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Links With Finite n-Quandles

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Abstract

We prove a conjecture of Przytycki which asserts that the n-quandle of a link L in the 3-sphere is finite if and only if the fundamental group of the n-fold cyclic branched cover of the 3-sphere, branched over L, is finite.

1   Introduction

While the algebraic study of racks and quandles dates back to the late 1800's, Fenn and Rourke in [1] credit Conway and Wraith with introducing the concepts in 1959 as an algebraic approach to study knots and links in 3-manifolds. During the same time period, several mathematicians were studying similar concepts under names such as kei, distributive groupoids, crystals, and automorphic sets. In 1982, Joyce [5] published a ground-breaking work which included introducing the term quandle, giving both topological and algebraic descriptions of the fundamental quandle of a link, and proving that the fundamental quandle of a knot is a complete invariant. In this article, we consider a quotient of the fundamental quandle of a link called the n-quandle, defined for any natural number n. Whereas the quandle of a link is usually infinite and somewhat untractable, there are many examples of knots and links for which the n-quandle is finite for some n. In his Ph.D. dissertation, Winker [10] developed a method to produce the analog of the Cayley diagram for a quandle. In addition, Winker established a relationship between the n-quandle of the link L and the fundamental group of $\widetilde{M}_n(L)$, the n-fold cyclic branched cover of the 3-sphere, branched over L. When combined with previous work of Joyce, this implied that if the n-quandle of a link L is finite, then so is $\pi_1(\widetilde{M}_n(L))$. Przytycki then conjectured that this condition is both

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necessary and sufficient, which we prove to be true in this paper. Our proof involves first generalizing a key result of Joyce: the cosets of the peripheral subgroup of a knot group can be given a quandle structure making it isomorphic to the fundamental quandle of the knot. We extend this result to the n-quandle of a knot, showing that it can also be viewed as the set of cosets of the peripheral subgroup in a certain quotient of the knot group. This result allows Winker's diagraming method to be replaced by the well known Todd-Coxeter method of coset enumeration.

We assume the reader is familiar with the theory of racks and quandles, but include basic definitions for completeness. The reader is referred to [1], [5], [6], and [10] for more information. A *quandle* is a set Q together with two binary operations $\triangleright$ and $\triangleright^{-1}$ which satisfies the following three axioms.

Q1. $x \triangleright x = x$ for all $x \in Q$.

Q2. $(x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y$ for all $x, y \in Q$.

Q3. $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ for all $x, y, z \in Q$.

A *rack* is more general, requiring only Q2 and Q3. It is important to note that, in general, the quandle operations are not associative. In fact, using axioms Q2 and Q3 it is easy to show that

$$x \triangleright (y \triangleright z) = \left((x \triangleright^{-1} z) \triangleright y\right) \triangleright z. \tag{1}$$

This property allows one to write any expression involving $\triangleright$ and $\triangleright^{-1}$ in a unique left-associated form (see [10]). Henceforth, expressions without parenthesis are assumed to be left-associated.

Given a quandle Q, each element $q \in Q$ defines a map $S_q : Q \to Q$ by $S_q(p) = p \triangleright q$. It follows from axiom Q2 that S_q is a bijection and $S_q^{-1}(p) = p \triangleright^{-1} q$. From axiom Q3, it follows that S_q is a quandle homomorphism. The automorphism S_q is called the *point symmetry at q* and the set of all point symmetries generate the *inner automorphism group* $\mathrm{Inn}(Q)$. A quandle Q is *algebraically connected* if $\mathrm{Inn}(Q)$ acts transitively on Q.

Joyce discusses two functors from the category of groups to the category of quandles. These functors and their adjoints will be of importance in this paper. The first, denoted Conj, takes a group G to a quandle $Q = \mathrm{Conj}(G)$ defined as the set G with operations given by conjugation. Specifically, $x \triangleright y = y^{-1}xy$ and $x \triangleright^{-1} y = yxy^{-1}$. Its adjoint, denoted Adconj takes the quandle Q to the group $\mathrm{Adconj}(Q)$ generated by the elements of Q and defined by the group presentation

$$\mathrm{Adconj}(Q) = \langle \overline{q} \text{ for all } q \text{ in } Q \mid \overline{p \triangleright q} = \overline{q}^{-1}\overline{p}\,\overline{q} \text{ for all } p \text{ and } q \text{ in } Q \rangle.$$

A quandle Q is called an *n-quandle* if each point symmetry S_q has order dividing n. It is convenient to write $x \triangleright^k y$ for $S_y^k(x)$, the k-th power of S_y evaluated at x. Thus Q is an

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n-quandle if for all x and y in Q, we have $x \triangleright^n y = x$. A second functor from groups to n-quandles is defined for each natural number n and is denoted Q_n. Given a group G, the n-quandle $\mathrm{Q}_n(G)$ is the set

$$\mathrm{Q}_n(G) = \{x \in G \mid x^n = 1\}$$

again with the operations given by conjugation. The adjoint of this functor is AdQ_n. If Q is any n-quandle, the group $\mathrm{AdQ}_n(Q)$ is defined by the presentation

$$\mathrm{AdQ}_n(Q) = \langle \bar{q} \text{ for all } q \text{ in } Q \mid \bar{q}^n = 1, \overline{p \triangleright q} = \bar{q}^{-1}\bar{p}\,\bar{q} \text{ for all } p \text{ and } q \text{ in } Q \rangle.$$

