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Hom Quandles

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Abstract

If A is an abelian quandle and Q is a quandle, the hom set $\mathrm{Hom}(Q, A)$ of quandle homomorphisms from Q to A has a natural quandle structure. We exploit this fact to enhance the quandle counting invariant, providing an example of links with the same counting invariant values but distinguished by the hom quandle structure. We generalize the result to the case of biquandles, collect observations and results about abelian quandles and the hom quandle, and show that the category of abelian quandles is symmetric monoidal closed.

Keywords: Quandles, biquandles, abelian quandles, abelian biquandles, enhancements of counting invariants

2010 MSC: 57M27, 57M25

1 Introduction

Ronnie Brown said it best when declaring, "One of the irritations of group theory is that the set $\mathrm{Hom}(H,K)$ of homomorphisms between groups H and K does not have a natural group structure." Of course, when H and K are both commutative, we know that $\mathrm{Hom}(H, K)$ is also a commutative group. Since quandles are algebraic structures having groups as their primordial example, it is natural to wonder when, if ever, the set of quandle homomorphisms from a quandle X to a quandle X', $\mathrm{Hom}(X, X')$, possesses additional structure. It was shown in [3] that if $Q(K)$ is the fundamental quandle of a knot and X is an Alexander quandle, the set $\mathrm{Hom}(Q(K), X)$ has an Alexander quandle structure. We generalize this result to show that the set $\mathrm{Hom}(Q(K), X)$ has a quandle structure provided the target quandle X is an 'abelian', or 'medial', quandle. Moreover, for links with two or more components, the resulting quandle structure is not determined by the cardinality of the target quandle and is thus a stronger invariant than the counting invariant $|\mathrm{Hom}(Q(K), X)|$.

This paper is organized as follows. In Section 3 we recall the basics of quandles, including definitions and examples. In Section 4 we turn our focus to the case of abelian quandles, also called medial quandles. Then, in Section 5, specifically in Proposition 4, we define a natural quandle structure on the set of quandle homomorphisms from an arbitrary quandle to a finite abelian quandle. We continue in this section by illustrating properties of the hom quandle, $\mathbf{Hom}(Q,A)$, including those inherited from the quandle A. In Section 6 we apply the results of Section 5 to define an enhanced invariant of links associated to finite abelian quandles and in Section 7 we generalize the results of previous sections to the case of biquandles. In Section 8 we consider the category of abelian quandles and show that it is symmetric monoidal closed, and we conclude in Section 9 with some questions for future work.

2 Acknowledgements

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3 Quandle Basics

We begin with a definition from [4].

Definition 1 A *quandle* is a set X equipped with a binary operation $\triangleright : Q \times X \to X$ satisfying

(i) (idempotence) for all $x \in X$, $x \triangleright x = x$,

(ii) (inverse) for all $x, y \in X$, there is a unique $z \in X$ with $x = z \triangleright y$, and

(iii) (self-distributivity) for all $x, y, z \in X$, $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

The quandle axioms capture the essential properties of group conjugation, and correspond to the Reidemeister moves on oriented link diagrams with elements corresponding to arcs and the quandle operation $\triangleright$ corresponding to crossings with $x \triangleright y$ the result of x crossing under y from right to left. In particular, the operation is distinctly non-symmetrical – in $x \triangleright y$, y is acting on x and not conversely – and thus it makes sense to use a non-symmetrical symbol like $\triangleright$.

Axiom (i) says that every element of X is idempotent under $\triangleright$. Axiom (ii) says that the action of y on X defined by $f_y(x) = x \triangleright y$ is bijective for every $y \in X$. Hence there are inverse actions, denoted by $f_y^{-1}(x) = x \triangleright^{-1} y$, and Axiom (ii) is equivalent to

(ii$'$) there is an *inverse*, or *dual*, operation $\triangleright^{-1} : X \times X \to X$ satisfying for all $x, y \in X$

$$(x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y.$$

Thus, we can eliminate the existential quantifier in Axiom (ii) at the cost of adding a second operation. It is a straightforward exercise to show that the inverse operation is also idempotent and self-distributive, so X is a quandle under $\triangleright^{-1}$, known as the *dual quandle* of $(X, \triangleright)$. One can also show that the two triangle operations distribute over each other, i.e. we have

$$(x \triangleright y) \triangleright^{-1} z = (x \triangleright^{-1} z) \triangleright (y \triangleright^{-1} z) \quad \text{and} \quad (x \triangleright^{-1} y) \triangleright z = (x \triangleright z) \triangleright^{-1} (y \triangleright z).$$

Axiom (iii) says that the quandle operation is self-distributive. This axiom then implies that the action maps $f_y : X \to X$ are endomorphisms of the quandle structure:

$$f_z(x \triangleright y) = (x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z) = f_z(x) \triangleright f_z(y).$$

Indeed, Axioms (ii) and (iii) together say that the action of any element $y \in X$ is always an automorphism of X. If $(X, \triangleright)$ satisfies only (ii) and (iii), then X is called a *rack* or *automorphic set*; a quandle is then a rack in which every element is a fixed point of its own action.

A quandle is *involutory* if $\triangleright = \triangleright^{-1}$, i.e. if for all x and y in X we have $(x \triangleright y) \triangleright y = x$. Involutory quandles are also known as *kei* or 圭; see [8, 10] for more details.

