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## Intrinsic linking and knotting of graphs in arbitrary 3-manifolds

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## Intrinsic linking and knotting of graphs in arbitrary 3–manifolds

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We prove that a graph is intrinsically linked in an arbitrary 3–manifold  $M$  if and only if it is intrinsically linked in  $S^3$ . Also, assuming the Poincaré Conjecture, we prove that a graph is intrinsically knotted in  $M$  if and only if it is intrinsically knotted in  $S^3$ .

[05C10](#), [57M25](#)

### 1 Introduction

The study of intrinsic linking and knotting began in 1983 when Conway and Gordon [1] showed that every embedding of  $K_6$  (the complete graph on six vertices) in  $S^3$  contains a non-trivial link, and every embedding of  $K_7$  in  $S^3$  contains a non-trivial knot. Since the existence of such a non-trivial link or knot depends only on the graph and not on the particular embedding of the graph in  $S^3$ , we say that  $K_6$  is *intrinsically linked* and  $K_7$  is *intrinsically knotted*.

At roughly the same time as Conway and Gordon's result, Sachs [12; 11] independently proved that  $K_6$  and  $K_{3,3,1}$  are intrinsically linked, and used these two results to prove that any graph with a minor in the *Petersen family* (Figure 1) is intrinsically linked. Conversely, Sachs conjectured that any graph which is intrinsically linked contains a minor in the Petersen family. In 1995, Robertson, Seymour and Thomas [10] proved Sachs' conjecture, and thus completely classified intrinsically linked graphs.

Examples of intrinsically knotted graphs other than  $K_7$  are now known, see Foisy [2], Kohara and Suzuki [3] and Shimabara [13]. Furthermore, a result of Robertson and Seymour [9] implies that there are only finitely many intrinsically knotted graphs that are minor-minimal with respect to intrinsic knottedness. However, as of yet, intrinsically knotted graphs have not been classified.

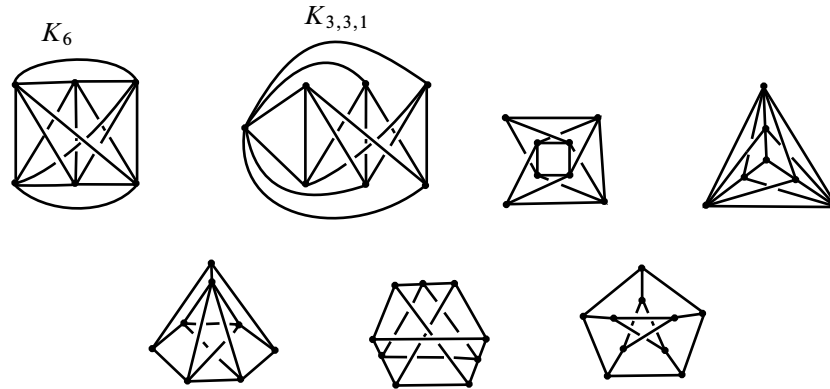


Figure 1: The Petersen family of graphs

In this paper we consider the properties of intrinsic linking and knotting in *arbitrary* 3-manifolds. We show that these properties are truly *intrinsic* to a graph in the sense that they do not depend on either the ambient 3-manifold or the particular embedding of the graph in the 3-manifold. Our proof in the case of intrinsic knotting assumes the Poincaré Conjecture.

We will use the following terminology. By a *graph* we shall mean a finite graph, possibly with loops and repeated edges. Manifolds may have boundary and do not have to be compact. All spaces are piecewise linear; in particular, we assume that the image of an *embedding* of a graph in a 3-manifold is a piecewise linear subset of the 3-manifold. An embedding of a graph  $G$  in a 3-manifold  $M$  is *unknotted* if every circuit in  $G$  bounds a disk in  $M$ ; otherwise, the embedding is *knotted*. An embedding of a graph  $G$  in a 3-manifold  $M$  is *unlinked* if it is unknotted and every pair of disjoint circuits in  $G$  bounds disjoint disks in  $M$ ; otherwise, the embedding is *linked*. A graph is *intrinsically linked* in  $M$  if every embedding of the graph in  $M$  is linked; and a graph is *intrinsically knotted* in  $M$  if every embedding of the graph in  $M$  is knotted. (So by definition an intrinsically knotted graph must be intrinsically linked, but not vice-versa.)