Quandles may be presented in terms of generators and relators in much the same way as groups. See [1] for a rigorous development of this topic. If the quandle Q is given by the finite presentation

$$Q = \langle q_1, q_2, \ldots, q_i \mid r_1, r_2, \ldots r_j \rangle,$$

then Winker proves in [10] that $\mathrm{Adconj}(Q)$ and $\mathrm{AdQ}_n(Q)$ can be finitely presented as

$$\mathrm{Adconj}(Q) = \langle \bar{q}_1, \bar{q}_2, \ldots, \bar{q}_i \mid \bar{r}_1, \bar{r}_2, \ldots, \bar{r}_j \rangle \tag{2}$$

and

$$\mathrm{AdQ}_n(Q) = \langle \bar{q}_1, \bar{q}_2, \ldots, \bar{q}_i \mid \bar{q}_1^n = 1, \bar{q}_2^n = 1, \ldots, \bar{q}_i^n = 1, \bar{r}_1, \bar{r}_2, \ldots, \bar{r}_j \rangle. \tag{3}$$

Here, each quandle relation r_i is an equation between two quandle elements each expressed using the generators, the operations $\triangleright$ and $\triangleright^{-1}$, and parenthesis to indicate the order of operations. The associated group relation $\bar{r}_i$ must now be formed in a corresponding way using conjugation. For example, if r is the quandle relation $x = y \triangleright (z \triangleright^{-1} w)$, then $\bar{r}$ is the relation $\bar{x} = \bar{w}\,\bar{z}^{-1}\,\bar{w}^{-1}\,\bar{y}\,\bar{w}\,\bar{z}\,\bar{w}^{-1}$.

Associated to every oriented knot or link L in the 3-sphere $\mathbb{S}^3$ is its *fundamental quandle* $Q(L)$ which is defined by means of a presentation derived from a regular diagram D of L as follows. First assign quandle generators $x_1, x_2, \ldots, x_n$ to each arc of D. Next, introduce a relation at each crossing of D as shown in Figure 1. It is easy to check that the three axioms, Q1, Q2, and Q3, are exactly what is needed to prove that $Q(L)$ is preserved by Reidemeister moves and hence is an invariant of the link. Passing from this presentation of $Q(L)$ to a presentation for $\mathrm{Adconj}(Q(L))$ by using Winker's formula (2), we obtain the well-known Wirtinger presentation of $\pi_1(\mathbb{S}^3 - L)$. Thus for any link L, $\pi_1(\mathbb{S}^3 - L) \cong \mathrm{Adconj}(Q(L))$.

Joyce proves in [5] that $Q(L)$ is a complete invariant of knots. A less sensitive, but presumably more tractable invariant, is the *fundamental n-quandle $Q_n(L)$* which can be defined for each natural number n by starting with the presentation for $Q(L)$ just described and adding the relations $x_i \triangleright^n x_j = x_i$ for all pairs of generators x_i and x_j. Passing from this presentation of $Q_n(L)$ to a presentation for $\mathrm{AdQ}_n(Q_n(L))$ by using Winker's formula (3), we see that $\mathrm{AdQ}_n(Q_n(L))$ is a quotient of $\mathrm{Adconj}(Q(L))$. In particular, we may present $\mathrm{AdQ}_n(Q_n(L))$ by starting with the Wirtinger presentation of $\pi_1(\mathbb{S}^3 - L)$ and then adjoining the relations $x^n = 1$ for each Wirtinger generator x. While the fundamental quandle of a
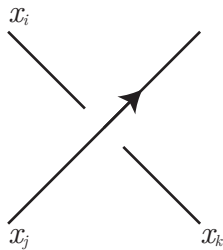
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Figure 1: The relation $x_i = x_k \triangleright x_j$ is associated to a crossing with arcs labeled as shown.

nontrivial knot is always infinite, the associated n-quandle is sometimes finite. Determining when this occurs is the focus of this paper.

If L is a link of more than one component, then both $Q(L)$ and $Q_n(L)$ are algebraically disconnected with one component $Q^i(L)$ and $Q_n^i(L)$, respectively, corresponding to each component K_i of L.

If K is a knot, let P be the peripheral subgroup of $G = \pi_1(\mathbb{S}^3 - K)$ generated by the meridian μ and longitude λ of K. In [5], Joyce defines a quandle structure on the set of right cosets $P \backslash G$ by declaring $Pg \triangleright^{\pm 1} Ph = Pgh^{-1}\mu^{\pm 1}h$. He denotes this quandle as $(P \backslash G; \mu)$ and then proves that it is isomorphic to $Q(K)$. This is the key step in Joyce's proof that the quandle is a complete knot invariant. It shows that the knot quandle not only determines the knot group, but also the peripheral structure. This implies that the order of $Q(K)$ is the index of P in G and hence that $Q(K)$ is infinite when K is nontrivial. The key result of this paper is the following theorem which extends Joyce's result to the case of $Q_n(L)$.

Theorem 1 *If $L = \{K_1, K_2, \ldots, K_s\}$ is a link in $\mathbb{S}^3$ and P_i is the subgroup of $\mathrm{AdQ}_n(Q_n(L))$ generated by the meridian μ_i and longitude λ_i of K_i, then the quandle $(P_i \backslash \mathrm{AdQ}_n(Q_n(L)); \mu_i)$ is isomorphic to the algebraic component $Q_n^i(L)$ of $Q_n(L)$.*

Section 2 is devoted to proving Theorem 1. In Section 3 we use this result, as well as a theorem of Joyce, to prove the conjecture of Przytycki stated in the Abstract. Theorem 1 implies that the Todd-Coxeter process for coset enumeration can be used to describe $Q_n^i(L)$ provided it is finite. In Section 4 we describe this in greater detail and give examples.

