i , using the color i . In the pictures we use to describe the various diagrams that we produce in what follows, we will just be showing how the diagrams differ from D_a^i . This will involve showing how C_a^i has been altered, and which line segments have been added or removed. At each stage, we will eliminate without comment those diagrams which are either boring or are trivial modulo (*) by our first inductive hypothesis (i.e. have components of degree greater than n). We obtain the relation:

$$D_a^i + m(D_a^i; i, a)D_a^i + m(D_a^i; i, 2)D_{a2}^i + \sum_{\substack{n+2 \leq b \leq k \\ b \neq a}} m(D_a^i; i, b)D_{ab}^i = 0$$

where:

$$D_{a2}^i : \begin{array}{c} \bar{C}_a^i \\ | \\ i \text{ --- } 2 \end{array} \quad (i, 2) \rightarrow (i, a)$$

$$D_{ab}^i : \begin{array}{c} \bar{C}_a^i \\ | \\ i \text{ --- } b \end{array} \quad (i, b) \rightarrow (i, a)$$

Notice that D_{a2}^i has a component of degree n with endpoints colored $1, 2, \dots, n+1$, and $m(D_{a2}^i; 1, a) = m(D_a; 1, a) = m(D; 1, a) - 1$ (since $i \neq 1$), so D_{a2}^i is trivial by the second inductive hypothesis. So we can rewrite the relation as:

$$(1 + m(D_a^i; i, a)D_a^i) + \sum_{\substack{n+2 \leq b \leq k \\ b \neq a}} m(D_a^i; i, b)D_{ab}^i = 0$$

Next we apply (*) to D_{ab}^i , using the color b , and find that:

$$D_{ab}^i + m(D_{ab}^i; i, b)D_{ab}^i + m(D_{ab}^i; 2, b)D_{ab2}^i + \sum_{\substack{n+2 \leq c \leq k \\ c \neq b}} m(D_{ab}^i; b, c)D_{abc}^i = 0$$

where:

$$D_{ab2}^i : \begin{array}{c} \bar{C}_a^i \\ | \\ 2 \text{ --- } b \end{array} \quad \begin{array}{l} (i, b) \rightarrow (i, a) \\ (2, b) \rightarrow (i, b) \end{array} \Rightarrow (2, b) \rightarrow (i, a)$$

$$D_{abc}^i : \begin{array}{c} \bar{C}_a^i \\ | \\ c \text{ --- } b \end{array} \quad \begin{array}{l} (i, b) \rightarrow (i, a) \\ (c, b) \rightarrow (i, b) \end{array} \Rightarrow (c, b) \rightarrow (i, a)$$

Now we apply (*) to D_{ab2}^i using the color 2, and to D_{abc}^i , using the color c . We get two relations:

$$D_{ab2}^i + m(D_{ab2}^i; 2, b)D_{ab2}^i + m(D_{ab2}^i; 2, i)D_{ab2i}^i + \sum_{\substack{n+2 \leq c \leq k \\ c \neq b}} m(D_{ab2}^i; 2, c)D_{ab2c}^i = 0$$

$$D_{abc}^i + m(D_{abc}^i; b, c)D_{abc}^i + m(D_{abc}^i; 2, c)D_{abc2}^i + m(D_{abc}^i; i, c)D_{abci}^i + \sum_{\substack{n+2 \leq d \leq k \\ d \neq b, c}} m(D_{abc}^i; c, d)D_{abcd}^i = 0$$

where:

$$D_{ab2i}^i : \begin{array}{c} \bar{C}_a^i \\ | \\ 2 \text{ --- } i \end{array} \quad \begin{array}{l} (2, b) \rightarrow (i, a) \\ (2, i) \rightarrow (2, b) \end{array} \Rightarrow (2, i) \rightarrow (i, a)$$

$$D_{ab2c}^i : \begin{array}{c} \bar{C}_a^i \\ | \\ 2 \text{ --- } c \end{array} \quad \begin{array}{l} (2, b) \rightarrow (i, a) \\ (2, c) \rightarrow (2, b) \end{array} \Rightarrow (2, c) \rightarrow (i, a)$$

