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From Loop Groups to 2-Groups

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Abstract

We describe an interesting relation between Lie 2-algebras, the Kac–Moody central extensions of loop groups, and the group $\mathrm{String}(n)$. A Lie 2-algebra is a categorified version of a Lie algebra where the Jacobi identity holds up to a natural isomorphism called the 'Jacobiator'. Similarly, a Lie 2-group is a categorified version of a Lie group. If G is a simply-connected compact simple Lie group, there is a 1-parameter family of Lie 2-algebras $\mathfrak{g}_k$ each having $\mathfrak{g}$ as its Lie algebra of objects, but with a Jacobiator built from the canonical 3-form on G. There appears to be no Lie 2-group having $\mathfrak{g}_k$ as its Lie 2-algebra, except when $k = 0$. Here, however, we construct for integral k an infinite-dimensional Lie 2-group $\mathcal{P}_k G$ whose Lie 2-algebra is *equivalent* to $\mathfrak{g}_k$. The objects of $\mathcal{P}_k G$ are based paths in G, while the automorphisms of any object form the level-k Kac–Moody central extension of the loop group ΩG. This 2-group is closely related to the kth power of the canonical gerbe over G. Its nerve gives a topological group $|\mathcal{P}_k G|$ that is an extension of G by $K(\mathbb{Z}, 2)$. When $k = \pm 1$, $|\mathcal{P}_k G|$ can also be obtained by killing the third homotopy group of G. Thus, when $G = \mathrm{Spin}(n)$, $|\mathcal{P}_k G|$ is none other than $\mathrm{String}(n)$.

1 Introduction

The theory of simple Lie groups and Lie algebras has long played a central role in mathematics. Starting in the 1980s, a wave of research motivated by physics has revitalized this theory, expanding it to include structures such as quantum groups, affine Lie algebras, and central extensions of loop groups. All these structures rely for their existence on the left-invariant closed 3-form ν naturally possessed by any compact simple Lie group G:

$$\nu(x,y,z) = \langle x, [y,z]\rangle \qquad x,y,z \in \mathfrak{g},$$

or its close relative, the left-invariant closed 2-form ω on the loop group ΩG:

$$\omega(f,g) = 2\int_{S^1} \langle f(\theta), g'(\theta)\rangle \, d\theta \qquad f,g \in \Omega\mathfrak{g}.$$

Moreover, all these new structures fit together in a grand framework that can best be understood with ideas from physics — in particular, the Wess–Zumino–Witten model and Chern–Simons theory. Since these ideas arose from work on string theory, which replaces point particles by higher-dimensional extended objects, it is not surprising that their study uses concepts from higher-dimensional algebra, such as gerbes [5, 7, 8].

More recently, work on higher-dimensional algebra has focused attention on Lie 2-groups [1] and Lie 2-algebras [2]. A '2-group' is a category equipped with operations analogous to those of a group, where all the usual group axioms hold only up to specified natural isomorphisms satisfying certain coherence laws of their own. A 'Lie 2-group' is a 2-group where the set of objects and the set of morphisms are smooth manifolds, and all the operations and natural isomorphisms are smooth. Similarly, a 'Lie 2-algebra' is a category equipped with operations analogous to those of a Lie algebra, satisfying the usual laws up to coherent natural isomorphisms. Just as Lie groups and Lie algebras are important in gauge theory, Lie 2-groups and Lie 2-algebras are important in 'higher gauge theory', which describes the parallel transport of higher-dimensional extended objects [3, 4].

The question naturally arises whether every finite-dimensional Lie 2-algebra comes from a Lie 2-group. The answer is surprisingly subtle, as illustrated by a class of Lie 2-algebras coming from simple Lie algebras. Suppose G is a simply-connected compact simple Lie group G, and let $\mathfrak{g}$ be its Lie algebra. For any real number k, there is a Lie 2-algebra $\mathfrak{g}_k$ for which the space of objects is $\mathfrak{g}$, the space of endomorphisms of any object is $\mathbb{R}$, and the 'Jacobiator'

$$J_{x,y,z}\colon [[x,y],z] \xrightarrow{\sim} [x,[y,z]] + [[x,z],y]$$

is given by

$$J_{x,y,z} = k\,\nu(x,y,z)$$

where ν is as above. If we normalize the invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ so that the de Rham cohomology class of the closed form $\nu/2\pi$ generates the third integral cohomology of G, then there is a 2-group G_k corresponding to $\mathfrak{g}_k$ in a certain sense explained below whenever k is an integer. The construction of this 2-group is very interesting, because it uses Chern–Simons theory in an essential way. However, for $k \neq 0$ there is no good way to make this 2-group into a Lie 2-group! The set of objects is naturally a smooth manifold, and so is the set of morphisms, and the group operations are smooth, but the associator

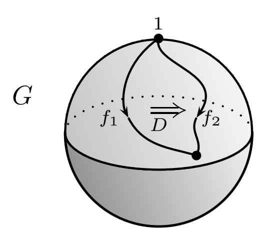
$$a_{x,y,z}\colon (xy)z \xrightarrow{\sim} x(yz)$$

cannot be made everywhere smooth, or even continuous.

It would be disappointing if such a fundamental Lie 2-algebra as $\mathfrak{g}_k$ failed to come from a Lie 2-group even when k was an integer. Here we resolve this dilemma by finding a Lie 2-algebra *equivalent* to $\mathfrak{g}_k$ that *does* come from a Lie 2-group — albeit an *infinite-dimensional* one.

The point is that the natural concept of 'sameness' for categories is a bit subtle: not isomorphism, but equivalence. Two categories are 'equivalent' if there are functors going back and forth between them that are inverses *up to natural isomorphism*. Categories that superficially look quite different can turn out to be equivalent. The same is true for 2-groups and Lie 2-algebras. Taking advantage of this, we show that while the finite-dimensional Lie 2-algebra $\mathfrak{g}_k$ has no corresponding Lie 2-group, it is equivalent to an infinite-dimensional Lie 2-algebra $\mathcal{P}_k\mathfrak{g}$ which comes from an infinite-dimensional Lie 2-group $\mathcal{P}_kG$.