Finally, in a separate paper, or papers, we plan to enumerate all links that have finite n-quandles for some n. Alternatively, this project may be viewed as a tabulation of all finite quandles that appear as an n-quandle of some link.

2 Relating $Q_n(L)$ to Cosets in $\mathrm{AdQ}_n(Q_n(L))$

To prove Theorem 1 we make use of topological descriptions of both the fundamental quandle $Q(L)$ and the n-quandle $Q_n(L)$. We begin by recalling Fenn and Rourke's formulation of $Q(L)$ given in [1] and then extend it to $Q_n(L)$. (Their formulation is actually for the rack associated to a framed link.) Let $X = \mathbb{S}^3 - \mathring{N}(L)$ be the exterior of L and choose a basepoint b in X. Define $T(L)$ to be the set of all homotopy classes of paths $\alpha : [0,1] \to X$ such that $\alpha(0) = b$ and $\alpha(1) \in \partial X$. Moreover, we require that any homotopy be through a sequence of paths each of which starts at b and ends at ∂X. Define the two binary operations, $\triangleright$ and $\triangleright^{-1}$, on $T(L)$ by

$$\alpha \triangleright^{\pm 1} \beta = \beta m^{\mp 1} \beta^{-1} \alpha \tag{4}$$

where m is a meridian of L. Namely, m is a loop in $\partial N(L)$ that begins and ends at $\beta(1)$, is essential in $\partial N(L)$, is nullhomotopic in $N(L)$, and has linking number $+1$ with L. Thus the arc $\alpha \triangleright \beta$ is formed by starting at the basepoint b, going along β to $\partial N(L)$, traveling around m^{-1}, following β^{-1} back to the base point, and finally following α to its endpoint in $\partial N(L)$. See Figure 2. Note that the component $T^i(L)$ corresponding to the i-th component K_i of L consists of those paths ending at $\partial N(K_i)$. The equivalence of $Q(L)$ and $T(L)$ is proven in [1]. A similar description using "nooses" is given in [5]. In order to give a topological
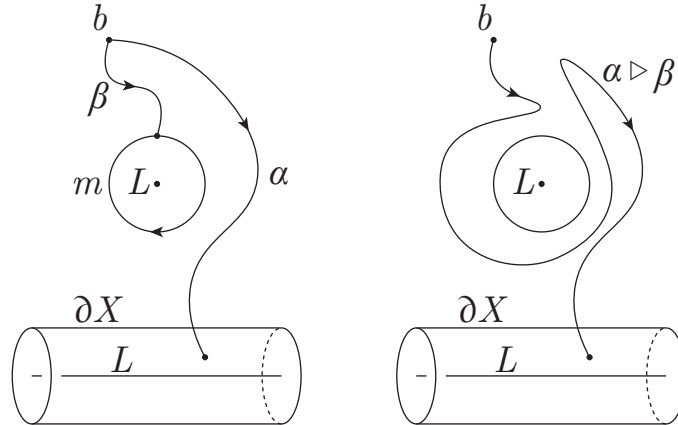
Figure 2: The topological definition of $\alpha \triangleright \beta$.

description of $Q_n(L)$ we introduce the following definition.

Definition 2 Suppose α is a path in X with $\alpha(0) = b$ and $\alpha(1) \in \{b\} \cup \partial X$. Suppose further that there exists t_0 with $0 \le t_0 \le 1$ such that $\alpha(t_0) \in \partial N(L)$. Let $\sigma_1(t) = \alpha(tt_0)$ and $\sigma_2(t) = \alpha((1-t)t_0 + t)$. We say that the path $\sigma_1 m^{\pm n} \sigma_2$ is obtained from α by a $\pm n$-*meridian move*. Two paths are called n-*meridionally equivalent* if they are related by a sequence of $\pm n$-meridian moves and homotopies.

We now define the n-quandle $T_n(L)$ as the set of n-meridional equivalence classes of paths with the quandle operations defined by (4). Again, paths that end at $\partial N(K_i)$ give the component $T_n^i(L)$ of $T_n(L)$.

Theorem 3 *The n-quandles $Q_n(L)$ and $T_n(L)$ are quandle-isomorphic.*

Proof. In [1], the topological and algebraic-presentation definitions of the rack of a framed link are proven to be quandle isomorphic by constructing homomorphisms $f : T \to Q$ and $g : Q \to T$ and then showing that both $f \circ g$ amd $g \circ f$ are the identity. The same maps can be used to show that $T_n(L)$ and $Q_n(L)$ are isomorphic. Rather than repeating and extending Fenn and Rourke's proof here, we simply enumerate the differences from which the interested reader can easily fill in the details of the proof.

- In [1] homotopies in T allow the endpoint of a path to move around on the chosen longitude of L given by the framing, while we allow homotopies in T_n to move the endpoint around in $\partial N(L)$. For our maps to be well-defined, this requires the idempotency axiom Q1 which is not present in a rack.