Example 1 Any module over $\mathbb{Z}[t^{\pm 1}]$ is a quandle with operations

$$x \triangleright y = tx + (1-t)y \quad \text{and} \quad x \triangleright^{-1} y = t^{-1}x + (1-t^{-1})y.$$

Such a quandle is called an *Alexander quandle*. Any module over $\mathbb{Z}[t]/(t^2 - 1)$ is a kei under

$$x \triangleright y = tx + (1-t)y$$

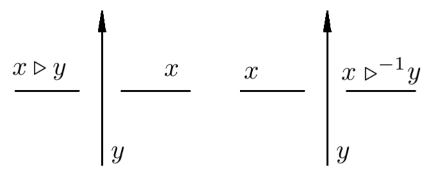
known as an *Alexander kei*.

Example 2 Any vector space V over a field k is a quandle under the operations

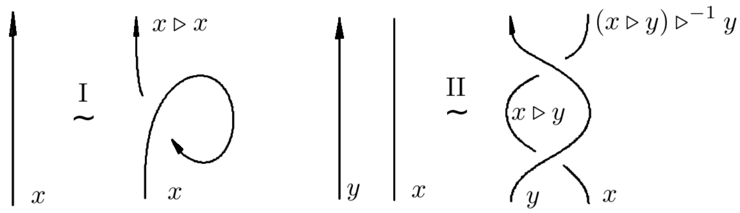
$$\vec{x} \triangleright \vec{y} = \vec{x} + \langle \vec{x}, \vec{y} \rangle \vec{y} \quad \text{and} \quad \vec{x} \triangleright^{-1} \vec{y} = \vec{x} - \langle \vec{x}, \vec{y} \rangle \vec{y}$$

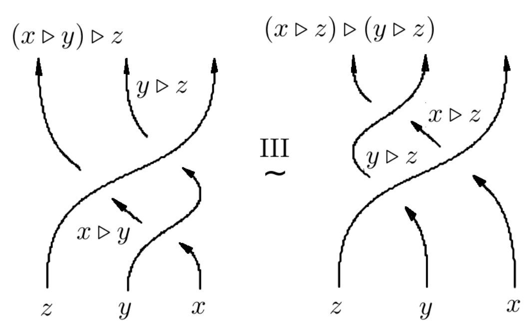
where $\langle , \rangle : V \times V \to k$ is an antisymmetric bilinear form (if the characterisitic of k is 2, then we also require $\langle \vec{x}, \vec{x} \rangle = 0$ for all $\vec{x} \in V$). Such a quandle is called a *symplectic quandle* [7].

As briefly mentioned already, the quandle axioms can be understood as arising from the oriented Reidemeister moves where quandle elements are associated to arcs in a knot or link diagram and the quandle operation $x \triangleright y$ is interpreted as arc x crossing under arc y from the right. We note that the orientation of the undercrossing arc is not relevant, but only the orientation of the overcrossing arc. The inverse triangle operation from Axiom (ii′) can be interpreted as the understrand crossing backwards from left to right as illustrated below:



Then the quandle axioms are exactly the conditions required for the diagrams to match up one-to-one before and after the Reidemeister moves.



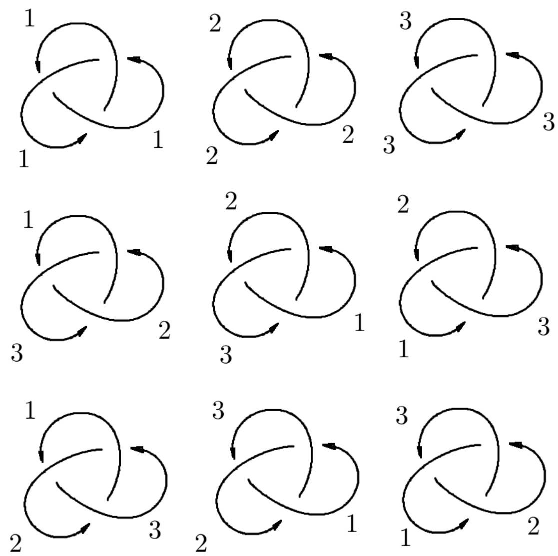


Given an oriented knot or link K, the *knot quandle*, denoted $Q(K)$, is the quandle with generators corresponding to arcs in a diagram of K and relations given by the crossings. More precisely, the elements of the knot quandle are equivalence classes of quandle words in the generators under the equivalence relation generated by the crossing relations and the quandle axioms.

We will find it convenient to specify quandle structures on a finite sets $X = \{x_1, x_2, \ldots, x_n\}$ using an $n \times n$ matrix encoding the quandle operation table. In particular, the entry in row i column j of the quandle matrix is k where $x_k = x_i \triangleright x_j$. Then, for example, the Alexander quandle structure on $\mathbb{Z}_3 = \{1, 2, 3\}$ (we use 3 for the class of zero so we can number our rows and columns starting with 1) with quandle operation $x \triangleright y = 2x + 2y$ has matrix

$$M = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

Given a knot K and a finite quandle X, the cardinaility of the set of quandle homomorphisms (maps $f : Q(K) \to X$ such that $f(x \triangleright y) = f(x) \triangleright f(y)$) is a computable knot invariant known as the *quandle counting invariant*. A quandle homomorphism $f : Q(K) \to X$ corresponds to a labeling of the arcs in a diagram of K with elements of X such that the crossing relations are all satisfied. For instance, the trefoil knot 3_1 has nine colorings by the quandle structure on $\mathbb{Z}_3$ listed above, as shown below:



Hence, the quandle counting invariant here is $|\mathrm{Hom}(Q(3_1), \mathbb{Z}_3)| = 9$.[1]

4 Abelian Quandles

We now turn our attention to a special class of quandles known as 'abelian' quandles. The reader should be aware that, unlike in the group case, the adjective "abelian" is not synonymous with "commutative." Abelian quandles satisfy the condition below whereas commutative quandles satisfy $a \triangleright b = b \triangleright a$.