The main results of this paper are that a graph is intrinsically linked in an arbitrary 3-manifold if and only if it is intrinsically linked in  $S^3$  ([Theorem 1](#)); and (assuming the Poincaré Conjecture) that a graph is intrinsically knotted in an arbitrary 3-manifold if and only if it is intrinsically knotted in  $S^3$  ([Theorem 2](#)). We use Robertson, Seymour, and Thomas' classification of intrinsically linked graphs in  $S^3$  for our proof of [Theorem 1](#). However, because there is no analogous classification of intrinsically knotted graphs in  $S^3$ , we need to take a different approach to prove [Theorem 2](#). In particular, the proof of [Theorem 2](#) uses [Proposition 2](#) (every compact subset of a simply connected 3-manifold is homeomorphic to a subset of  $S^3$ ), whose proof in turn relies on the Poincaré

Conjecture. Our assumption of the Poincaré Conjecture seems reasonable, because Perelman [7; 8] has announced a proof of Thurston’s Geometrization Conjecture, which implies the Poincaré Conjecture [4]. (See also Morgan and Tian [5].)

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## 2 Intrinsically linked graphs

In this section, we prove that intrinsic linking is independent of the 3-manifold in which a graph is embedded. We begin by showing (Lemma 1) that any unlinked embedding of a graph  $G$  in a 3-manifold lifts to an unlinked embedding of  $G$  in the universal cover. In the universal cover, the linking number can be used to analyze intrinsic linking (Lemma 2), as in the proofs of Conway and Gordon [1] and Sachs [12; 11]. After we’ve shown that  $K_6$  and  $K_{3,3,1}$  are intrinsically linked in any 3-manifold (Proposition 1), we use the classification of intrinsically linked graphs in  $S^3$ , Robertson, Seymour, and Thomas [10], to conclude that any graph that is intrinsically linked in  $S^3$  is intrinsically linked in every 3-manifold (Theorem 1).

We call a circuit of length 3 in a graph a *triangle* and a circuit of length 4 a *square*.

**Lemma 1** *Any unlinked embedding of a graph  $G$  in a 3-manifold  $M$  lifts to an unlinked embedding of  $G$  in the universal cover  $\tilde{M}$ .*

**Proof** Let  $f: G \rightarrow M$  be an unlinked embedding.  $\pi_1(G)$  is generated by the circuits of  $G$  (attached to a basepoint). Since  $f(G)$  is unknotted, every cycle in  $f(G)$  bounds a disk in  $M$ . So  $f_*(\pi_1(G))$  is trivial in  $\pi_1(M)$ .

Thus, an unlinked embedding of  $G$  into  $M$  lifts to an embedding of  $G$  in the universal cover  $\tilde{M}$ . Since the embedding into  $M$  is unlinked, cycles of  $G$  bound disks in  $M$  and pairs of disjoint cycles of  $G$  bound disjoint disks in  $M$ . All of these disks in  $M$  lift to disks in  $\tilde{M}$ , so the embedding of the graph in  $\tilde{M}$  is also unlinked.  $\square$

Recall that if  $M$  is a 3-manifold with  $H_1(M) = 0$ , then disjoint oriented loops  $J$  and  $K$  in  $M$  have a well-defined linking number  $\text{lk}(J, K)$ , which is the algebraic intersection number of  $J$  with any oriented surface bounded by  $K$ . Also, the linking number is symmetric:  $\text{lk}(J, K) = \text{lk}(K, J)$ .

It will be convenient to have a notation for the linking number modulo 2: Define  $\omega(J, K) = \text{lk}(J, K) \bmod 2$ . Notice that  $\omega(J, K)$  is defined for a pair of *unoriented* loops. Since linking number is symmetric, so is  $\omega(J, K)$ . If  $J_1, \dots, J_n$  are loops in an embedded graph such that in the list  $J_1, \dots, J_n$  every edge appears an even number of times, and if  $K$  is another loop, disjoint from the  $J_i$ , then  $\sum \omega(J_i, K) = 0 \bmod 2$ .

If  $G$  is a graph embedded in a simply connected 3-manifold, let

$$\omega(G) = \sum \omega(J, K) \bmod 2,$$

where the sum is taken over all *unordered* pairs  $(J, K)$  of disjoint circuits in  $G$ . Notice that if  $\omega(G) \neq 0$ , then the embedding is linked (but the converse is not true).