- In T_n we allow n-meridional moves that are not present in T. In order for our maps to be well-defined this requires the addition of the corresponding relations $q_i \triangleright^n q_j = q_i$ to Q_n. $\square$

We are now prepared to prove Theorem 1.

Theorem 1 *If $L = \{K_1, K_2, \ldots, K_s\}$ is a link in $\mathbb{S}^3$ and P_i is the subgroup of $\mathrm{AdQ}_n(Q_n(L))$ generated by the meridian μ_i and longitude λ_i of K_i, then the quandle $(P_i \backslash \mathrm{AdQ}_n(Q_n(L)); \mu_i)$ is isomorphic to the algebraic component $Q_n^i(L)$ of $Q_n(L)$.*

Proof. Suppose that $L = \{K_1, K_2, \ldots, K_s\}$. Without loss of generality, we shall prove the theorem for the first component K_1. We begin by fixing some element $\nu \in Q_n(L)$ which we think of as a path from the basepoint b in X to $\partial N(K_1)$. We now define a map $\tau : \mathrm{AdQ}_n(Q_n(L)) \to Q_n(L)$ by $\tau(\alpha) = \alpha^{-1}\nu$.

Claim 1: The map τ is onto $Q_n^1(L)$.

Proof: Let σ be a path representing any element of $Q_n^1(L)$. Move σ by a homotopy until $\sigma(1) = \nu(1)$ and let α be the loop $\alpha = \nu\sigma^{-1}$. Now $\tau(\alpha) = \alpha^{-1}\nu = \sigma\nu^{-1}\nu = \sigma$.

Let P_ν be the subgroup of $\mathrm{AdQ}_n(Q_n(L))$ generated by the meridian $\mu_1 = \nu m \nu^{-1}$ and longitude $\lambda_1 = \nu \ell \nu^{-1}$ of K_1.

Claim 2: $\tau^{-1}(\nu) = P_\nu$.

Proof: Notice first that $\tau^{-1}(\nu)$ is a subgroup of $\mathrm{AdQ}_n(Q_n(L))$. For suppose that $\alpha, \beta \in \tau^{-1}(\nu)$. Now $\tau(\alpha\beta^{-1}) = \beta\alpha^{-1}\nu = \beta\nu = \nu$ because $\alpha^{-1}\nu = \nu$ and $\beta^{-1}\nu = \nu$ implies $\nu = \beta\nu$. Thus to show that $P_\nu \subset \tau^{-1}(\nu)$ we need only show that $\mu, \lambda \in \tau^{-1}(\nu)$. But $\tau(\lambda) = \lambda^{-1}\nu = (\nu\ell\nu^{-1})^{-1}\nu = \nu\ell^{-1}\nu^{-1}\nu = \nu\ell^{-1} = \nu$ because $\ell \subset \partial X$. Similarly, $\mu \in \tau^{-1}(\nu)$.

Now suppose that $\alpha \in \tau^{-1}(\nu)$. This means that $\alpha^{-1}\nu$ can be taken to ν be a sequence of n-meridian moves separated by homotopies. We illustrate the situation in Figure 3. The first homotopy begins at $\alpha^{-1}\nu$ and ends at the path $\sigma_1\rho_1$ where $\sigma_1(1) = \rho_1(0)$ is a point in ∂X. We then do an n-meridian move, replacing $\sigma_1\rho_1$ with the path $\sigma_1 m^{\pm n}\rho_1$. This path is then homotopic to the path $\sigma_2\rho_2$ and so on until finally the last homotopy ends at ν. For simplicity, the Figure illustrates the case of three homotopies separated by two n-meridian moves. Notice that the "right edge" of the i-th homotopy defines a path in $\partial N(K_1)$ which we call β_i. These homotopies can be reparameterized so that the polygonal paths indicated in each homotopy depict the new level sets. The first homotopy can now be thought of as one between the loop α and the loop $\nu\beta_1\rho_1^{-1}\sigma_1^{-1}$. We then perform an n-meridian move to this loop and continue through the second homotopy, ending at the loop $\nu\beta_1\beta_2\rho_2^{-1}\sigma_2^{-1}$. Eventually we arrive at the loop $\nu\beta_1\beta_2\ldots\beta_k\nu^{-1}$, an element of P_ν. Thus α represents an element of P_ν and hence $\tau^{-1}(\nu) \subset P_\nu$.
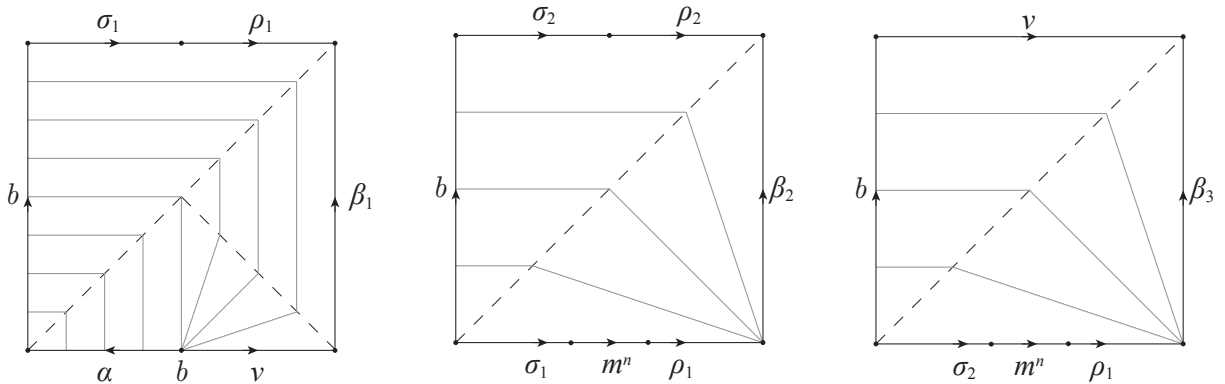
Figure 3: Homotopies separated by n-meridian moves.