Definition 2 A quandle Q is *abelian* if for all $x, y, z, w \in Q$ we have

$$(x \triangleright y) \triangleright (z \triangleright w) = (x \triangleright z) \triangleright (y \triangleright w).$$

Abelian quandles are also called *medial* quandles.

Example 3 Alexander quandles are abelian:

$$\begin{aligned}
(x \triangleright y) \triangleright (z \triangleright w) &= t(tx + (1-t)y) + (1-t)(tz + (1-t)w) \\
&= t^2 x + t(1-t)y + t(1-t)z + (1-t)^2 w \\
&= t(tx + (1-t)z) + (1-t)(ty + (1-t)w) \\
&= (x \triangleright z) \triangleright (y \triangleright w)
\end{aligned}$$

Example 4 However, not all abelian quandles are Alexander. The quandle Q_2 with operation table

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 3 & 3 & 3 \end{bmatrix}$$

is abelian as can be verified by checking that $(a \triangleright b) \triangleright (c \triangleright d) = (a \triangleright c) \triangleright (b \triangleright d)$ for $a, b, c, d \in \{1, 2, 3\}$; however, Q_2 is not isomorphic to any Alexander quandle by the following lemma.

[1]The reader may recognize these as Fox 3-colorings.

Lemma 1 *If Q is an Alexander quandle containing an element $y \in Q$ which acts trivially on Q, i.e. if $x \triangleright y = x$ for all $x \in Q$, then Q is isomorphic to the trivial quandle on $|Q|$ elements.*

Proof. Suppose y acts trivially on Q, so that $x \triangleright y = x$ for all $x \in Q$. Then

$$0 = x - (x \triangleright y) = x - tx - (1-t)y = (1-t)(x-y)$$

for all $x \in Q$. In particular, since every $x \in Q$ has the form $x = (x+y) - y$, the map $(1-t) : Q \to Q$ is the zero map, so $t : Q \to Q$ is the identity map. Then

$$x \triangleright z = tx + (1-t)z = 1x + 0z = x$$

and the quandle operation on Q is trivial. $\square$

Example 5 Unlike Alexander quandles, symplectic quandles are generally non-abelian. Consider the symplectic quandle structure on $(\mathbb{Z}_2)^2$ defined by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \triangleright \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left(\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

This four-element quandle has operation matrix

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & 3 \\ 3 & 4 & 3 & 2 \\ 4 & 3 & 2 & 4 \end{bmatrix}$$

where

$$x_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad x_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It is easy to see from the table that this quandle is not abelian; for instance, we have

$$(2 \triangleright 4) \triangleright (1 \triangleright 2) = 3 \triangleright 1 = 3$$

but

$$(2 \triangleright 1) \triangleright (4 \triangleright 2) = 2 \triangleright 3 = 4 \neq 3.$$

We also note that this quandle is a kei, so this example also shows that kei need not be abelian.

Lemma 2 *In addition to right-distributivity, the operation in an abelian quandle is left-distributive.*

Proof. If A is an abelian quandle, then for all $x, y, z \in A$ we have

$$x \triangleright (y \triangleright z) = (x \triangleright x) \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z).$$

$\square$

5 Hom Quandles

We now wish to study the structure of the set of quandle homomorphisms $\mathrm{Hom}(Q, A)$ where Q is any quandle and A is an abelian quandle.

Theorem 3 *Let Q and A be quandles. If A is abelian, then the set of quandle homomorphisms* $\mathrm{Hom}(Q, A)$ *is a quandle under the pointwise operation $(f \triangleright g)(q) = f(q) \triangleright g(q)$. Moreover,* $\mathrm{Hom}(Q, A)$ *is abelian.*

Proof.

For Axiom (i), we have

$$(f \triangleright f)(q) = f(q) \triangleright f(q) = f(q).$$

For Axiom (ii), define $(f \triangleright^{-1} g)(q) = f(q) \triangleright^{-1} g(q)$ in A. Then we have

$$((f \triangleright g) \triangleright^{-1} g)(q) = (f(q) \triangleright g(q)) \triangleright^{-1} g(q) = f(q).$$

For Axiom (iii), we have

$$\begin{aligned}
((f \triangleright g) \triangleright h)(q) &= (f \triangleright g)(q) \triangleright h(q) \\
&= (f(q) \triangleright g(q)) \triangleright h(q) \\
&= (f(q) \triangleright h(q)) \triangleright (g(q) \triangleright h(q)) \\
&= (f \triangleright h)(q) \triangleright (g \triangleright h)(q) \\
&= [(f \triangleright h) \triangleright (g \triangleright h)](q).
\end{aligned}$$

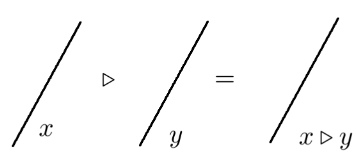
To show that $\mathrm{Hom}(Q, A)$ is abelian, let $f, g, h, k \in \mathrm{Hom}(Q, A)$. Then

$$\begin{aligned}
[(f \triangleright g) \triangleright (h \triangleright k)](q) &= (f \triangleright g)(q) \triangleright (h \triangleright k)(q) \\
&= (f(q) \triangleright g(q)) \triangleright (h(q) \triangleright k(q)) \\
&= (f(q) \triangleright h(q)) \triangleright (g(q) \triangleright k(q)) \\
&= (f \triangleright h)(q) \triangleright (g \triangleright k)(q) \\
&= [(f \triangleright h) \triangleright (g \triangleright k)](q)
\end{aligned}$$