$$D_{abc2}^i : \begin{array}{c} \bar{C}_a^i \\ | \\ c \text{ --- } 2 \end{array} \quad \begin{array}{l} (c, b) \rightarrow (i, a) \\ (2, c) \rightarrow (c, b) \end{array} \Rightarrow (2, c) \rightarrow (i, a)$$

$$D_{abci}^i : \begin{array}{c} \bar{C}_a^i \\ | \\ c \text{ --- } i \end{array} \quad \begin{array}{l} (c, b) \rightarrow (i, a) \\ (i, c) \rightarrow (c, b) \end{array} \Rightarrow (i, c) \rightarrow (i, a)$$

$$D_{abcd}^i : \begin{array}{c} \bar{C}_a^i \\ | \\ c \text{ --- } d \end{array} \quad \begin{array}{l} (c, b) \rightarrow (i, a) \\ (c, d) \rightarrow (c, b) \end{array} \Rightarrow (c, d) \rightarrow (i, a)$$

We make several observations, by antisymmetry:

- $D_{ab2i}^i = -D_{a2}^i = 0$.
- $D_{ab2c}^i = -D_{abc2}^i$.

- $D_{abc i}^i = -D_{ac}^i$.
- $D_{abcd}^i = D_{adc}^i = -D_{acd}^i$.

Now that we have these recursive relations, we can plug them into our various equations. We will use the following equalities:

$$\begin{aligned} m(D_{ab}^i; i, b) + 1 &= m(D_a^i; i, b) \\ m(D_{ab2}^i; 2, b) + 1 &= m(D_{ab}^i; 2, b) = m(D_a^i; 2, b) \\ m(D_{abc}^i; b, c) + 1 &= m(D_a^i; b, c) \end{aligned}$$

And for all the other coefficients we have:

$$m(D_*^i; x, y) = m(D_a^i; x, y)$$

For convenience, we will write $m(x, y) = m(D_a^i; x, y)$ in what follows:

$$\begin{aligned} (m(i, a) + 1)D_a^i &= \sum_{\substack{n+2 \leq b \leq k \\ b \neq a}} -m(i, b)D_{ab}^i \\ &= \sum_{\substack{n+2 \leq b \leq k \\ b \neq a}} -(m(D_{ab}^i; i, b) + 1)D_{ab}^i \\ &= \sum_{\substack{n+2 \leq b \leq k \\ b \neq a}} \left(m(2, b)D_{ab2}^i + \sum_{\substack{n+2 \leq c \leq k \\ c \neq b}} m(b, c)D_{abc}^i \right) \end{aligned}$$

Note that:

$$\begin{aligned} m(2, b)D_{ab2}^i &= (m(D_{ab2}^i; 2, b) + 1)D_{ab2}^i \\ &= \sum_{\substack{n+2 \leq c \leq k \\ c \neq b}} -m(2, c)D_{ab2c}^i \\ m(b, c)D_{abc}^i &= (m(D_{abc}^i; b, c) + 1)D_{abc}^i \\ &= -m(2, c)D_{abc2}^i - m(i, c)D_{abci}^i - \sum_{\substack{n+2 \leq d \leq k \\ d \neq c, b}} m(c, d)D_{abcd}^i \end{aligned}$$

Therefore:

$$\begin{aligned} &m(2, b)D_{ab2}^i + \sum_{\substack{n+2 \leq c \leq k \\ c \neq b}} m(b, c)D_{abc}^i = \\ &\sum_{\substack{n+2 \leq c \leq k \\ c \neq b}} \left(-m(2, c)(D_{ab2c}^i + D_{abc2}^i) - m(i, c)D_{abci}^i - \sum_{\substack{n+2 \leq d \leq k \\ d \neq c, b}} m(c, d)D_{abcd}^i \right) = \\ &\sum_{\substack{n+2 \leq c \leq k \\ c \neq b}} \left(m(i, c)D_{ac}^i + \sum_{\substack{n+2 \leq d \leq k \\ d \neq c, b}} m(c, d)D_{acd}^i \right) \end{aligned}$$