**Lemma 2** *Let  $\tilde{M}$  be a simply connected 3-manifold, and let  $H$  be an embedding of  $K_6$  or  $K_{3,3,1}$  in  $\tilde{M}$ . Let  $e$  be an edge of  $H$ , and let  $e'$  be an arc in  $\tilde{M}$  with the same endpoints as  $e$ , but otherwise disjoint from  $H$ . Let  $H'$  be the graph  $(H - e) \cup e'$ . Then  $\omega(H') = \omega(H)$ .*

**Proof** Let  $D = e \cup e'$ .

First consider the case that  $H$  is an embedding of  $K_6$ . We will count how many terms in the sum defining  $\omega(H)$  change when  $e$  is replaced by  $e'$ . Let  $K_1, K_2, K_3$  and  $K_4$  be the four triangles in  $H$  disjoint from  $e$  (hence also disjoint from  $e'$  in  $H'$ ), and for each  $i$  let  $J_i$  be the triangle complementary to  $K_i$ . The  $J_i$  all contain  $e$ . For each  $i$ , let  $J'_i = (J_i - e) \cup e'$ , and notice that

$$(1) \quad \omega(J'_i, K_i) = \omega(J_i, K_i) + \omega(D, K_i) \bmod 2.$$

Because each edge appears twice in the list  $K_1, K_2, K_3, K_4$ , we have  $\omega(K_1, D) + \omega(K_2, D) + \omega(K_3, D) + \omega(K_4, D) = 0 \bmod 2$ . Thus,  $\omega(K_i, D)$  is nonzero for an even number of  $i$ . It follows from Equation (1) that there are an even number of  $i$  such that  $\omega(J'_i, K_i) \neq \omega(J_i, K_i)$ . Thus,  $\sum_{i=1}^4 \omega(J'_i, K_i) = \sum_{i=1}^4 \omega(J_i, K_i) \bmod 2$ , and

$$\begin{aligned} \omega(H') &= \sum_{\substack{J, K \subseteq H' \\ \ni e' \notin J, K}} \omega(J, K) + \sum_{i=1}^4 \omega(J'_i, K_i) \bmod 2 \\ &= \sum_{\substack{J, K \subseteq H \\ \ni e \notin J, K}} \omega(J, K) + \sum_{i=1}^4 \omega(J_i, K_i) \bmod 2 \\ &= \omega(H) \end{aligned}$$

Next consider the case that  $H$  is an embedding of  $K_{3,3,1}$ . Let  $x$  be the vertex of valence six in  $H$  (and in  $H'$ ).

**Case 1**  $e$  contains  $x$ . Then  $e$  is not in any square in  $H$  that has a complementary disjoint triangle. Let  $K_1, K_2$  and  $K_3$  be the three squares in  $H$  disjoint from  $e$ , and let  $J_1, J_2$  and  $J_3$  be the corresponding complementary triangles, all of which contain  $e$ . As in the  $K_6$  case, let  $J'_i = (J_i - e) \cup e'$  for each  $i$ ; again we have Equation (1). Every edge in the list  $K_1, K_2, K_3$  appears exactly twice, so  $\omega(K_1, D) + \omega(K_2, D) + \omega(K_3, D) = 0 \pmod 2$ . Thus,  $\omega(K_i, D)$  is nonzero for an even number of  $i$ ; and for an even number of  $i$ ,  $\omega(J'_i, K_i) \neq \omega(J_i, K_i)$ . The other pairs of circuits contributing to  $\omega(H)$  do not involve  $e$ . As in the  $K_6$  case, it follows that  $\omega(H') = \omega(H)$ .

**Case 2**  $e$  doesn't contain  $x$ . Let  $J_0$  be the triangle containing  $e$ , and let  $K_0$  be the complementary square. Let  $J_1$  through  $J_4$  be the four squares that contain  $e$ , but not  $x$  (so that they have complementary triangles); and let  $K_1$  through  $K_4$  be the complementary triangles. With  $J'_i$  defined as in the other cases, we again have Equation (1). Every edge appears an even number of times in the list  $K_0, K_1, K_2, K_3, K_4$ , so  $\sum_{i=0}^4 \omega(K_i, D) = 0 \pmod 2$ , and  $\omega(K_i, D) \neq 0$  for an even number of  $i$ . As in the other cases, it follows that for an even number of  $i$ ,  $\omega(J'_i, K_i) \neq \omega(J_i, K_i)$ ; and an even number of the terms in the sum defining  $\omega(H)$  change when  $e$  is replaced by  $e'$ ; and  $\omega(H') = \omega(H)$ .  $\square$