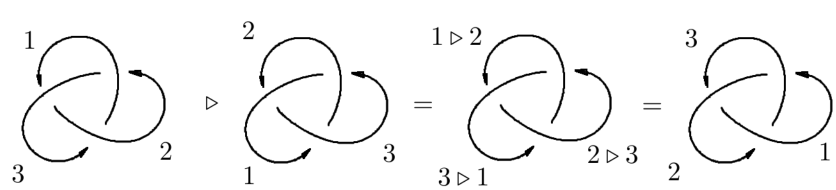
□

Remark 6 Henceforth, we will use $\mathrm{Hom}(Q, A)$ to denote the set of quandle homomorphisms and $\mathbf{Hom}(Q, A)$ to denote the quandle.

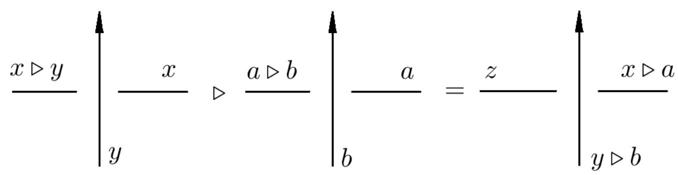
When the domain quandle Q is a knot quandle, we can interpret the pointwise quandle operation in terms of knot diagrams. In particular, $\mathbf{Hom}(Q(K), A)$ can be represented as the set of A-labelings of a fixed diagram D of A; the $\triangleright$ operation on diagrams is then given by the pointwise operation on the arc labels, i.e.



Example 7 In $\mathbf{Hom}(Q(3_1), \mathbb{Z}_3)$ we have



Then at any crossing, we have



On the one hand, we have $z = (x \triangleright y) \triangleright (a \triangleright b)$ by definition of the $\triangleright$ operation on diagrams; on the other hand, for the labeling to be a valid quandle labeling, we must have $z = (x \triangleright a) \triangleright (y \triangleright b)$. Thus, we have

Proposition 4 *The set* $\mathrm{Hom}(Q(K), A)$ *forms a quandle under the pointwise* $\triangleright$ *operation if and only if* $(x \triangleright y) \triangleright (a \triangleright b) = (x \triangleright a) \triangleright (y \triangleright b)$*, i.e., if and only if* A *is abelian.*

The hom quandle $\mathbf{Hom}(Q, A)$ inherits many properties held by the target quandle A.

Theorem 5 *Let* Q *and* A *be quandles, where* A *is abelian. If* A *is commutative or involutory, then* $\mathbf{Hom}(Q, A)$ *is also commutative or involutory.*

Proof. This follows from a straightforward computation. □

An additional observation:

Theorem 6 *Let* Q *be a quandle and* $A \cong A'$ *be abelian quandles. Then* $\mathbf{Hom}(Q, A) \cong \mathbf{Hom}(Q, A')$.

Proof. This follows from a straightforward computation. □

Theorem 7 *Let* Q *be a finitely generated quandle and* A *a finite abelian quandle. Then* $\mathbf{Hom}(Q, A)$ *contains a subquandle isomorphic to* A.

Proof. Define maps $f_a : Q \to A$ by $f_a(x) = a$ for all $x \in Q$ and consider the map $\phi : A \to \mathbf{Hom}(Q, A)$ defined by $\phi(a) = f_a$. First, note that ϕ is a homomorphism of quandles since for any $a, b \in A$ we have

$$\phi(a \triangleright b) = f_{a \triangleright b} \quad \text{and} \quad \phi(a) \triangleright \phi(b) = f_a \triangleright f_b$$

Then for any $x \in Q$, we have

$$(f_a \triangleright f_b)(x) = a \triangleright b = f_{a \triangleright b}(x)$$

as required. Further, ϕ is injective since $\phi(a) = \phi(b)$ implies $f_a = f_b$ which implies $a = b$. Then the image subquandle $\mathrm{Im}(\phi) \subset \mathbf{Hom}(Q, A)$ is isomorphic to A. □

More generally, we have

Theorem 8 *Let* Q *be a finitely generated quandle and* A *an abelian quandle. Then* $\mathbf{Hom}(Q, A)$ *is isomorphic to a subquandle of* A^c *where* c *is minimal number of generators of* Q.

Proof. Let $q_1, \ldots, q_c$ be a set of generators of Q with minimal cardinality. Any homomoprhism $f : Q \to A$ must send each q_k to an element $f(q_k)$ in A, and such an assignment of images to generators defines a quandle homomorphism if and only if the relations in Q are satisfied by the assignment, i.e. if and only if

$$f(q_j \triangleright q_k) = f(q_j) \triangleright f(q_k)$$

for all $1 \leq j, k \leq c$. Then the elements of $\mathbf{Hom}(Q, A)$ can be identified with the subset of A^c consisting of c-tuples of images of generators under f satisfying the relations in Q, i.e.