The 2-group $\mathcal{P}_kG$ is easy to describe, in part because it is 'strict': all the usual group axioms hold as equations. The basic idea is easiest to understand using some geometry. Apart from some technical fine print, an object of $\mathcal{P}_kG$ is just a path in G starting at the identity. A morphism from the path f_1 to the path f_2 is an equivalence class of pairs (D, z) consisting of a disk D going from f_1 to f_2 together with a unit complex number z:



Given two such pairs (D_1, z_1) and (D_2, z_2), we can always find a 3-ball B whose boundary is $D_1 \cup D_2$, and we say the pairs are equivalent when

$$z_2/z_1 = e^{ik\int_B \nu}$$

where ν is the left-invariant closed 3-form on G given as above. Note that $\exp(ik\int_B \nu)$ is independent of the choice of B, because the integral of ν over any 3-sphere is 2π times an integer. There is an obvious way to compose morphisms in $\mathcal{P}_k G$, and the resulting category inherits a Lie 2-group structure from the Lie group structure of G.

The above description of $\mathcal{P}_k G$ is modeled after Murray's construction [16] of a gerbe from an integral closed 3-form on a manifold with a chosen basepoint. Indeed, $\mathcal{P}_k G$ is just another way of talking about the kth power of the canonical gerbe on G, and the 2-group structure on $\mathcal{P}_k G$ is a reflection of the fact that this gerbe is 'multiplicative' in the sense of Brylinski [6]. The 3-form $k\nu$, which plays the role of the Jacobiator in $\mathfrak{g}_k$, is the 3-curvature of a connection on this gerbe.

In most of this paper we take a slightly different viewpoint. Let $P_0 G$ be the space of smooth paths $f\colon [0, 2\pi] \to G$ that start at the identity of G. This becomes an infinite-dimensional Lie group under pointwise multiplication. The map $f \mapsto f(2\pi)$ is a homomorphism from $P_0 G$ to G whose kernel is precisely ΩG. For any $k \in \mathbb{Z}$, the loop group ΩG has a central extension

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{\Omega_k G} \xrightarrow{p} \Omega G \longrightarrow 1$$

which at the Lie algebra level is determined by the 2-cocycle $ik\omega$, with ω defined as above. This is called the 'level-k Kac–Moody central extension' of G. The infinite-dimensional Lie 2-group $\mathcal{P}_k G$ has $P_0 G$ as its group of objects, and given $f_1, f_2 \in P_0 G$, a morphism $\hat{\ell}\colon f_1 \to f_2$ is an element $\hat{\ell} \in \widehat{\Omega_k G}$ such that

$$f_2/f_1 = p(\hat{\ell}).$$

In this description, composition of morphisms in $\mathcal{P}_k G$ is multiplication in $\widehat{\Omega_k G}$, while again $\mathcal{P}_k G$ becomes a Lie 2-group using the Lie group structure of G.

To better understand the significance of the Lie 2-algebra $\mathfrak{g}_k$ and the 2-group G_k it is helpful to recall the classification of 2-groups and Lie 2-algebras. In [2] it is shown that Lie 2-algebras are classified up to equivalence by quadruples consisting of:

- a Lie algebra $\mathfrak{g}$,
- an abelian Lie algebra $\mathfrak{h}$,
- a representation ρ of $\mathfrak{g}$ on $\mathfrak{h}$,
- an element $[j] \in H^3(\mathfrak{g}, \mathfrak{h})$ of the Lie algebra cohomology of $\mathfrak{g}$.

Given a Lie 2-algebra $\mathfrak{c}$, we obtain this data by choosing a 'skeleton' $\mathfrak{c}_0$ of $\mathfrak{c}$: that is, an equivalent Lie 2-algebra in which any pair of isomorphic objects are equal. The objects in this skeleton form the Lie algebra $\mathfrak{g}$, while the endomorphisms of any object form the abelian Lie algebra $\mathfrak{h}$. The representation of $\mathfrak{g}$ on $\mathfrak{h}$ comes from the bracket in $\mathfrak{c}_0$, and the element $[j]$ comes from the Jacobiator.

Similarly, in [1] we give a proof of the already known fact that 2-groups are classified up to equivalence by quadruples consisting of:

- a group G,

- an abelian group H,
- an action α of G as automorphisms of H,
- an element $[a] \in H^3(G, H)$ of group cohomology of G.

Given a 2-group C, we obtain this data by choosing a skeleton C_0: that is, an equivalent 2-group in which any pair of isomorphic objects are equal. The objects in this skeleton form the group G, while the automorphisms of any object form the abelian group H. The action of G on H comes from conjugation in C_0, and the element $[a]$ comes from the associator.

These strikingly parallel classifications suggest that 2-groups should behave like Lie 2-algebras to the extent that group cohomology resembles Lie algebra cohomology. But this is where the subtleties begin!

Suppose G is a simply-connected compact simple Lie group, and let $\mathfrak{g}$ be its Lie algebra. If ρ is the trivial representation of $\mathfrak{g}$ on $\mathfrak{u}(1)$, we have

$$H^3(\mathfrak{g}, \mathfrak{u}(1)) \cong \mathbb{R}$$

because this cohomology group can be identified with the third de Rham cohomology group of G, which has the class $[\nu]$ as a basis. Thus, for any $k \in \mathbb{R}$ we obtain a skeletal Lie 2-algebra $\mathfrak{g}_k$ having $\mathfrak{g}$ as its Lie algebra of objects and $\mathfrak{u}(1)$ as the endomorphisms of any object, where the Jacobiator in $\mathfrak{g}_k$ is given by

$$J_{x,y,z} = k\nu(x, y, z).$$

To build a 2-group G_k analogous to this Lie 2-algebra $\mathfrak{g}_k$, we need to understand the relation between $H^3(G, \mathrm{U}(1))$ and $H^3(\mathfrak{g}, \mathfrak{u}(1))$. They are not isomorphic. However, $H^3(\mathfrak{g}, \mathfrak{u}(1))$ contains a lattice Λ consisting of the integer multiples of $[\nu]$. The papers of Chern–Simons [10] and Cheeger–Simons [9] construct an inclusion

$$\iota\colon \Lambda \hookrightarrow H^3(G, \mathrm{U}(1)).$$

Thus, when k is an integer, we can build a skeletal 2-group G_k having G as its group of objects, $\mathrm{U}(1)$ as the group of automorphisms of any object, the trivial action of G on $\mathrm{U}(1)$, and $[a] \in H^3(G, \mathrm{U}(1))$ given by $k\,\iota[\nu]$.