**Proposition 1**  $K_6$  and  $K_{3,3,1}$  are intrinsically linked in any 3-manifold  $M$ .

**Proof** Let  $G$  be either  $K_6$  or  $K_{3,3,1}$ , and let  $f: G \rightarrow M$  be an embedding. Suppose for the sake of contradiction that  $f(G)$  is unlinked. Let  $\tilde{M}$  be the universal cover of  $M$ . By Lemma 1,  $f$  lifts to an unlinked embedding  $\tilde{f}: G \rightarrow \tilde{M}$ .

Let  $\tilde{G} = \tilde{f}(G) \subseteq \tilde{M}$ , and let  $\tilde{H}$  be a copy of  $G$  embedded in a ball in  $\tilde{M}$ . Isotope  $\tilde{G}$  so that  $\tilde{H}$  and  $\tilde{G}$  have the same vertices, but do not otherwise intersect. Then  $\tilde{G}$  can be transformed into  $\tilde{H}$  by changing one edge at a time – replace an edge of  $\tilde{G}$  by the corresponding edge of  $\tilde{H}$ , once for every edge. By repeated applications of Lemma 2,  $\omega(\tilde{G}) = \omega(\tilde{H})$ . Since  $\tilde{H}$  is inside a ball in  $\tilde{M}$ , Conway and Gordon's proof [1], and Sachs' proof [12; 11], that  $K_6$  and  $K_{3,3,1}$  are intrinsically linked in  $S^3$ , show that  $\omega(\tilde{H}) = 1$ .

Thus,  $\omega(\tilde{G}) = 1$ , and there must be disjoint circuits  $J$  and  $K$  in  $\tilde{G}$  that do not bound disjoint disks in  $\tilde{M}$ , contradicting that  $\tilde{f}$  is an unlinked embedding. Thus,  $f(G)$  is linked in  $M$ .  $\square$

Let  $G$  be a graph which contains a triangle  $\Delta$ . Remove the three edges of  $\Delta$  from  $G$ . Add three new edges, connecting the three vertices of  $\Delta$  to a new vertex. The

resulting graph,  $G'$ , is said to have been obtained from  $G$  by a “ $\Delta - Y$  move” (Figure 2). The seven graphs that can be obtained from  $K_6$  and  $K_{3,3,1}$  by  $\Delta - Y$  moves are the *Petersen family* of graphs (Figure 1).

If a graph  $G'$  can be obtained from a graph  $G$  by repeatedly deleting edges and isolated vertices of  $G$ , and/or contracting edges of  $G$ , then  $G'$  is a *minor* of  $G$ .

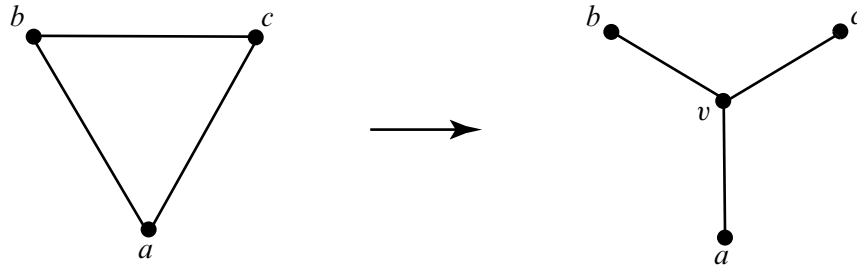


Figure 2: A  $\Delta - Y$  Move

The following facts were first proved, in the  $S^3$  case, by Motwani, Raghunathan and Saran [6]. Here we generalize the proofs to any 3-manifold  $M$ .