$$f \leftrightarrow (f(q_1), f(q_2), \ldots, f(q_c)).$$

The pointwise operation in $\mathbf{Hom}(Q, A)$ agrees with the componentwise operation in the Cartesian product A^c,

$$\begin{aligned}((f \triangleright g)(q_1), \ldots, (f \triangleright g)(q_c)) &= (f(q_1) \triangleright g(q_1), \ldots, f(q_c) \triangleright g(q_c)) \\ &= (f(q_1), \ldots, f(q_c)) \triangleright (g(q_1), \ldots, g(q_c))\end{aligned}$$

so $\mathbf{Hom}(Q, A)$ is isomorphic to the subquandle of A^c consisting of c-tuples satisfying the relations of Q. $\square$

In the simplest case, we can identify the structure of the hom quandle. Recall that the *trivial quandle of n elements* is a set T_n of cardinality n with quandle operation $x \triangleright y = x$ for all $x, y \in X$. That is, the trivial quandle has quandle matrix

$$M_{T_n} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 2 & 2 & \ldots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ n & n & \ldots & n \end{bmatrix}.$$

Lemma 9 *Any map between trivial quandles is a quandle homomorphism.*

Proof. Let $f : T_n \to T_m$ be a map between trivial quandles T_n and T_m. Then for any $x_i, x_j \in X$, we have $f(x_i \triangleright x_j) = f(x_i)$ and $f(x_i) \triangleright f(x_j) = f(x_i)$. Thus

$$f(x_i \triangleright x_j) = f(x_i) = f(x_i) \triangleright f(x_j)$$

and f is a quandle homomoprhism. $\square$

Theorem 10 *Let T_n and T_m be the trivial quandles of orders n and m, respectively. Then* $\mathbf{Hom}(T_n, T_m) \cong T_{m^n}$.

Proof. Let $T_n = \{x_1, \ldots, x_n\}$ and $T_m = \{y_1, \ldots, y_m\}$. Any map $f : T_n \to T_m$ can be encoded as a vector

$$f = (f(x_1), f(x_2), \ldots f(x_n))$$

and there are m^n such maps. Indeed, every such map is quandle homomorphism by Lemma 9.

Now, define $\phi : \mathbf{Hom}(T_n, T_m) \to T_{m^n}$ by

$$\phi(y_1, \ldots, y_n) = \sum_{k=1}^{n} y_k m^{k-1}.$$

Then ϕ is a bijection; the inverse map rewrites $x \in \{1, 2, \ldots, m^n\}$ in base-m. The quandle structure on $\mathbf{Hom}(T_n, T_m)$ is defined by

$$(f \triangleright g)(x) = f(x) \triangleright g(x) = f(x),$$

so we have $f \triangleright g = f$ for all $f, g \in \mathbf{Hom}(T_n, T_m)$ and $\mathbf{Hom}(T_n, T_m)$ is a trivial quandle. Then by Lemma 9, ϕ is an isomorphism of quandles. $\square$

Remark 8 The condition that $\mathbf{Hom}(Q, A) \cong A^c$ where c is the minimal number of generators of A is not limited to trivial quandles. For instance, the quandle $\mathbf{Hom}(Q(3_1), R_3)$ is isomorphic to $(R_3)^2$, where $Q(3_1)$ is the fundamental quandle of the trefoil knot and $R_3 = \mathbb{Z}_3[t]/(t-2)$ is the connected quandle of 3 elements; we note that $Q(3_1)$ has a presentation with two generators (as do all 2-bridge knots). In the next section, we show that $\mathbf{Hom}(Q, A)$ need not be isomorphic to A^c for links with quandle generator index c.

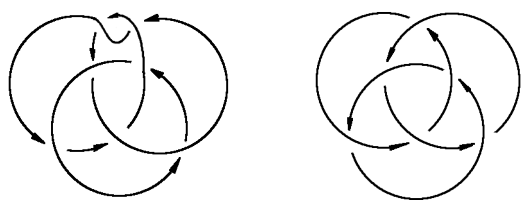
6 Hom Quandle Enhancement

Recall that for any oriented knot K and finite quandle A, the cardinality of the hom set $\mathrm{Hom}(Q(K), A)$ is a computable knot invariant. As we have seen, if A is abelian then the hom set is not just a set but a quandle. The natural question is then whether the hom quandle is a stronger invariant than the counting invariant. In general, an invariant which determines the counting invariant is an *enhancement* of the counting invariant, and if there are examples in which the enhancement distinguishes knots or links which have the same counting invariant, we say the enhancement is a *proper enhancement*. Thus, we would like to know whether the hom quandle is a proper enhancement. It turns out, the answer is yes:

Example 9 Let A be the quandle defined by the quandle matrix

$$M_a = \begin{bmatrix} 1 & 4 & 4 & 1 \\ 3 & 2 & 2 & 3 \\ 2 & 3 & 3 & 2 \\ 4 & 1 & 1 & 4 \end{bmatrix}$$

and consider the links $L6a1$ and $L6a5$ on the Thistlethewaite link table on the knot atlas [1].