The question naturally arises whether G_k can be made into a Lie 2-group. The problem is that there is no continuous representative of the cohomology class $k\,\iota[\nu]$ unless $k = 0$. Thus, for k nonzero, we cannot make G_k into a Lie 2-group in any reasonable way. More precisely, we have this result [1]:

Theorem 1. *Let G be a simply-connected compact simple Lie group. Unless $k = 0$, there is no way to give the 2-group G_k the structure of a Lie 2-group for which the group G of objects and the group $\mathrm{U}(1)$ of endomorphisms of any object are given their usual topology.*

The goal of this paper is to sidestep this 'no-go theorem' by finding a Lie 2-algebra equivalent to $\mathfrak{g}_k$ which does come from an (infinite-dimensional) Lie group when $k \in \mathbb{Z}$. We show:

Theorem 2. *Let G be a simply-connected compact simple Lie group. For any $k \in \mathbb{Z}$, there is a Fréchet Lie 2-group $\mathcal{P}_k G$ whose Lie 2-algebra $\mathcal{P}_k \mathfrak{g}$ is equivalent to $\mathfrak{g}_k$.*

We also study the relation between $\mathcal{P}_k G$ and the topological group $\hat{G}$ obtained by killing the third homotopy group of G. When $G = \mathrm{Spin}(n)$, this topological group is famous under the name of $\mathrm{String}(n)$, since it plays a role in string theory [17, 21, 22]. More generally, any compact simple Lie group G has $\pi_3(G) = \mathbb{Z}$, but after killing $\pi_1(G)$ by passing to the universal cover of G, one can then kill $\pi_3(G)$ by passing to $\hat{G}$, which is defined as the homotopy fiber of the canonical map from G to the Eilenberg–Mac Lane space $K(\mathbb{Z}, 3)$. This specifies $\hat{G}$ up to homotopy, but there is still the interesting problem of finding nice geometrical models for $\hat{G}$.

Before presenting their solution to this problem, Stolz and Teichner [21] wrote: "To our best knowledge, there has yet not been found a canonical construction for $\mathrm{String}(n)$ which has reasonable 'size' and a geometric interpretation." Here we present another solution. There is a way to turn any topological 2-group C into a topological group $|C|$, which we explain in Section 4.2. Applying this to $\mathcal{P}_k G$ when $k = \pm 1$, we obtain $\hat{G}$:

Theorem 3. *Let G be a simply-connected compact simple Lie group. Then $|\mathcal{P}_k G|$ is an extension of G by a topological group that is homotopy equivalent to $K(\mathbb{Z}, 2)$. Moreover, $|\mathcal{P}_k G| \simeq \hat{G}$ when $k = \pm 1$.*

While this construction of $\hat{G}$ uses simplicial methods and is thus arguably less 'geometric' than that of Stolz and Teichner, it avoids their use of type III_1 von Neumann algebras, and has a simple relation to the Kac–Moody central extension of G.

2 Review of Lie 2-Algebras and Lie 2-Groups

We begin with a review of Lie 2-algebras and Lie 2-groups. More details can be found in our papers HDA5 [1] and HDA6 [2]. Our notation largely follows that of these papers, but the reader should be warned that here we denote the composite of morphisms $f \colon x \to y$ and $g \colon y \to z$ as $g \circ f \colon x \to z$.

2.1 Lie 2-algebras

The concept of 'Lie 2-algebra' blends together the notion of a Lie algebra with that of a category. Just as a Lie algebra has an underlying vector space, a Lie 2-algebra has an underlying 2-vector space: that is, a category where everything is *linear*. More precisely, a **2-vector space** L is a category for which:

- the set of objects $\mathrm{Ob}(L)$,
- the set of morphisms $\mathrm{Mor}(L)$

are both vector spaces, and:

- the maps $s, t\colon \mathrm{Mor}(L) \to \mathrm{Ob}(L)$ sending any morphism to its source and target,
- the map $i\colon \mathrm{Ob}(L) \to \mathrm{Mor}(L)$ sending any object to its identity morphism,
- the map $\circ$ sending any composable pair of morphisms to its composite

are all linear. As usual, we write a morphism as $f\colon x \to y$ when $s(f) = x$ and $t(f) = y$, and we often write $i(x)$ as 1_x.

To obtain a Lie 2-algebra, we begin with a 2-vector space and equip it with a bracket functor, which satisfies the Jacobi identity up to a natural isomorphism called the 'Jacobiator'. Then we require that the Jacobiator satisfy a new coherence law of its own: the 'Jacobiator identity'.

Definition 4. *A* **Lie 2-algebra** *consists of:*

- *a 2-vector space L*

equipped with:

- *a functor called the* **bracket**
$$[\cdot,\cdot]\colon L \times L \to L,$$
bilinear and skew-symmetric as a function of objects and morphisms,
- *a natural isomorphism called the* **Jacobiator***,*
$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y],$$
trilinear and antisymmetric as a function of the objects $x, y, z \in L$,

such that:

- *the* **Jacobiator identity** *holds: the following diagram commutes for all objects $w, x, y, z \in L$:*

$$\begin{tikzcd}
 & {[[[w,x],y],z]} \arrow[dl, "{[J_{w,x,y},z]}"'] \arrow[dr, "{J_{[w,x],y,z}}"] & \\
{[[[w,y],x],z]+[[w,[x,y]],z]} \arrow[d, "{J_{[w,y],x,z}+J_{w,[x,y],z}}"'] & & {[[[w,x],z],y]+[[w,x],[y,z]]} \arrow[d, "{[J_{w,x,z},y]+1}"] \\
{\begin{array}{c}[[[w,y],z],x]+[[w,y],[x,z]]\\ +[w,[[x,y],z]]+[[w,z],[x,y]]\end{array}} \arrow[d, "{[J_{w,y,z},x]+1}"'] & & {\begin{array}{c}[[w,[x,z]],y]\\ +[[w,x],[y,z]]+[[[w,z],x],y]\end{array}} \arrow[d, "{J_{w,[x,z],y}+J_{[w,z],x,y}+J_{w,x,[y,z]}}"] \\
{\begin{array}{c}[[[w,z],y],x]+[[w,[y,z]],x]\\ +[[w,y],[x,z]]+[w,[[x,y],z]]+[[w,z],[x,y]]\end{array}} \arrow[rr, "{[w,J_{x,y,z}]+1}"'] & & {\begin{array}{c}[[[w,z],y],x]+[[w,z],[x,y]]+[[w,y],[x,z]]\\ +[w,[[x,z],y]]+[[w,[y,z]],x]+[w,[x,[y,z]]]\end{array}}
\end{tikzcd}$$

A homomorphism between Lie 2-algebras is a linear functor preserving the bracket, but only up to a specified natural isomorphism satisfying a suitable coherence law. More precisely:

Definition 5. *Given Lie* 2*-algebras* L *and* L'*, a* **homomorphism** $F\colon L \to L'$ *consists of:*

- *a functor* F *from the underlying* 2*-vector space of* L *to that of* L'*, linear on objects and morphisms,*
- *a natural isomorphism*

$$F_2(x,y)\colon [F(x),F(y)] \to F[x,y],$$

bilinear and skew-symmetric as a function of the objects $x, y \in L$*,*

such that:

- *the following diagram commutes for all objects* $x, y, z \in L$*:*

$$\begin{tikzcd}
{[F(x),[F(y),F(z)]]} \arrow[rr, "J_{F(x),F(y),F(z)}"] \arrow[d, "{[1,F_2]}"'] & & {[[F(x),F(y)],F(z)] + [F(y),[F(x),F(z)]]} \arrow[d, "{[F_2,1]+[1,F_2]}"] \\
{[F(x),F[y,z]]} \arrow[d, "F_2"'] & & {[F[x,y],F(z)] + [F(y),F[x,z]]} \arrow[d, "F_2+F_2"] \\
{F[x,[y,z]]} \arrow[rr, "F(J_{x,y,z})"] & & {F[[x,y],z] + F[y,[x,z]]}
\end{tikzcd}$$

Here and elsewhere we omit the arguments of natural transformations such as F_2 and G_2 when these are obvious from context.

Similarly, a '2-homomorphism' is a linear natural isomorphism that is compatible with the bracket structure:

Definition 6. *Let* $F, G\colon L \to L'$ *be Lie 2-algebra homomorphisms. A* **2-homomorphism** $\theta\colon F \Rightarrow G$ *is a natural transformation*

$$\theta_x\colon F(x) \to G(x),$$

linear as a function of the object $x \in L$, such that the following diagram commutes for all $x, y \in L$:

$$\begin{tikzcd}
{[F(x), F(y)]} \arrow[r, "F_2"] \arrow[d, "{[\theta_x, \theta_y]}"'] & F[x,y] \arrow[d, "\theta_{[x,y]}"] \\
{[G(x), G(y)]} \arrow[r, "G_2"] & G[x,y]
\end{tikzcd}$$

In HDA6 we showed:

Proposition 7. *There is a strict 2-category* **Lie2Alg** *with Lie 2-algebras as objects, homomorphisms between these as morphisms, and 2-homomorphisms between those as 2-morphisms.*

2.2 L_∞-algebras

Just as the concept of Lie 2-algebra blends the notions of Lie algebra and category, the concept of 'L_∞-algebra' blends the notions of Lie algebra and chain complex. More precisely, an L_∞-algebra is a chain complex equipped with a bilinear skew-symmetric bracket operation that satisfies the Jacobi identity up to a chain homotopy, which in turn satisfies a law of its own up to chain homotopy, and so on *ad infinitum*. In fact, L_∞-algebras were defined long before Lie 2-algebras, going back to a 1985 paper by Schlessinger and Stasheff [19]. They are also called 'strongly homotopy Lie algebras', or 'sh Lie algebras' for short.

Our conventions regarding L_∞-algebras follow those of Lada and Markl [12]. In particular, for graded objects $x_1, \dots, x_n$ and a permutation $\sigma \in S_n$ we define the **Koszul sign** $\epsilon(\sigma)$ by the equation

$$x_1 \wedge \cdots \wedge x_n = \epsilon(\sigma)\, x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)},$$

which must be satisfied in the free graded-commutative algebra on $x_1, \dots, x_n$. Furthermore, we define

$$\chi(\sigma) = \mathrm{sgn}(\sigma)\, \epsilon(\sigma; x_1, \dots, x_n).$$

Thus, $\chi(\sigma)$ takes into account the sign of the permutation in S_n as well as the Koszul sign. Finally, if n is a natural number and $1 \le j \le n-1$ we say that $\sigma \in S_n$ is an $(j, n-j)$**-unshuffle** if

$$\sigma(1) \le \sigma(2) \le \cdots \le \sigma(j) \quad \text{and} \quad \sigma(j+1) \le \sigma(j+2) \le \cdots \le \sigma(n).$$

Readers familiar with shuffles will recognize unshuffles as their inverses.

Definition 8. *An* $\boldsymbol{L_\infty}$**-algebra** *is a graded vector space V equipped with a system $\{l_k | 1 \leq k < \infty\}$ of linear maps $l_k \colon V^{\otimes k} \to V$ with $\deg(l_k) = k - 2$ which are totally antisymmetric in the sense that*

$$l_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = \chi(\sigma) l_k(x_1, \ldots, x_n) \tag{1}$$

for all $\sigma \in S_n$ and $x_1, \ldots, x_n \in V$, and, moreover, the following generalized form of the Jacobi identity holds for $0 \leq n < \infty$:

$$\sum_{i+j=n+1} \sum_{\sigma} \chi(\sigma)(-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0, \tag{2}$$

where the summation is taken over all $(i, n-i)$-unshuffles with $i \geq 1$.

In this definition the map l_1 makes V into a chain complex, since this map has degree -1 and Equation (2) says its square is zero. In what follows, we denote l_1 as d. The map l_2 resembles a Lie bracket, since it is skew-symmetric in the graded sense by Equation (1). The higher l_k maps are related to the Jacobiator and the Jacobiator identity.

To make this more precise, we make the following definition:

Definition 9. *A* $\boldsymbol{k}$**-term** $\boldsymbol{L_\infty}$**-algebra** *is an L_∞-algebra V with $V_n = 0$ for $n \geq k$.*

A 1-term L_∞-algebra is simply an ordinary Lie algebra, where $l_3 = 0$ gives the Jacobi identity. However, in a 2-term L_∞-algebra, we no longer have $l_3 = 0$. Instead, Equation (2) says that the Jacobi identity for $x, y, z \in V_0$ holds up to a term of the form $dl_3(x, y, z)$. We do, however, have $l_4 = 0$, which provides us with the coherence law that l_3 must satisfy. It follows that a 2-term L_∞-algebra consists of:

- vector spaces V_0 and V_1,
- a linear map $d \colon V_1 \to V_0$,
- bilinear maps $l_2 \colon V_i \times V_j \to V_{i+j}$, where $0 \leq i + j \leq 1$,
- a trilinear map $l_3 \colon V_0 \times V_0 \times V_0 \to V_1$

satisfying a list of equations coming from Equations (1) and (2) and the fact that $l_4 = 0$. This list can be found in HDA6, but we will not need it here.

In fact, 2-vector spaces are equivalent to 2-term chain complexes of vector spaces: that is, chain complexes of the form

$$V_1 \xrightarrow{d} V_0.$$

To obtain such a chain complex from a 2-vector space L, we let V_0 be the space of objects of L. However, V_1 is not the space of morphisms. Instead, we define the **arrow part** $\vec{f}$ of a morphism $f \colon x \to y$ by

$$\vec{f} = f - i(s(f)),$$

and let V_1 be the space of these arrow parts. The map $d\colon V_1 \to V_0$ is then just the target map $t\colon \mathrm{Mor}(L) \to \mathrm{Ob}(L)$ restricted to $V_1 \subseteq \mathrm{Mor}(L)$.