**Fact 1** *If a graph  $G$  is intrinsically linked in  $M$ , and  $G'$  is obtained from  $G$  by a  $\Delta - Y$  move, then  $G'$  is intrinsically linked in  $M$ .*

**Proof** Suppose to the contrary that  $G'$  has an unlinked embedding  $f: G' \rightarrow M$ . Let  $a, b, c$  and  $v$  be the embedded vertices of the  $Y$  illustrated in Figure 2. Let  $B$  denote a regular neighborhood of the embedded  $Y$  such that  $a, b$  and  $c$  are on the boundary of  $B$ ,  $v$  is in the interior of  $B$ , and  $B$  is otherwise disjoint from  $f(G')$ . Now add edges  $ab, bc$  and  $ac$  in the boundary of  $B$  so that the resulting embedding of the  $K_4$  with vertices  $a, b, c$ , and  $v$  is *pannelled* in  $B$  (ie, every cycle bounds a disk in the complement of the graph). We now remove vertex  $v$  (and its incident edges) to get an embedding  $h$  of  $G$  such that if  $e$  is any edge of  $G \cap G'$  then  $h(e) = f(e)$  and the triangle  $abc$  is in  $\partial B$ .

Observe that if  $K$  is any circuit in  $h(G)$  other than the triangle  $abc$ , then  $K$  is isotopic to a circuit in  $G'$ . The triangle  $abc$  bounds a disk in  $B$ , and since  $f(G')$  is unknotted, every circuit in  $f(G')$  bounds a disk in  $M$ . Thus  $h(G)$  is unknotted. Also if  $J$  and  $K$  are disjoint circuits in  $h(G)$  neither of which is  $abc$ , then  $J \cup K$  is isotopic to a pair of disjoint circuits  $J' \cup K'$  in  $f(G')$ . Since  $f(G')$  is unlinked,  $J'$  and  $K'$  bound disjoint disks in  $M$ . Hence  $J$  and  $K$  also bound disjoint disks in  $M$ . Finally if  $K$  is a circuit in  $h(G)$  which is disjoint from  $abc$ , then  $K$  is contained in  $f(G')$ . Since  $f(G')$  is unknotted,  $K$  bounds a disk  $D$  in  $M$ . Furthermore, since  $B$  is a ball, we

can isotope  $D$  to a disk which is disjoint from  $B$ . Now  $abc$  and  $K$  bound disjoint disks in  $M$ . So  $h(G)$  is unlinked, contradicting the hypothesis that  $G$  is intrinsically linked in  $M$ . We conclude that  $G'$  is also intrinsically linked in  $M$ .  $\square$

**Fact 2** *If a graph  $G$  has an unlinked embedding in  $M$ , then so does every minor of  $G$ .*

**Proof** The proof is identical to the proof for  $S^3$ .  $\square$

**Theorem 1** *Let  $G$  be a graph, and let  $M$  be a 3-manifold. The following are equivalent:*

- (1)  $G$  is intrinsically linked in  $M$ ,
- (2)  $G$  is intrinsically linked in  $S^3$ ,
- (3)  $G$  has a minor in the Petersen family of graphs.

**Proof** Robertson, Seymour and Thomas [10] proved that (2) and (3) are equivalent. We see as follows that (1) implies (2): Suppose there is an unlinked embedding of  $G$  in  $S^3$ . Then the embedded graph and its system of disks in  $S^3$  are contained in a ball, which embeds in  $M$ .

We will complete the proof by checking that (3) implies (1).  $K_6$  and  $K_{3,3,1}$  are intrinsically linked in  $M$  by Proposition 1. Thus, by Fact 1, all the graphs in the Petersen family are intrinsically linked in  $M$ . Therefore, if  $G$  has a minor in the Petersen family, then it is intrinsically linked in  $M$ , by Fact 2.  $\square$

### 3 Compact subsets of a simply connected space

In this section, we assume the Poincaré Conjecture, and present some known results about 3-manifolds, which will be used in Section 4 to prove that intrinsic knotting is independent of the 3-manifold (Theorem 2).

**Fact 3** *Assume that the Poincaré Conjecture is true. Let  $\tilde{M}$  be a simply connected 3-manifold, and suppose that  $B \subseteq \tilde{M}$  is a compact 3-manifold whose boundary is a disjoint union of spheres. Then  $B$  is a ball with holes (possibly zero holes).*

**Proof** By the Seifert–Van Kampen theorem,  $B$  itself is simply connected. Cap off each boundary component of  $B$  with a ball, and the result is a closed simply connected 3-manifold. By the Poincaré Conjecture, this must be the 3-sphere.  $\square$



**Fact 4** Let  $\tilde{M}$  be a simply connected 3-manifold, and suppose that  $N \subseteq \tilde{M}$  is a compact 3-manifold whose boundary is nonempty and not a union of spheres. Then there is a compression disk  $D$  in  $\tilde{M}$  for a component of  $\partial N$  such that  $D \cap \partial N = \partial D$ .