$L6a1$ $L6a5$

Our `Python` computations show that while both hom quandles have the same cardinality, namely 16, the hom quandles are not isomorphic.

$$M_{\mathbf{Hom}(Q(L6a1),A)} = \begin{bmatrix}
1 & 1 & 2 & 2 & 16 & 16 & 15 & 15 & 15 & 15 & 16 & 16 & 2 & 2 & 1 & 1 \\
2 & 2 & 1 & 1 & 15 & 15 & 16 & 16 & 16 & 16 & 15 & 15 & 1 & 1 & 2 & 2 \\
4 & 4 & 3 & 3 & 13 & 13 & 14 & 14 & 14 & 14 & 13 & 13 & 3 & 3 & 4 & 4 \\
3 & 3 & 4 & 4 & 14 & 14 & 13 & 13 & 13 & 13 & 14 & 14 & 4 & 4 & 3 & 3 \\
12 & 12 & 11 & 11 & 5 & 5 & 6 & 6 & 6 & 6 & 5 & 5 & 11 & 11 & 12 & 12 \\
11 & 11 & 12 & 12 & 6 & 6 & 5 & 5 & 5 & 5 & 6 & 6 & 12 & 12 & 11 & 11 \\
9 & 9 & 10 & 10 & 8 & 8 & 7 & 7 & 7 & 7 & 8 & 8 & 10 & 10 & 9 & 9 \\
10 & 10 & 9 & 9 & 7 & 7 & 8 & 8 & 8 & 8 & 7 & 7 & 9 & 9 & 10 & 10 \\
7 & 7 & 8 & 8 & 10 & 10 & 9 & 9 & 9 & 9 & 10 & 10 & 8 & 8 & 7 & 7 \\
8 & 8 & 7 & 7 & 9 & 9 & 10 & 10 & 10 & 10 & 9 & 9 & 7 & 7 & 8 & 8 \\
6 & 6 & 5 & 5 & 11 & 11 & 12 & 12 & 12 & 12 & 11 & 11 & 5 & 5 & 6 & 6 \\
5 & 5 & 6 & 6 & 12 & 12 & 11 & 11 & 11 & 11 & 12 & 12 & 6 & 6 & 5 & 5 \\
14 & 14 & 13 & 13 & 3 & 3 & 4 & 4 & 4 & 4 & 3 & 3 & 13 & 13 & 14 & 14 \\
13 & 13 & 14 & 14 & 4 & 4 & 3 & 3 & 3 & 3 & 4 & 4 & 14 & 14 & 13 & 13 \\
15 & 15 & 16 & 16 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 16 & 16 & 15 & 15 \\
16 & 16 & 15 & 15 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 15 & 15 & 16 & 16
\end{bmatrix}$$

$$M_{\mathbf{Hom}(Q(L6a5),A)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 4 & 4 & 4 & 4 \\ 12 & 12 & 12 & 12 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 12 & 12 & 12 & 12 \\ 11 & 11 & 11 & 11 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 11 & 11 & 11 & 11 \\ 10 & 10 & 10 & 10 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 10 & 10 & 10 & 10 \\ 9 & 9 & 9 & 9 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 9 & 9 & 9 & 9 \\ 8 & 8 & 8 & 8 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 8 & 8 & 8 & 8 \\ 7 & 7 & 7 & 7 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 7 & 7 & 7 & 7 \\ 6 & 6 & 6 & 6 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 6 & 6 & 6 & 6 \\ 5 & 5 & 5 & 5 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 5 & 5 & 5 & 5 \\ 13 & 13 & 13 & 13 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 13 & 13 & 13 & 13 \\ 14 & 14 & 14 & 14 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 14 & 14 & 14 & 14 \\ 15 & 15 & 15 & 15 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 15 & 15 & 15 & 15 \\ 16 & 16 & 16 & 16 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 16 & 16 & 16 & 16 \end{bmatrix}$$

Recall from [9] that if X is a quandle, then the polyomial

$$\phi(X) = \sum_{x \in X} s^{r(x)} t^{c(x)}$$

where $r(x) = |\{y \in X \ : \ x \triangleright y = x\}|$ and $c(x) = |\{y \in X \ : \ y \triangleright x = y\}|$ is an invariant of quandle isomorphism type known as the *quandle polynomial* of X. Then we have

$$\phi(\mathbf{Hom}(Q(L6a1), A)) = 16s^4t^4 \neq 16s^8t^8 = \phi(\mathbf{Hom}(Q(L6a5), A))$$

and the quandles are not isomorphic.

7 Abelian Biquandles

If we think of quandles as the algebraic structure encoding the oriented Reidemeister moves where the inbound overcrossing arc act on the inbound undercrossing arc at a crossing, it natural to ask what algebraic structure results when we allow both inbound *semiarcs*, i.e. portions of the knot divided at both over and undercrossings, to act on each other at a crossing. The resulting algebraic structure is known as a *biquandle*; see [5]. More precisely, we have

Definition 3 Let X be a set and define $\Delta : X \to X \times X$ by $\Delta(x) = (x, x)$. A *biquandle map* on X is an invertible map $B : X \times X \to X \times X$ denoted

$$B(x, y) = (B_1(x, y), B_2(x, y)) = (y^x, x_y)$$

such that

(i) There exists a unique invertible *sideways map* $S : X \times X \to X \times X$ such that for all $x, y \in X$, we have
$$S(B_1(x, y), x) = (B_2(x, y), y);$$

(ii) The component map $(S\Delta)_1 = (S\Delta)_2 : X \to X$ is bijective, and

(iii) B satisfies the *set-theoretic Yang-Baxter equation*
$$(B \times I)(I \times B)(B \times I) = (I \times B)(B \times I)(I \times B).$$

A *biquandle* is a set X with a choice of biquandle map B.

Example 10 Examples of biquandles include

- *Constant Action Biquandles.* For any set X and bijection $\sigma : X \to X$, the map $B(x, y) = (\sigma(y), \sigma^1(x))$ is a biquandle map.