To understand this construction a bit better, note that given any morphism $f\colon x \to y$, its arrow part is a morphism $\vec{f}\colon 0 \to y - x$. Thus, taking the arrow part has the effect of 'translating f to the origin'. We can always recover any morphism from its source together with its arrow part, since $f = \vec{f} + i(s(f))$. It follows that any morphism $f\colon x \to y$ can be identified with the ordered pair $(x, \vec{f})$ consisting of its source and arrow part. So, we have $\mathrm{Mor}(L) \cong V_0 \oplus V_1$.

We can actually recover the whole 2-vector space structure of L from just the chain complex $d\colon V_1 \to V_0$. To do this, we take:

$$\begin{array}{rcl} \mathrm{Ob}(L) & = & V_0 \\ \mathrm{Mor}(L) & = & V_0 \oplus V_1, \end{array}$$

with source, target and identity-assigning maps defined by:

$$\begin{array}{rcl} s(x, \vec{f}) & = & x \\ t(x, \vec{f}) & = & x + d\vec{f} \\ i(x) & = & (x, 0) \end{array}$$

and with the composite of $f\colon x \to y$ and $g\colon y \to z$ defined by:

$$g \circ f = (x, \vec{f} + \vec{g}).$$

So, 2-vector spaces are equivalent to 2-term chain complexes.

Given this, it should not be surprising that Lie 2-algebras are equivalent to 2-term L_∞-algebras. Since we make frequent use of this fact in the calculations to come, we recall the details here.

Suppose V is a 2-term L_∞-algebra. We obtain a 2-vector space L from the underlying chain complex of V as above. We continue by equipping L with additional structure that makes it a Lie 2-algebra. It is sufficient to define the bracket functor $[\cdot,\cdot]\colon L \times L \to L$ on a pair of objects and on a pair of morphisms where one is an identity morphism. So, we set:

$$\begin{array}{rcl} [x, y] & = & l_2(x, y), \\ [1_z, f] & = & (l_2(z, x), l_2(z, \vec{f})), \\ [f, 1_z] & = & (l_2(x, z), l_2(\vec{f}, z)), \end{array}$$

where $f\colon x \to y$ is a morphism in L and z is an object. Finally, we define the Jacobiator for L in terms of its source and arrow part as follows:

$$J_{x,y,z} = ([[x, y], z], l_3(x, y, z)).$$

For a proof that L defined this way is actually a Lie 2-algebra, see HDA6.

In our calculations we shall often describe Lie 2-algebra homomorphisms as homomorphisms between the corresponding 2-term L_∞-algebras:

Definition 10. *Let V and V' be 2-term L_∞-algebras. An* ***L_∞-homomorphism*** *$\phi\colon V \to V'$ consists of:*

- *a chain map $\phi\colon V \to V'$ consisting of linear maps $\phi_0\colon V_0 \to V_0'$ and $\phi_1\colon V_1 \to V_1'$,*
- *a skew-symmetric bilinear map $\phi_2\colon V_0 \times V_0 \to V_1'$,*

such that the following equations hold for all $x, y, z \in V_0$ and $h \in V_1$:

$$d(\phi_2(x,y)) = \phi_0(l_2(x,y)) - l_2(\phi_0(x), \phi_0(y)) \tag{3}$$

$$\phi_2(x, dh) = \phi_1(l_2(x,h)) - l_2(\phi_0(x), \phi_1(h)) \tag{4}$$

$$\begin{aligned} &l_3(\phi_0(x), \phi_0(y), \phi_0(z)) - \phi_1(l_3(x,y,z)) = \\ &\phi_2(x, l_2(y,z)) + \phi_2(y, l_2(z,x)) + \phi_2(z, l_2(x,y)) + \\ &l_2(\phi_0(x), \phi_2(y,z)) + l_2(\phi_0(y), \phi_2(z,x)) + l_2(\phi_0(z), \phi_2(x,y)) \end{aligned} \tag{5}$$

Equations (3) and (4) say that ϕ_2 defines a chain homotopy from $l_2(\phi(\cdot), \phi(\cdot))$ to $\phi(l_2(\cdot,\cdot))$, where these are regarded as chain maps from $V \otimes V$ to V'. Equation (5) is just a chain complex version of the commutative diagram in Definition 5.

Without providing too many details, let us sketch how to obtain the Lie 2-algebra homomorphism F corresponding to a given L_∞-homomorphism $\phi\colon V \to V'$. We define the chain map $F\colon L \to L'$ in terms of ϕ using the fact that objects of a 2-vector space are 0-chains in the corresponding chain complex, while morphisms are pairs consisting of a 0-chain and a 1-chain. To make F into a Lie 2-algebra homomorphism we must equip it with a skew-symmetric bilinear natural transformation F_2 satisfying the conditions in Definition 5. We do this using the skew-symmetric bilinear map $\phi_2\colon V_0 \times V_0 \to V_1'$. In terms of its source and arrow parts, we let

$$F_2(x,y) = (l_2(\phi_0(x), \phi_0(y)), \phi_2(x,y)).$$

We should also know how to compose L_∞-homomorphisms. We compose a pair of L_∞-homomorphisms $\phi\colon V \to V'$ and $\psi\colon V' \to V''$ by letting the chain map $\psi \circ \phi\colon V \to V''$ be the usual composite:

$$V \xrightarrow{\phi} V' \xrightarrow{\psi} V''$$

while defining $(\psi \circ \phi)_2$ as follows:

$$(\psi \circ \phi)_2(x,y) = \psi_2(\phi_0(x), \phi_0(y)) + \psi_1(\phi_2(x,y)). \tag{6}$$

This is just a chain complex version of how we compose homomorphisms between Lie 2-algebras. Note that the identity homomorphism $1_V\colon V \to V$ has the identity chain map as its underlying map, together with $(1_V)_2 = 0$.