**Proof** Since  $\partial N$  is nonempty, and not a union of spheres, there is a boundary component  $F$  with positive genus. Because  $\tilde{M}$  is simply connected,  $F$  is not incompressible in  $\tilde{M}$ . Thus,  $F$  has a compression disk.

Among all compression disks for boundary components of  $N$  (intersecting  $\partial N$  transversely), let  $D$  be one such that  $D \cap \partial N$  consists of the fewest circles. Suppose, for the sake of contradiction, that there is a circle of intersection in the interior of  $D$ . Let  $c$  be a circle of intersection which is innermost in  $D$ , bounding a disk  $D'$  in  $D$ . Either  $c$  is nontrivial in  $\pi_1(\partial N)$ , in which case  $D'$  is itself a compression disk; or  $c$  is trivial, bounding a disk on  $\partial N$ , which can be used to remove the circle  $c$  of intersection from  $D \cap \partial N$ . In either case, there is a compression disk for  $\partial N$  which has fewer intersections with  $\partial N$  than  $D$  has, contradicting minimality. Thus,  $D \cap \partial N = \partial D$ .  $\square$

We are now ready to prove the main result of this section. Because its proof uses [Fact 3](#), it relies on the Poincaré Conjecture.

**Proposition 2** Assume that the Poincaré Conjecture is true. Then every compact subset  $K$  of a simply connected 3-manifold  $\tilde{M}$  is homeomorphic to a subset of  $S^3$ .

**Proof** We may assume without loss of generality that  $K$  is connected. Let  $N \subseteq \tilde{M}$  be a closed regular neighborhood of  $K$  in  $\tilde{M}$ . Then  $N$  is a compact connected 3-manifold with boundary. It suffices to show that  $N$  embeds in  $S^3$ .

Let  $g(S)$  denote the genus of a connected closed orientable surface  $S$ . Define the complexity  $c(S)$  of a closed orientable surface  $S$  to be the sum of the squares of the genera of the components  $S_i$  of  $S$ , so  $c(S) = \sum_{S_i} g(S_i)^2$ . Our proof will proceed by induction on  $c(\partial N)$ . We make two observations about the complexity function.

- (1)  $c(S) = 0$  if and only if  $S$  is a union of spheres.
- (2) If  $S'$  is obtained from  $S$  by surgery along a non-trivial simple closed curve  $\gamma$ , then  $c(S') < c(S)$ .

We prove Observation (2) as follows. It is enough to consider the component  $S_0$  of  $S$  containing  $\gamma$ . If  $\gamma$  separates  $S_0$ , then  $S_0 = S_1 \# S_2$ , where  $S_1$  and  $S_2$  are not spheres, and  $S'$  is the result of replacing  $S_0$  by  $S_1 \cup S_2$  in  $S$ . In this case,  $c(S_0) = g(S_0)^2 = (g(S_1) + g(S_2))^2 = c(S_1) + c(S_2) + 2g(S_1)g(S_2) > c(S_1) + c(S_2)$ ,

since  $g(S_1)$  and  $g(S_2)$  are nonzero. On the other hand, if  $\gamma$  does not separate  $S_0$ , then surgery along  $\gamma$  reduces the genus of the surface. Then the square of the genus is also smaller, and hence again  $c(S') < c(S)$ .

If  $c(\partial N) = 0$ , then by [Fact 3](#)  $N$  is a ball with holes, and so embeds in  $S^3$ , establishing our base case. If  $c(\partial N) > 0$ , then by [Fact 4](#) there is a compression disk  $D$  for  $\partial N$  such that  $D \cap \partial N = \partial D$ . There are three cases to consider.

**Case 1**  $D \cap N = \partial D$ . Let  $N' = N \cup \text{nbd}(D)$ . Since  $\partial N'$  is the result of surgery on  $\partial N$  along a non-trivial simple closed curve,  $c(\partial N') < c(\partial N)$ , so by induction  $N'$  embeds in  $S^3$ . Hence  $N$  embeds in  $S^3$ .

**Case 2**  $D \cap N = D$ , and  $D$  separates  $N$ . Then cutting  $N$  along  $D$  (ie removing  $D \times (-1, 1)$ ) yields two connected manifolds  $N_1$  and  $N_2$ , with  $c(\partial N_1) < c(\partial N)$  and  $c(\partial N_2) < c(\partial N)$ . So  $N_1$  and  $N_2$  each embed in  $S^3$ . Consider two copies of  $S^3$ , one containing  $N_1$  and the other containing  $N_2$ .