Claim 3: Let ϕ_1 be the automorphism of $\mathrm{AdQ}_n(Q_n(L))$ given by conjugation by μ_1. Then ϕ_1 fixes every element of P_ν.

Proof: Suppose that $\nu\beta\nu^{-1} \in P_\nu$. Now

$$
\begin{aligned}
\phi_1(\nu\beta\nu^{-1}) &= \mu_1^{-1}\nu\beta\nu^{-1}\mu_1 \\
&= (\nu m\nu^{-1})^{-1}\nu\beta\nu^{-1}(\nu m\nu^{-1}) \\
&= \nu m^{-1}\beta m\nu^{-1} \\
&= \nu m^{-1}m\beta\nu^{-1} \\
&= \nu\beta\nu^{-1}
\end{aligned}
$$

because loops in $\partial N(K_1)$ commute.

We can now turn the set of right cosets $P_\nu \backslash \mathrm{AdQ}_n(Q_n(L))$ into a quandle, which we denote as $(P_\nu \backslash \mathrm{AdQ}_n(Q_n(L)); \mu_1)$ by defining

$$
\begin{aligned}
P_\nu \alpha \triangleright^{\pm 1} P_\nu \beta &= P_\nu \phi_1^{\pm 1}(\alpha \beta^{-1})\beta \\
&= P_\nu \mu_1^{\mp 1}\alpha\beta^{-1}\mu_1^{\pm 1}\beta \\
&= P_\nu \alpha\beta^{-1}\mu_1^{\pm 1}\beta
\end{aligned}
\tag{5}
$$

because $\mu_1 \in P_\nu$.

Claim 4: The quandle operations defined in (5) are well-defined.

Proof: Suppose that $P_\nu \alpha = P_\nu a$ and $P_\nu \beta = P_\nu b$. Then

$$
\begin{aligned}
\alpha\beta^{-1}\mu_1^{\pm 1}\beta(ab^{-1}\mu_1^{\pm 1}b)^{-1} &= \alpha\beta^{-1}\mu_1^{\pm 1}\beta b^{-1}\mu_1^{\mp 1}ba^{-1} \\
&= \alpha\beta^{-1}\beta b^{-1}ba^{-1} \\
&= \alpha a^{-1} \in P_\nu
\end{aligned}
$$

because conjugation by $\mu_1^{\pm 1}$ fixes βb^{-1}, an element of P_ν. Hence $P_\nu \alpha \triangleright^{\pm 1} P_\nu \beta = P_\nu a \triangleright^{\pm 1} P_\nu b$.

Claim 5: The map τ determines a quandle isomorphism between $(P_\nu \backslash \mathrm{AdQ}_n(Q_n(L)); \mu_1)$ and $Q_n^1(L)$.

Proof: Define $\overline{\tau} : (P_\nu \backslash \mathrm{AdQ}_n(Q_n(L)); \mu) \to Q_n^1(L)$ as $\overline{\tau}(P_\nu \alpha) = \tau(\alpha)$. Because $\tau^{-1}(\nu) = P_\nu$, it follows easily that $\overline{\tau}$ is both well-defined and injective. Because τ is onto $Q^1(L)$, we also have that $\overline{\tau}$ is onto $Q_n^1(L)$. Thus $\overline{\tau}$ is a bijection. However, $\overline{\tau}$ is also a quandle homomorphism because

$$
\begin{aligned}
\overline{\tau}(P_\nu \alpha \triangleright P_\nu \beta) &= \overline{\tau}(P_\nu \alpha\beta^{-1}\mu_1^{-1}\beta) \\
&= \tau(\alpha\beta^{-1}\mu_1^{-1}\beta) \\
&= \beta^{-1}\mu_1^{-1}\beta\alpha^{-1}\nu \\
&= (\beta^{-1}\nu)m^{-1}(\beta^{-1}\nu)^{-1}(\alpha^{-1}\nu) \\
&= \tau(\alpha) \triangleright \tau(\beta) \\
&= \overline{\tau}(P_\nu \alpha) \triangleright \overline{\tau}(P_\nu \beta).
\end{aligned}
$$

$\square$

3 Przytycki's Conjecture

In this section we prove the conjecture of Przytycki stated in the abstract.