We also have ‘2-homomorphisms’ between homomorphisms:

Definition 11. *Let V and V' be 2-term L_∞-algebras and let $\phi, \psi\colon V \to V'$ be L_∞-homomorphisms. An **L_∞-2-homomorphism** $\tau\colon \phi \Rightarrow \psi$ is a chain homotopy τ from ϕ to ψ such that the following equation holds for all $x, y \in V_0$:*

$$\phi_2(x,y) - \psi_2(x,y) = l_2(\phi_0(x), \tau(y)) + l_2(\tau(x), \psi_0(y)) - \tau(l_2(x,y)) \tag{7}$$

Given an L_∞-2-homomorphism $\tau\colon \phi \Rightarrow \psi$ between L_∞-homomorphisms $\phi, \psi\colon V \to V'$, there is a corresponding Lie 2-algebra 2-homomorphism θ whose source and arrow part are:

$$\theta(x) = (\phi_0(x), \tau(x))$$

for any object x. Checking that this really is a Lie 2-algebra 2-homomorphism is routine. In particular, Equation (7) is just a chain complex version of the commutative diagram in the Definition 6.

In HDA6, we showed:

Proposition 12. *There is a strict* 2*-category* $\mathbf{2TermL_\infty}$ *with* 2*-term* L_∞*-algebras as objects,* L_∞*-homomorphisms as morphisms, and* L_∞*-*2*-homomorphisms as* 2*-morphisms.*

Using the equivalence between 2-vector spaces and 2-term chain complexes, we established the equivalence between Lie 2-algebras and 2-term L_∞-algebras:

Theorem 13. *The* 2*-categories* Lie2Alg *and* $\mathrm{2TermL_\infty}$ *are* 2*-equivalent.*

We use this result extensively in Section 5. Instead of working in Lie2Alg, we do calculations in $\mathrm{2TermL_\infty}$. The reason is that defining Lie 2-algebra homomorphisms and 2-homomorphisms would require specifying both source and arrow parts of morphisms, while defining the corresponding L_∞-morphisms and 2-morphisms only requires us to specify the arrow parts. Manipulating the arrow parts rather than the full-fledged morphisms leads to less complicated computations.

2.3 The Lie 2-Algebra $\mathfrak{g}_k$

Another benefit of the equivalence between Lie 2-algebras and L_∞-algebras is that it gives some important examples of Lie 2-algebras. Instead of thinking of a Lie 2-algebra as a category equipped with extra structure, we may work with a 2-term chain complex endowed with the structure described in Definition 8. This is especially simple when the differential d vanishes. Thanks to the formula

$$d\vec{f} = t(f) - s(f),$$

this implies that the source of any morphism in the Lie 2-algebra equals its target. In other words, the corresponding Lie 2-algebra is 'skeletal':

Definition 14. *A category is* **skeletal** *if isomorphic objects are always equal.*

Every category is equivalent to a skeletal one formed by choosing one representative of each isomorphism class of objects [13]. As shown in HDA6, the same sort of thing is true in the context of Lie 2-algebras:

Proposition 15. *Every Lie* 2*-algebra is equivalent, as an object of* Lie2Alg*, to a skeletal one.*

This result helps us classify Lie 2-algebras up to equivalence. We begin by reminding the reader of the relationship between L_∞-algebras and Lie algebra cohomology described in HDA6:

Theorem 16. *There is a one-to-one correspondence between isomorphism classes of L_∞-algebras consisting of only two nonzero terms V_0 and V_n with $d = 0$, and isomorphism classes of quadruples $(\mathfrak{g}, V, \rho, [l_{n+2}])$ where $\mathfrak{g}$ is a Lie algebra, V is a vector space, ρ is a representation of $\mathfrak{g}$ on V, and $[l_{n+2}]$ is an element of the Lie algebra cohomology group $H^{n+2}(\mathfrak{g}, V)$.*

Here the representation ρ comes from $\ell_2 \colon V_0 \times V_n \to V_n$.

Because L_∞-algebras are equivalent to Lie 2-algebras, which all have equivalent skeletal versions, Theorem 16 implies:

Corollary 17. *Up to equivalence, Lie 2-algebras are classified by isomorphism classes of quadruples $(\mathfrak{g}, \rho, V, [\ell_3])$ where:*

- *$\mathfrak{g}$ is a Lie algebra,*
- *V is a vector space,*
- *ρ is a representation of $\mathfrak{g}$ on V,*
- *$[\ell_3]$ is an element of $H^3(\mathfrak{g}, V)$.*

This classification of Lie 2-algebras is just another way of stating the result mentioned in the Introduction. And, as mentioned there, this classification lets us construct a 1-parameter family of Lie 2-algebras $\mathfrak{g}_k$ for any simple real Lie algebra $\mathfrak{g}$:

Example 18. *Suppose $\mathfrak{g}$ is a simple real Lie algebra and $k \in \mathbb{R}$. Then there is a skeletal Lie 2-algebra $\mathfrak{g}_k$ given by taking $V_0 = \mathfrak{g}$, $V_1 = \mathbb{R}$, ρ the trivial representation, and $l_3(x, y, z) = k\langle x, [y, z]\rangle$.*

Here $\langle \cdot, \cdot \rangle$ is a suitably rescaled version of the Killing form $\mathrm{tr}(\mathrm{ad}(\cdot)\mathrm{ad}(\cdot))$. The precise rescaling factor will only become important in Section 3.1. The equation saying that l_3 is a 3-cocycle is equivalent to the equation saying that the left-invariant 3-form ν on G with $\nu(x, y, z) = \langle x, [y, z]\rangle$ is *closed*.

2.4 The Lie 2-Algebra of a Fréchet Lie 2-Group

Just as Lie groups have Lie algebras, 'strict Lie 2-groups' have 'strict Lie 2-algebras'. Strict Lie 2-groups and Lie 2-algebras are categorified versions of Lie groups and Lie algebras in which all laws hold 'on the nose' as equations, rather than up to isomorphism. All the Lie 2-groups discussed in this paper are strict. However, most of them are infinite-dimensional 'Fréchet' Lie 2-groups.

Since the concept of a Fréchet Lie group is easy to explain but perhaps not familiar to all readers, we begin by recalling this. For more details we refer the interested reader to the survey article by Milnor [14], or Pressley and Segal's book on loop groups [18].

A **Fréchet space** is a vector space equipped with a topology given by a countable family of seminorms $\|\cdot\|_n$, or equivalently by the metric

$$d(x,y) = \sum_n 2^{-n} \frac{\|x-y\|_n}{\|x-y\|_n + 1},$$

where we require that this metric be complete. A classic example is the space of smooth maps from the interval or circle to a finite-dimensional normed vector space, where $\|f\|_n$ is the supremum of the norm of the nth derivative of f. In particular, the space of smooth paths or loops in a finite-dimensional simple Lie algebra is a Fréchet space. This is the sort of example we shall need.