Let  $C_1$  be the component of  $S^3 - N_1$  whose boundary contains  $D \times \{1\}$ , and  $C_2$  be the component of  $S^3 - N_2$  whose boundary contains  $D \times \{-1\}$ . Remove small balls  $B_1$  and  $B_2$  from  $C_1$  and  $C_2$ , respectively. Then glue together the balls  $\text{cl}(S^3 - B_1)$  and  $\text{cl}(S^3 - B_2)$  along their boundaries. The result is a 3-sphere containing both  $N_1$  and  $N_2$ , in which  $D \times \{1\}$  and  $D \times \{-1\}$  lie in the boundary of the same component of  $S^3 - (N_1 \cup N_2)$ . So we can embed the arc  $\{0\} \times (-1, 1)$  (the core of  $D \times (-1, 1)$ ) in  $S^3 - (N_1 \cup N_2)$ , which means we can extend the embedding of  $N_1 \cup N_2$  to an embedding of  $N$ .

**Case 3**  $D \cap N = D$ , but  $D$  does not separate  $N$ . Then cutting  $N$  along  $D$  yields a new connected manifold  $N'$  with  $c(\partial N') < c(\partial N)$ , so  $N'$  embeds in  $S^3$ . As in the last case, we also need to embed the core  $\gamma$  of  $D$ . Suppose for the sake of contradiction that  $\gamma$  has endpoints on two different boundary components  $F_1$  and  $F_2$  of  $N'$ . Let  $\beta$  be a properly embedded arc in  $N'$  connecting  $F_1$  and  $F_2$ . Then  $\gamma \cup \beta$  is a loop in  $\tilde{M}$  that intersects the closed surface  $F_1$  in exactly one point. But because  $H_1(\tilde{M}) = 0$ , the algebraic intersection number of  $\gamma \cup \beta$  with  $F_1$  is zero. This is impossible since  $\gamma \cup \beta$  meets  $F_1$  in a single point. Thus, both endpoints of  $\gamma$  lie on the same boundary component of  $N'$ , and so  $\gamma$  can be embedded in  $S^3 - N'$ . So the embedding of  $N'$  can be extended to an embedding of  $N$  in  $S^3$ .  $\square$

## 4 Intrinsically knotted graphs

In this section, we use [Proposition 2](#) to prove that the property of a graph being intrinsically knotted is independent of the 3-manifold it is embedded in. Notice that

since [Proposition 2](#) relies on the Poincaré Conjecture, so does the intrinsic knotting result.

**Theorem 2** *Assume that the Poincaré Conjecture is true. Let  $M$  be a 3–manifold. A graph is intrinsically knotted in  $M$  if and only if it is intrinsically knotted in  $S^3$ .*

**Proof** Suppose that a graph  $G$  is not intrinsically knotted in  $S^3$ . Then it embeds in  $S^3$  in such a way that every circuit bounds a disk embedded in  $S^3$ . The union of the embedding of  $G$  with these disks is compact, hence is contained in a ball  $B$  in  $S^3$ . Any embedding of  $B$  in  $M$  yields an unknotted embedding of  $G$  in  $M$ .

Conversely, suppose there is an unknotted embedding  $f: G \rightarrow M$ . Let  $\tilde{M}$  be the universal cover of  $M$ . By using the same argument as in the proof of Lemma 1, we can lift  $f$  to an unknotted embedding  $\tilde{f}: G \rightarrow \tilde{M}$ . Let  $K$  be the union of  $\tilde{f}(G)$  with the disks bounded by its circuits. Then  $K$  is compact, so by [Proposition 2](#), there is an embedding  $g: K \rightarrow S^3$ . Now  $g \circ \tilde{f}(G)$  is an embedding of  $G$  in  $S^3$ , in which every circuit bounds a disk. Hence  $g \circ f(G)$  is an unknotted embedding of  $G$  in  $S^3$ .  $\square$

**Remark** The proof of [Theorem 2](#) can also be used, almost verbatim, to show that intrinsic *linking* is independent of the 3–manifold. Of course, this argument relies on the Poincaré Conjecture; so the proof given in [Section 2](#) is more elementary.

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