- *Alexander Biquandles.* For any module X over $\mathbb{Z}[t^{\pm 1}, r^{\pm 1}]$, the map

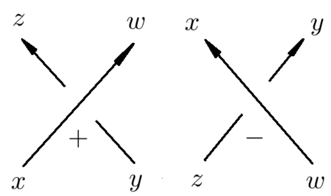
$$B(\vec{x}, \vec{y}) = ((1 - tr)\vec{x} + t\vec{y}, r\vec{x})$$

  is a biquandle map.

- *Fundamental Biquandle of an oriented Link.* Given an oriented link diagram L, let G be a set of generators corresponding bijectively with semiarcs (portions of the link divided at both over and undercrossing points) in L, and define the set of *biquandle words* W recursively by the rules

  - $G \subset W$ and
  - If $x, y \in W$ then $B^{\pm 1}_{1,2}(x, y), S^{\pm 1}_{1,2}(x, y) \in W$.

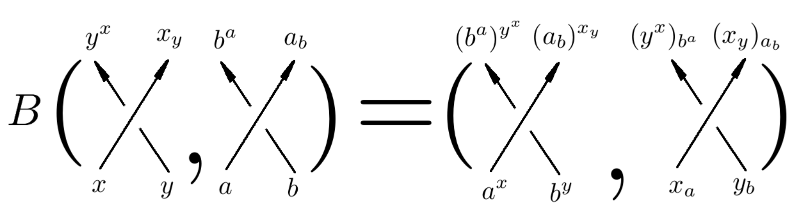
  Then the *fundamental biquandle* of L, denoted $B(L)$, is the set of equivalence classes of W under the equivalence relation generated by the biquandle axioms and the *crossing relations* in L:



$z = y^x, \ w = x_y$

- An $n \times 2n$ matrix M with entries in $X = \{1, 2, \dots, n\}$ can be interpreted as a pair of operation tables, say with $B(i, j) = (M[j, i], M[i, j + n])$. Then such a matrix defines a biquandle structure on X provided the entries satisfy the biquandle axioms.

We can generalize the abelian property from quandles to biquandles X by requiring that the set of biquandle homomorphsims Hom : $B(L) \to X$ forms a biquandle under the diagrammatic operations analogous to the quandle case.



More precisely, we have

Definition 4 We say a biquandle map $B : X \times X \to X \times X$ is *abelian* if for all $a, b, x, y \in X$ we have

$$(b^a)^{y^x} = (b^y)^{a^x}, \quad (a_b)^{x_y} = (a^x)_{b^y}, \qquad \text{and} \quad (x_y)_{a_b} = (x_a)_{y_b}.$$

Note that two of the four conditions determined in the diagram, namely $(a_b)^{x_y} = (a^x)_{b^y}$ and $(y^x)_{b^a} = (y_b)^{x_a}$, are equivalent.

Example 11 Alexander biquandles are abelian, as we can verify directly.

$$\begin{aligned}(b^a)^{y^x} &= t(b^a)+(1-tr)(y^x)\\ &= t(tb+(1-tr)a)+(1-tr)(ty+(1-tr)x)\\ &= t^2b+t(1-tr)a+t(1-tr)y+(1-tr)^2x\\ &= t^2b+t(1-tr)y+t(1-tr)a+(1-tr)^2x\\ &= (b^y)^{a^x},\end{aligned}$$

$$\begin{aligned}(a_b)^{x_y} &= t(a_b)+(1-tr)(x_y)\\ &= tra+(1-tr)(rx)\\ &= r(ta+(1-tr)x)\\ &= (a^x)_{b^y},\end{aligned}$$

and

$$(x_y)_{a_b} = r(x_y) = r^2x = r(x_a) = (x_a)_{y_b}.$$

As with quandles, we have

Proposition 11 *If Y is a biquandle and X is an abelian biquandle, then the set of biquandle homomorphisms* $\mathrm{Hom}(Y,X)$ *has a biquandle structure defined by*

$$f^g(x)=f(x)^{g(x)} \quad \text{and} \quad g_f(x)=g(x)_{f(x)}.$$

In particular, if X is a finite abelian biquandle, then the hom biquandle $\mathbf{Hom}(B(L),X)$ is a link invariant which determines, but is not determined by, the biquandle counting invariant.

Example 12 Consider the biquandle with operation matrix

$$M_X = \left[\begin{array}{ccccc|ccccc} 3 & 3 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2\\ 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3\\ 5 & 5 & 5 & 5 & 5 & 4 & 4 & 4 & 5 & 5\\ 4 & 4 & 4 & 4 & 4 & 5 & 5 & 5 & 4 & 4\end{array}\right].$$

Our `Python` computations indicate that both the $(4,2)$-torus link $L4a1$ and the Whitehead link $L5a1$ have biquandle counting invariant value $|\mathrm{Hom}(B(L4a1),X)| = 81 = |\mathrm{Hom}(B(L5a1),X)|$ with respect to X, but $\mathbf{Hom}(B(L4a1),X)$ and $\mathbf{Hom}(B(L5a1),X)$ are not isomorphic as biquandles. Indeed, they have distinct upper biquandle polynomials

$$\phi_1(\mathbf{Hom}(B(L4a1),X)) = 12s^{77}t^{77}+4s^{77}t^{68}+40s^{71}t^{71}+9s^{68}t^{72}+16s^{65}t^{65}$$

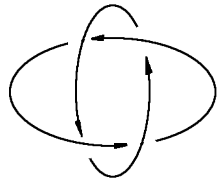
and

$$\phi_1(\mathbf{Hom}(B(L5a1),X)) = 16s^{77}t^{77}+40s^{71}t^{71}+4s^{65}t^{56}+9s^{56}t^{60}+12s^{56}t^{56}$$

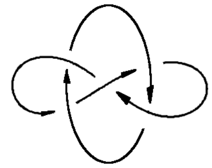
respectively where

$$\phi_1(X)=\sum_{x\in X} s^{r(x)}t^{c(x)}$$

where $r(x)=|\{y\in X \ : \ x^y=x\}|$ and $c(x)=|\{y\in X \ : \ y_x=y\}|$.