The theory of manifolds generalizes from the finite-dimensional case to the infinite-dimensional case by replacing $\mathbb{R}^n$ with a Fréchet space [11]. In particular, there is a concept of the 'Fréchet derivative' of a map between Fréchet spaces, and higher derivatives of such maps can also be defined. If V, W are Fréchet spaces and $U \subset V$ is an open set, a map $\phi\colon U \to W$ is called **smooth** if its nth derivative exists for all n. A **Fréchet manifold** modeled on the Fréchet space V is a paracompact Hausdorff space M that can be covered with open sets U_α equipped with homeomorphisms $\phi_\alpha\colon U_\alpha \to V$ called **charts** such that the maps $\phi_\alpha \circ \phi_\beta^{-1}$ are smooth where defined. In particular, the space of smooth paths or loops in a compact simple Lie group G is naturally a Fréchet manifold modeled on the Fréchet space of smooth paths or loops in the Lie algebra $\mathfrak{g}$.

A map between Fréchet manifolds is **smooth** if composing it with charts and their inverses in the usual way, we get functions between Fréchet spaces that are smooth where defined. A **Fréchet Lie group** is a Fréchet manifold G such that the multiplication map $m\colon G \times G \to G$ and the inverse map $\mathrm{inv}\colon G \to G$ are smooth. A **homomorphism** of Fréchet Lie groups is a group homomorphism that is also smooth.

Finally:

Definition 19. *A* **strict Fréchet Lie 2-group** C *is a category such that:*

- *the set of objects* $\mathrm{Ob}(C)$ *and*
- *the set of morphisms* $\mathrm{Mor}(C)$

are both Fréchet Lie groups, and:

- *the maps* $s, t\colon \mathrm{Mor}(C) \to \mathrm{Ob}(L)$ *sending any morphism to its source and target,*
- *the map* $i\colon \mathrm{Ob}(C) \to \mathrm{Mor}(C)$ *sending any object to its identity morphism,*
- *the map* $\circ\colon \mathrm{Mor}(C) \times_{\mathrm{Ob}(C)} \mathrm{Mor}(C) \to \mathrm{Mor}(C)$ *sending any composable pair of morphisms to its composite*

are all Fréchet Lie group homomorphisms.

Here $\mathrm{Mor}(C) \times_{\mathrm{Ob}(C)} \mathrm{Mor}(C)$ is the set of composable pairs of morphisms, which we require to be a Fréchet Lie group.

Just as for ordinary Lie groups, taking the tangent space at the identity of a Fréchet Lie group gives a Lie algebra. Using this, it is not hard to see that strict Fréchet Lie 2-groups give rise to Lie 2-algebras. These Lie 2-algebras are actually 'strict':

Definition 20. *A Lie 2-algebra is* **strict** *if its Jacobiator is the identity.*

This means that the map l_3 vanishes in the corresponding L_∞-algebra. Alternatively:

Proposition 21. *A strict Lie 2-algebra is the same as a 2-vector space L such that:*

- $\mathrm{Ob}(L)$ *is equipped with the structure of a Lie algebra,*
- $\mathrm{Mor}(L)$ *is equipped with the structure of a Lie algebra,*

and:

- *the source and target maps* $s, t\colon \mathrm{Mor}(L) \to \mathrm{Ob}(L)$*,*
- *the identity-assigning map* $i\colon \mathrm{Ob}(L) \to \mathrm{Mor}(L)$*, and*
- *the composition map* $\circ\colon \mathrm{Mor}(L) \times_{\mathrm{Ob}(L)} \mathrm{Mor}(L) \to \mathrm{Mor}(L)$

are Lie algebra homomorphisms.

Proof. - A straightforward verification; see also HDA6. $\square$

Proposition 22. *Given a strict Fréchet Lie* 2*-group* C*, there is a strict Lie* 2*-algebra* $\mathfrak{c}$ *for which:*

- $\mathrm{Ob}(\mathfrak{c})$ *is the Lie algebra of the Fréchet Lie group* $\mathrm{Ob}(C)$*,*
- $\mathrm{Mor}(\mathfrak{c})$ *is the Lie algebra of the Fréchet Lie group* $\mathrm{Mor}(C)$*,*

and the maps:

- $s, t\colon \mathrm{Mor}(\mathfrak{c}) \to \mathrm{Ob}(\mathfrak{c})$*,*
- $i\colon \mathrm{Ob}(\mathfrak{c}) \to \mathrm{Mor}(\mathfrak{c})$*, and*
- $\circ\colon \mathrm{Mor}(\mathfrak{c}) \times_{\mathrm{Ob}(\mathfrak{c})} \mathrm{Mor}(\mathfrak{c}) \to \mathrm{Mor}(\mathfrak{c})$

are the differentials of the corresponding maps for C*.*

Proof. This is a generalization of a result in HDA6 for ordinary Lie 2-groups, which is straightforward to show directly. $\square$

In what follows all Fréchet Lie 2-groups are strict, so we omit the term 'strict'.

3 Review of Loop Groups

Next we give a brief review of loop groups and their central extensions. More details can be found in the canonical text on the subject, written by Pressley and Segal [18].

3.1 Definitions and Basic Properties

Let G be a simply-connected compact simple Lie group. We shall be interested in the **loop group** ΩG consisting of all smooth maps from $[0, 2\pi]$ to G with $f(0) = f(2\pi) = 1$. We make ΩG into a group by pointwise multiplication of loops: $(fg)(\theta) = f(\theta)g(\theta)$. Equipped with its C^∞ topology, ΩG naturally becomes an infinite-dimensional Fréchet manifold. In fact ΩG is a Fréchet Lie group, as defined in Section 2.4.

As remarked by Pressley and Segal, the behaviour of the group ΩG is "untypical in its simplicity," since it turns out to behave remarkably like a compact Lie group. For example, it has an exponential map that is locally one-to-one and onto, and it has a well-understood highest weight theory of representations. One striking difference between ΩG and G, though, is the existence of nontrivial central extensions of ΩG by the circle $\mathrm{U}(1)$:

$$1 \to \mathrm{U}(1) \to \widehat{\Omega G} \xrightarrow{p} \Omega G \to 1. \tag{8}$$

It is important to understand that these extensions are nontrivial, not merely in that they are classified by a nonzero 2-cocycle, but also *topologically*. In other words, $\widehat{\Omega G}$ is a nontrivial principal $\mathrm{U}(1)$-bundle over ΩG with the property that $\widehat{\Omega G}$ is a Fréchet Lie group, and $\mathrm{U}(1)$ sits inside $\widehat{\Omega G}$ as a central subgroup in such a way that the quotient $\widehat{\Omega G}/\mathrm{U}(1)$ can be identified with ΩG. Perhaps the best analogy is with the double cover of $\mathrm{SO}(3)$: there $\mathrm{SU}(2)$ fibers over $\mathrm{SO}(3)$ as a 2-sheeted covering and $\mathrm{SU}(2)$ is not homeomorphic to $SO(3) \times \mathbb{Z}/2\mathbb{Z}$. $\widehat{\Omega G}$ is called the **Kac–Moody group**.