Theorem 4 *Let L be an oriented link in $\mathbb{S}^3$ and let $\widetilde{M}_n(L)$ be the n-fold cyclic branched cover of $\mathbb{S}^3$, branched over L. Then $Q_n(L)$ is finite, if and only if $\pi_1(\widetilde{M}_n(L))$ is finite.*

Before giving the proof of Theorem 4, we point out the relationship between $\pi_1(\widetilde{M}_n(L))$ and a certain subgroup of $\mathrm{AdQ}_n(Q_n(L))$. The reader is referred to [10] for more details. If $M_n(L)$ is the n-fold cyclic cover of $\mathbb{S}^3 - L$, then $\pi_1(M_n(L))$ is isomorphic to the subgroup E^0 of $\pi_1(\mathbb{S}^3 - L) \cong \mathrm{Adconj}(Q(L))$ consisting of those loops in $\mathbb{S}^3 - L$ that lift to loops in the cover. Equivalently, E^0 consists of loops having total linking number zero with L, that is, those loops α such that the sum of the linking numbers of α with each component of L is zero. The subgroup E^0 can also be described as those elements of $\pi_1(\mathbb{S}^3 - L)$ which, when written as words in the Wirtinger generators, have total exponent sum equal to zero. This concept is well-defined, and defines a subgroup, because each of the relators in the Wirtinger presentation has total exponent sum equal to zero. This last description extends to the quotient group $\mathrm{AdQ}_n(Q_n(L))$. Let E_n^0 be the subgroup of $\mathrm{AdQ}_n(Q_n(L))$ consisting of all elements with total exponent sum equal to zero modulo n. In order to obtain the fundamental group of the cyclic branched cover we must algebraically kill the n-th power of each Wirtinger generator in E^0, hence,

$$\pi_1(\widetilde{M}_n(L)) \cong E_n^0. \tag{6}$$

Notice further, that the index of E_n^0 in $\mathrm{AdQ}_n(Q_n(L))$ is n.

One direction of Theorem 4 follows from work that appears in the Ph.D. thesis of Joyce [6]. For completeness, and because this result does not appear in Joyce's paper [5], we reproduce his proof here (with some modification).

Theorem 5 (Joyce) *If Q_n is finite, then $|\mathrm{AdQ}_n(Q_n)| \leq n^{|Q_n|}$ and hence $\mathrm{AdQ}_n(Q_n)$ is finite.*

Proof: Suppose that Q_n is a finite n-quandle with elements $\{q_1, q_2, \ldots, q_k\}$. Now $\mathrm{AdQ}_n(Q_n)$ is generated by the ordered set of elements $\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_k$ so that every element in $\mathrm{AdQ}_n(Q_n)$ is a word in these generators and their inverses.

Claim 1: If $w = \bar{q}_{i_1}^{\epsilon_1} \bar{q}_{i_2}^{\epsilon_2} \ldots \bar{q}_{i_m}^{\epsilon_m}$, where each exponent is ± 1, then we may rewrite w as $w = \bar{q}_{j_1}^{\eta_1} \bar{q}_{j_2}^{\eta_2} \ldots \bar{q}_{j_m}^{\eta_m}$, where each exponent is ± 1, $j_1 = \min(j_1, j_2, \ldots, j_m)$ and $j_1 \leq \min(i_1, i_2, \ldots, i_m)$.

Proof: Suppose $\bar{q}_{i_k}^{\epsilon_k}$ is the first occurrence of the generator with smallest index and that $k > 1$. Now $q_{i_{k-1}} \triangleright^{\epsilon_k} q_{i_k} = q_t$ for some t and so $\bar{q}_{i_{k-1}}^{\epsilon_{k-1}} \bar{q}_{i_k}^{\epsilon_k} = \bar{q}_{i_k}^{\epsilon_k} \bar{q}_t^{\epsilon_{k-1}}$. If we replace $\bar{q}_{i_{k-1}}^{\epsilon_{k-1}} \bar{q}_{i_k}^{\epsilon_k}$

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with $\overline{q}_{i_k}^{\epsilon_k}\overline{q}_t^{\epsilon_{k-1}}$ in w, then either the first occurrence of the generator with smallest index has moved one place closer to the beginning of w, or a new generator of smaller index was introduced if $t < i_k$. Hence, after a finite number of steps of this kind, the first generator of w will have the smallest index and it will be no greater than any of the indices in the original word.

Claim 2: If $w = \overline{q}_{i_1}^{\epsilon_1}\overline{q}_{i_2}^{\epsilon_2}\ldots\overline{q}_{i_m}^{\epsilon_m}$, where each exponent is ± 1, then we may rewrite w as $w = \overline{q}_{j_1}^{\eta_1}\overline{q}_{j_2}^{\eta_2}\ldots\overline{q}_{j_m}^{\eta_m}$, where each exponent is ± 1 and $j_1 \le j_2 \le \cdots \le j_m$.

Proof: We proceed by induction on m. The case with $m = 2$ is a direct consequence of Claim 1. Assume now that the result is true for words of length m and suppose that $w = \overline{q}_{i_1}^{\epsilon_1}\overline{q}_{i_2}^{\epsilon_2}\ldots\overline{q}_{i_{m+1}}^{\epsilon_{m+1}}$. Applying the inductive hypothesis to the last m generators of w, we may assume that $i_2 \le i_3 \le \cdots \le i_{m+1}$. If $i_1 \le i_2$, we are done. If not, apply Claim 1 to w, which will strictly decrease the index of the first generator in w, and then again apply the inductive hypothesis to the last m generators. This cannot continue forever because the index of the first generator in w cannot decrease below 1.

We may now write any word in $\mathrm{AdQ}_n(Q_n)$ as $\overline{q}_1^{r_1}\overline{q}_2^{r_2}\ldots\overline{q}_k^{r_k}$ and, using the fact that $\overline{q}_i^n = 1$, we may assume that $0 \le r_i < n$ for each i. There are at most $n^k = n^{|Q_n|}$ words of this kind. $\square$

Proof of Theorem 4: Suppose L is an oriented link and $Q_n(L)$ is finite. By Theorem 5, it follows that $\mathrm{AdQ}_n(Q_n(L))$ is finite. Hence the subgroup E_n^0 of $\mathrm{AdQ}_n(Q_n(L))$ is finite and so $\pi_1(\widetilde{M}_n(L))$ is finite by (6).