We plug this back in above to find:

$$(m(i, a) + 1)D_a^i = \sum_{\substack{n+2 \leq b \leq k \\ b \neq a}} \sum_{\substack{n+2 \leq c \leq k \\ c \neq b}} \left(m(i, c)D_{ac}^i + \sum_{\substack{n+2 \leq d \leq k \\ d \neq c, b}} m(c, d)D_{acd}^i \right)$$

We notice that:

$$\begin{aligned} \sum_{\substack{n+2 \leq c \leq k \\ c \neq b}} \sum_{\substack{n+2 \leq d \leq k \\ d \neq c, b}} m(c, d)D_{acd}^i &= \frac{1}{2} \sum_{\substack{n+2 \leq c, d \leq k \\ c \neq d \\ c, d \neq b}} m(c, d)(D_{acd}^i + D_{adc}^i) \\ &= \frac{1}{2} \sum_{\substack{n+2 \leq c, d \leq k \\ c \neq d \\ c, d \neq b}} m(c, d)(D_{acd}^i - D_{adc}^i) \\ &= 0 \end{aligned}$$

Therefore:

$$(m(i, a) + 1)D_a^i = \sum_{\substack{n+2 \leq b \leq k \\ b \neq a}} \sum_{\substack{n+2 \leq c \leq k \\ c \neq b}} m(i, c)D_{ac}^i$$

Returning to our first equation above, we have (simply replacing b by c in the second equality):

$$\begin{aligned} (m(i, a) + 1)D_a^i &= \sum_{\substack{n+2 \leq b \leq k \\ b \neq a}} -m(i, b)D_{ab}^i \\ &= \sum_{\substack{n+2 \leq c \leq k \\ c \neq a}} -m(i, c)D_{ac}^i \\ &= \left(\sum_{\substack{n+2 \leq c \leq k \\ c \neq b}} -m(i, c)D_{ac}^i \right) + m(i, a)D_{aa}^i - m(i, b)D_{ab}^i \end{aligned}$$

Since $D_{aa}^i = D_a^i$, we can cancel and rearrange terms to write:

$$\sum_{\substack{n+2 \leq c \leq k \\ c \neq b}} m(i, c)D_{ac}^i = -D_a^i - m(i, b)D_{ab}^i$$

Therefore:

$$\begin{aligned} (m(i, a) + 1)D_a^i &= \sum_{\substack{n+2 \leq b \leq k \\ b \neq a}} -D_a^i - m(i, b)D_{ab}^i \\ &= -(k - n - 2)D_a^i + (m(i, a) + 1)D_a^i \end{aligned}$$

$$\Rightarrow (k - n - 2)D_a^i = 0$$

Since $n < k - 2$ (the cases when $n = k - 1, k - 2$ were dealt with in the lemmas), $k - n - 2 \neq 0$; so we can conclude that D_a^i is trivial modulo (*). Hence, D_a and,

ultimately, D are also trivial modulo $(*)$. This completes the induction and the proof. \square This theorem tells us that the only elements of B^{hsl} which are *not* in the kernel of the relation $(*)$ are univalent diagrams all of whose components are of degree 1 (i.e. line segments). Restricted to the space generated by these elements, $(*)$ is clearly trivial, so B^{hl} is in fact simply the polynomial algebra over the reals generated by these univalent diagrams (since $(*)$ is trivial on this space, B^{hl} inherits a multiplication from B^{hsl}). We formalize this as a corollary:

Corollary 1. $B^{hl}(k)$ (and hence $A^{hl}(k)$) is isomorphic to the algebra $\mathbb{R}[x_{ij}]$, where each x_{ij} is of degree 1, and $1 \leq i < j \leq k$.

It is well-known that the only finite-type link homotopy invariants of degree 1 are the pairwise linking numbers of the components, so we conclude:

Corollary 2. The pairwise linking numbers of the components of a link are the only real-valued finite type link homotopy invariants of the link.