$L4a1$



$L5a1$

8 Categorical Framework

Since we know that the collection of homomorphisms between two abelian quandles forms an abelian quandle, a natural question to ask is whether the category of abelian quandles is a 'symmetric monoidal closed' category. In this section, we show that the answer to this question is yes.

Recall that a symmetric monoidal category C is *closed* if for any object $Q \in C$, the functor $- \otimes Q : C \to C$ has a right adjoint. We will denote this right adjoint by $\mathbf{Hom}(Q, -)$ and the adjointness condition then says:

$$\mathrm{Hom}(P \otimes Q, R) = \mathrm{Hom}(P, \mathbf{Hom}(Q, R)) \qquad (*)$$

for all $P, Q, R \in C$ and where "=" means natural isomorphism. The right adjoint $\mathbf{Hom}(-,-)$ is uniquely determined by adjointness and defines a functor $C^{op} \times C \to C$.

8.1 Tensor Product of Abelian Quandles

Let Q and A be abelian quandles. We define the *tensor product*, $Q \otimes A$, to be the free quandle on the set $Q \times A$ quotiented out by the relations

$$(q, a_1) \rhd (q, a_2) = (q, a_1 \rhd a_2) \qquad \text{and} \qquad (q_1, a) \rhd (q_2, a) = (q_1 \rhd q_2, a)$$

for $q, q_1, q_2 \in Q$ and $a, a_1, a_2 \in A$. For abelian quandles X, a homomorphism $Q \otimes A \to X$ is essentially the same thing as a *bihomomorphism*, that is, a function $f : Q \times A \to X$ such that:

- $f(q, -) : A \to X$ is a homomorphism for each $q \in Q$, and
- $f(-, a) : Q \to X$ is a homomorphism for each $a \in A$.

The unit for this tensor product is the one-element quandle 1, which can be checked directly. We note that, in principle, the unit is actually the free quandle on a single generator, but that is, in fact, the one-element quandle due to the first quandle axiom.

We remark that this situation works very much as it does for modules. The key point is that both the theory of abelian quandles and the theory of modules are 'commutative' theories. We recall that a *commutative theory* is an algebraic theory such that each operation of the theory is a homomorphism. For example, the theory of abelian groups is commutative because for any abelian group G, the map $+ : G \times G \to G$ is a homomorphism. More explicitly, in a commutative theory, given any n-ary operation α and any m-ary operation β, the equation

$$\alpha(\beta(x_{11}, \ldots, x_{1m}), \ldots, \beta(x_{n1}, \ldots, x_{nm})) = \beta(\alpha(x_{11}, \ldots, x_{n1}), \ldots, \alpha(x_{1m}, \ldots, x_{nm}))$$

holds. Since the theory of quandles only has one operation $\rhd$, all we need is the equation

$$(x_{11} \rhd x_{12}) \rhd (x_{21} \rhd x_{22}) = (x_{11} \rhd x_{21}) \rhd (x_{12} \rhd x_{22}),$$

and this is precisely what the definition of abelian quandle guarantees.

8.2 Category of Abelian Quandles

By Theorem 4, we know that given abelian quandles Q and A, the set $\mathrm{Hom}(Q, A)$ becomes an abelian quandle under pointwise operations. Indeed, $\mathrm{Hom}(Q, A)$ is the underlying set of $\mathbf{Hom}(Q, A)$. It is not difficult to show that a homomorphism $Q \otimes A \to X$ is essentially the same thing as a homomorphism $Q \to \mathbf{Hom}(A, X)$. This fact, more formally, gives us the answer to the question raised at the beginning of this section:

Theorem 12 *The category of abelian quandles is symmetric monoidal closed under the tensor product $\otimes$ and closed structure* **Hom**(-,-) *defined above.*

Proof. This follows from the main theorem of Linton [6] since the theory of abelian quandles is commutative. $\square$

We note that this means the category of abelian quandles can be enriched over itself, and the adjointness condition $(*)$ is a natural isomorphism of quandles

$$\mathbf{Hom}(Q \otimes A, X) = \mathbf{Hom}(Q, \mathbf{Hom}(A, X)).$$

.

9 Questions for Future Research

In this section we collect a few questions for future research.

In [4], Joyce shows that the fundamental abelian quandle of a classical knot determines, and is determined by, the fundamental Alexander quandle of the knot. Is the analogous statement true for Alexander biquandles?

What other properties does the hom quandle, $\mathrm{Hom}(Q, A)$ inherit from the quandles Q and A? For example, does it inherit the properties of being connected, dihedral, Core or conjugation? Given connected quandles, is it true that the hom quandle structure is determined by the counting invariant?

Moreover, what is the relationship (if any) between the cardinalities of Q, A, and $\mathrm{Hom}(Q, A)$? How is the homology of $Q \otimes A$ related to the homologies of Q and A? Under the conjugation or Core functors, the braid group becomes a quandle. Is this 'braid quandle' abelian?

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