Now suppose that $\pi_1(\widetilde{M}_n(L))$ is finite. Because E_n^0 has finite index in $\mathrm{AdQ}_n(Q_n(L))$, it follows that $\mathrm{AdQ}_n(Q_n(L))$ is finite. Hence, for each component K_i of L, the set of cosets $P_i\backslash\mathrm{AdQ}_n(Q_n(L))$ is finite and therefore, by Theorem 1, each component $Q_n^i(L)$ of $Q_n(L)$ is finite. $\square$

4   Examples

From the proof of Theorem 4, all information about the knot invariant $Q_n(L)$ is encoded by the cosets of the subgroups P_i in the group $\mathrm{AdQ}_n(Q_n(L))$. For example, $Q_n(L)$ is finite if and only if

$$\sum_{i=1}^{s} [\mathrm{AdQ}_n(Q_n(L)) : P_i]$$

is finite. Algorithmically computing the index of P_i in the group $\mathrm{AdQ}_n(Q_n(L))$ from a presentation of the group is a well-known problem in computational group theory. The first process to accomplish this task was introduced by Todd and Coxeter in 1936 [9] and is now a fundamental method in computational group theory. In addition to determining the index

(if it is finite), the Todd-Coxeter process also provides a Cayley diagram that represents the action of right-multiplication on the cosets. In this section we will apply the Todd-Coxeter process to several examples and determine the quandle multiplication table from the Cayley diagram of the cosets. More detailed treatments of the Todd-Coxeter process can be found in [3] and [4].

Consider the right-hand trefoil knot K and fix $n = 3$. From the Wirtinger presentation we obtain the presentation

$$\mathrm{AdQ}_3(Q_3(K)) = \langle x, y \mid x^3 = 1, y^3 = 1, x^{-1}y^{-1}xyxy^{-1} = 1 \rangle.$$

A meridian for K is $\mu = x$ and a (nonpreferred) longitude is $\lambda = yxxy$. The Todd-Coxeter process produces a *coset table* whose rows are numbered by indices $\alpha \in \{1, 2, \ldots, \kappa\}$ that represent cosets of P. The columns are labeled by the generators and their inverses and encode the action of $\mathrm{AdQ}_3(Q_3(K))$ on the cosets by right-multiplication. An additional column will be added to give a representative $\phi(\alpha) \in \mathrm{AdQ}_3(Q_3(K))$ of coset α.

We initialize the coset table by letting 1 represent the trivial coset P, thus $\phi(1) = e$ is a representative of this coset (we use e here for the identity element of $\mathrm{AdQ}_3(Q_3(K))$ to avoid confusion). Since $\mu = x \in P$, we have $Px = P$, this information is encoded in a *helper table* where P is represented by index 1 and is encoded in the coset table as a relation $1x = 1$. Of course, it follows from this that $1x^{-1} = 1$ as well, so there are two defined entries in row 1 of the coset table.

	x	y	x^{-1}	y^{-1}	ϕ
1	1		1		e

	x	
	1	1

Since $\lambda = yxxy \in P$ we also produce a helper table to encode $1yxxy = 1$. Additional entries in the table are required to represent the cosets $1y$, $1yx$, and $1yxx$. These entries are defined by adding indices 2, 3, and 4, respectively, and adding additional information to the coset table for these indices coming from the helper table. For example, 2 is defined to be the coset $1y$ and, thus, $1y = 2$ and $2y^{-1} = 1$ are encoded in the coset table. At this point a *deduction* also occurs. Since $1yxxy = 1$, we see in the helper table that $4y = 1$.

	x	y	x^{-1}	y^{-1}	ϕ
1	1	2	1	4	e
2	3			1	y
3	4		2		yx
4		1	3		yx^2

	y	x	x	y	
	1	2	3	4	1

This completes the initial set up of the coset table and is referred to as *scanning* the generators of P. The Todd-Coxeter process next proceeds to scan the relations of $\mathrm{AdQ}_3(Q_3(K))$ for all indices. This encodes the fact that if α is any coset and $w = e \in \mathrm{AdQ}_3(Q_3(K))$, then $\alpha w = \alpha$ in the coset table since $P\phi(\alpha)w = P\phi(\alpha)$ in $\mathrm{AdQ}_3(Q_3(K))$. We scan the three

relations $x^3 = e$, $y^3 = e$, and $x^{-1}y^{-1}xyxy^{-1} = e$, in this order, for each index, defining new indices and obtaining new deductions along the way.