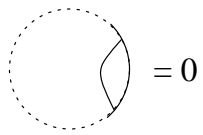
Remark: Bar-Natan [2] has shown that the Milnor μ -invariants are finite type homotopy invariants for *string* links. However, the analogous $\bar{\mu}$ -invariants for links have indeterminacies arising from the fact that many string links can close up to give the same link (up to homotopy). As a result, these invariants are only well-defined modulo the values of lower-order μ -invariants. This keeps us from being able to extend the invariants to singular links, since two links which differ by a crossing change may have entirely different lower-order invariants, and so their $\bar{\mu}$ -invariants may have values lying in completely different groups. So there is no way to interpret these invariants as finite type invariants in the usual way.

4. ACKNOWLEDGEMENTS

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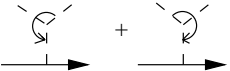
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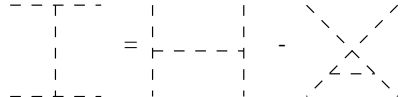
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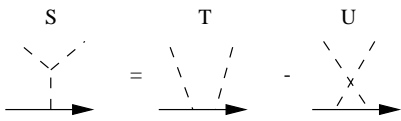


$$\text{Diagram 1} - \text{Diagram 2} + \text{Diagram 3} - \text{Diagram 4} = 0$$

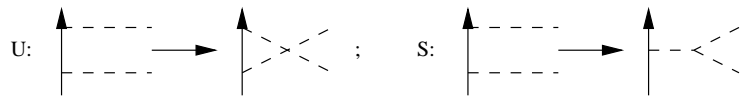
The image shows a mathematical equation involving four diagrams of a sphere with a dashed boundary and three internal lines meeting at a central point. The diagrams are arranged horizontally and separated by minus and plus signs. The first diagram has the lines meeting at the bottom. The second diagram has the lines meeting at the top. The third diagram has the lines meeting at the top, with the left and right lines curved outwards. The fourth diagram has the lines meeting at the bottom, with the left and right lines curved outwards. The entire expression is followed by an equals sign and the number 0.

AS:  = 0

IHX: 

$$\text{S} = \text{T} - \text{U}$$


The image shows three Feynman diagrams labeled S, T, and U, arranged in a subtraction equation. Diagram S is a tree-level diagram with an incoming horizontal arrow from the left, a vertical line connecting it to a vertex, and two outgoing dashed lines from that vertex. Diagram T is a tree-level diagram with an incoming horizontal arrow from the left, a vertex, and two outgoing dashed lines from that vertex. Diagram U is a tree-level diagram with an incoming horizontal arrow from the left, a vertex, and two outgoing dashed lines from that vertex, with an 'x' mark at the vertex. The diagrams are connected by an equals sign and a minus sign.



$$\Delta(\textcircled{\diagup}) = \textcircled{\diagup} \otimes \textcircled{\quad} + \textcircled{\diagdown} \otimes \textcircled{\diagup} + \textcircled{\diagup} \otimes \textcircled{\diagdown} + \textcircled{\quad} \otimes \textcircled{\diagup}$$

The image shows a mathematical identity between three Feynman diagrams. The first diagram on the left is a square with two diagonal lines crossing in the center, forming an 'X'. The second diagram in the middle is a square with a diagonal line from the bottom-left to the top-right, and a separate diagonal line from the top-left to the bottom-right. The third diagram on the right is a square with a diagonal line from the top-left to the bottom-right, and a separate diagonal line from the bottom-left to the top-right. An equals sign is placed between the first and second diagrams, and a minus sign is placed between the second and third diagrams.

$$\text{Diagram 1} = \text{Diagram 2} - \text{Diagram 3}$$

$$\begin{array}{c} \text{red} \\ \text{red} \\ \text{red} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \bullet \\ | \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \left(\right) + \begin{array}{c} \text{red} \\ \text{red} \\ \text{red} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array} \left(\right) + \begin{array}{c} \text{red} \\ \text{red} \\ \text{red} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \bullet \end{array} \left(\right) = 0 \quad (*)$$