Scanning x^3 for $\alpha = 1$ gives no new information. Scanning y^3 gives no new definitions but does produce the deduction $2y = 4$ and scanning $x^{-1}y^{-1}xyxy^{-1}$ defines the indices 5 and 6 as shown in the coset tables below.

	x	y	x^{-1}	y^{-1}	ϕ
1	1	2	1	4	e
2	3	4		1	y
3	4		2		yx
4		1	3	2	yx^2

y	y	y	
1	2	4	1

	x	y	x^{-1}	y^{-1}	ϕ
1	1	2	1	4	e
2	3	4	6	1	y
3	4		2		yx
4	5	1	3	2	yx^2
5		6	4		yx^3
6	2			5	yx^3y

	x^{-1}	y^{-1}	x	y	x	y^{-1}
1	1	4	5	6	2	1

At this point we see that the representative for coset 5 is $\phi(5) = yx^3$. Since $x^3 = e$ in the group $\phi(5) = yx^3 = y = \phi(2)$ and so the cosets 5 and 2 are the same. This information is determined by a *coincidence* which occurs when scanning x^3 for $\alpha = 2$. Filling in the entries of the helper table from left to right, $2x = 3$, $3x = 4$, $4x = 5$. However we require $2xxx = 2$ thus we see that $5 = 2$. In the coset table we process this coincidence by replacing all values of 5 with 2, merging the data from row 5 into row 2, and then deleting row 5. In merging the data from 5 to 2 we see a new coincidence, namely $6 = 4$ and so we repeat the coincidence procedure for $6 = 4$ before moving on to the next scan.

	x	y	x^{-1}	y^{-1}	ϕ
1	1	2	1	4	e
2	3	4	~~6~~4	1	y
3	4		2		yx
4	~~5~~2	1	3	2	yx^2
~~5~~		~~6~~	~~4~~		yx^3
~~6~~	~~2~~			~~5~~~~2~~	yx^3y

x	x	x	
2	3	4	$5 = 2$

Scanning $x^{-1}y^{-1}xyxy^{-1}$ for $\alpha = 2$ completes the table. The process terminates after the table is complete and all relations have been scanned for all indices. In our example, no additional coincidences occur and the completed table is shown below.

	x	y	x^{-1}	y^{-1}	ϕ
1	1	2	1	4	e
2	3	4	4	1	y
3	4	3	2	3	yx
4	2	1	3	2	yx^2

Table 1: Completed coset table for $P\backslash \mathrm{AdQ}_3(Q_3(K))$ where K is the trefoil knot

It is important to note that the operation encoded by the coset table is that of right-multiplication. It is not the operations of $\rhd^{\pm 1}$ in the quandle $P\backslash \mathrm{AdQ}_3(Q_3(K))$. The multiplication table for the quandle can be easily worked out, however, from the coset table and the definition of the operations $Pg \rhd^{\pm 1} Ph = Pgh^{-1}x^{\pm 1}h$ since $\mu = x$. From the completed coset table, the quandle $Q_3(K)$ has four elements P, Py, Pyx, and Pyx^2. So, for example, $Py \rhd Pyx = Pyx^{-1}y^{-1}xyx$. This coset is represented by $1yx^{-1}y^{-1}xyx = 4$ in the coset table. Therefore, $Py \rhd Pyx = Pyx^2$. The full multiplication table for $Q_3(K)$ is given below.

$\rhd$	P	Py	Pyx	Pyx^2
P	P	Pyx^2	Py	Pyx
Py	Pyx	Py	Pyx^2	P
Pyx	Pyx^2	P	Pyx	Py
Pyx^2	Py	Pyx	P	Pyx^2

Table 2: The multiplication table for $Q_3(K)$ where K is the trefoil knot.

Applying the Todd-Coxeter method in the case of the trefoil for $n = 2, 3, 4, 5$, enumerating the cosets of both the trivial subgroup as well as $P = \langle \mu, \lambda \rangle$, we obtain the data in Table 4. These calculations agree with the well known fact that $\pi_1(\widetilde{M}_n(K))$ for the trefoil with $n = 2, 3, 4$, or 5 is, respectively, the cyclic group of order 3, the quaternion group of order 8, the binary tetrahedral group of order 24, and the binary icosahedral group of order 120. See [7].

n	$|P|$	$|Q_n(K)|$	$|\mathrm{AdQ}_n(Q_n(K))|$	$|\pi_1(\widetilde{M}_n(K))|$
2	2	3	6	3
3	6	4	24	8
4	16	6	96	24
5	50	12	600	120

Table 3: The order of $\mathrm{AdQ}_n(Q_n(K))$ and index of P for the right-handed trefoil.

As another example, consider the $(2, 2, 3)$-pretzel link L and fix $n = 2$. Starting with the standard pretzel diagram with Wirtinger generators x, y, z, we obtain the following presen-

tation of $\mathrm{AdQ}_2(Q_2(L))$.

$$\langle x, y, z \mid x^{-1}z^{-1}xzxy^{-1}x^{-1}y = 1, y^{-1}x^{-1}yxyzyz^{-1}y^{-1}z^{-1} = 1,$$
$$y^{-1}x^{-1}yxyzyz^{-1}y^{-1}x^{-1}y^{-1}xyx^{-1}z^{-1}z = 1, x^2 = 1, y^2 = 1, z^2 = 1\rangle$$

with $P_1 = \langle x, x^{-1}zxy^{-1}\rangle$ and $P_2 = \langle y, y^{-1}x^{-1}yzyz^{-1}xzyzy^{-1}\rangle$. Applying the Todd-Coxeter process gives $|\mathrm{AdQ}_2(Q_2(L))| = 96$, $|Q_2^1(L)| = 8$, and $|Q_2^2(L)| = 24$. These calculations agree with those given in [2] where it is shown using Winker's diagramming method [10] that if L is the Montesinos link of the form $(1/2, 1/2, p/q; e)$ and q is odd, then $|Q_2^1(L)| = 2|(e-1)q - p|$ and $|Q_2^2(L)| = 2q|(e-1)q - p|$.

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