Associated to the central extension (8) there is a central extension of Lie algebras:

$$0 \to \mathfrak{u}(1) \to \widehat{\Omega \mathfrak{g}} \xrightarrow{p} \Omega \mathfrak{g} \to 0 \tag{9}$$

Here $\Omega\mathfrak{g}$ is the Lie algebra of ΩG, consisting of all smooth maps $f\colon S^1 \to \mathfrak{g}$ such that $f(0) = 0$. The bracket operation on $\Omega\mathfrak{g}$ is given by the pointwise bracket of functions: thus $[f, g](\theta) = [f(\theta), g(\theta)]$ if $f, g \in \Omega\mathfrak{g}$. $\widehat{\Omega\mathfrak{g}}$ is the simplest example of an affine Lie algebra.

The Lie algebra extension (9) is simpler than the group extension (8) in that it is determined up to isomorphism by a Lie algebra 2-cocycle $\omega(f, g)$, i.e. a skew bilinear map $\omega\colon \Omega\mathfrak{g} \times \Omega\mathfrak{g} \to \mathbb{R}$ satisfying the **2-cocycle condition**

$$\omega([f, g], h) + \omega([g, h], f) + \omega([h, f], g) = 0. \tag{10}$$

For G as above we may assume the cocycle ω equal, up to a scalar multiple, to the **Kac–Moody 2-cocycle**

$$\omega(f,g) = 2\int_0^{2\pi} \langle f(\theta), g'(\theta)\rangle \, d\theta \tag{11}$$

Here $\langle \cdot,\cdot \rangle$ is an invariant symmetric bilinear form on $\mathfrak{g}$. Thus, as a vector space $\widehat{\Omega\mathfrak{g}}$ is isomorphic to $\Omega\mathfrak{g} \oplus \mathbb{R}$, but the bracket is given by

$$[(f,\alpha),(g,\beta)] = ([f,g], \omega(f,g))$$

Since ω is a skew form on $\Omega\mathfrak{g}$, it defines a left-invariant 2-form ω on ΩG. The cocycle condition, Equation (10), says precisely that ω is closed. We quote the following theorem from Pressley and Segal, slightly corrected:

Theorem 23. *Suppose G is a simply-connected compact simple Lie group. Then:*

1. *The central extension of Lie algebras*

$$0 \to \mathfrak{u}(1) \to \widehat{\Omega\mathfrak{g}} \to \Omega\mathfrak{g} \to 0$$

defined by the cocycle ω above corresponds to a central extension of Fréchet Lie groups

$$1 \to \mathrm{U}(1) \to \widehat{\Omega G} \to \Omega G \to 1$$

in the sense that $i\omega$ is the curvature of a left-invariant connection on the principal $\mathrm{U}(1)$*-bundle $\widehat{\Omega G}$ iff the* 2*-form $\omega/2\pi$ on ΩG has integral periods.*

2. *The 2-form $\omega/2\pi$ has integral periods iff the invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ satisfies this integrality condition: $\langle h_\theta, h_\theta\rangle \in \frac{1}{2\pi}\mathbb{Z}$ for the coroot h_θ associated to the highest root θ of G.*

Since G is simple, all invariant symmetric bilinear forms on its Lie algebra are proportional, so there is a unique invariant inner product $(\cdot,\cdot)$ with $(h_\theta, h_\theta) = 2$. Pressley and Segal [18] call this inner product the **basic inner product** on $\mathfrak{g}$. In what follows, we always use $\langle\cdot,\cdot\rangle$ to stand for this basic inner product divided by 4π. This is the smallest inner product to satisfy the integrality condition in the above theorem.

More generally, for any integer k, the inner product $k\langle\cdot,\cdot\rangle$ satisfies the integrality condition in Theorem 23. It thus gives rise to a central extension

$$1 \to \mathrm{U}(1) \to \widehat{\Omega_k G} \to \Omega G \to 1$$

of ΩG. The integer k is called the **level** of the central extension $\widehat{\Omega_k G}$.

3.2 The Kac–Moody group $\widehat{\Omega_k G}$

In this section we begin by recalling an explicit construction of $\widehat{\Omega_k G}$ due to Murray and Stevenson [17], inspired by the work of Mickelsson [15]. We then use this to prove a result, Proposition 24, that will be crucial for constructing the 2-group $\mathcal{P}_k G$.

First, suppose that $\mathcal{G}$ is any Fréchet Lie group. Let $P_0\mathcal{G}$ denote the space of smooth based paths in $\mathcal{G}$:

$$P_0\mathcal{G} = \{f \in C^\infty([0,2\pi],\mathcal{G})\colon\ f(0) = 1\}$$

$P_0\mathcal{G}$ is a Fréchet Lie group under pointwise multiplication of paths, whose Lie algebra is

$$P_0 L = \{f \in C^\infty([0,2\pi], L)\colon\ f(0) = 0\}$$

where L is the Lie algebra of $\mathcal{G}$. Furthermore, the map $\pi\colon P_0\mathcal{G} \to \mathcal{G}$ which evaluates a path at its endpoint is a homomorphism of Fréchet Lie groups. The kernel of π is equal to

$$\Omega\mathcal{G} = \{f \in C^\infty([0,2\pi],\mathcal{G})\colon\ f(0) = f(1) = 1\,\}$$

Thus, $\Omega\mathcal{G}$ is a normal subgroup of $P_0\mathcal{G}$. Note that we are defining $\Omega\mathcal{G}$ in a somewhat nonstandard way, since its elements can be thought of as loops $f\colon S^1 \to \mathcal{G}$ that are smooth everywhere except at the basepoint, where both left and right derivatives exist to all orders, but need not agree. However, we need this for the sequence

$$1 \longrightarrow \Omega\mathcal{G} \longrightarrow P_0\mathcal{G} \xrightarrow{\pi} \mathcal{G} \longrightarrow 1$$

to be exact, and our $\Omega\mathcal{G}$ is homotopy equivalent to the standard one.

At present we are most interested in the case where $\mathcal{G} = \Omega G$. Then a point in $P_0\mathcal{G}$ gives a map $f\colon [0,2\pi] \times S^1 \to G$ with $f(0,\theta) = 1$ for all $\theta \in S^1$, $f(t,0) = 1$ for all $t \in [0,2\pi]$. It is an easy calculation [17] to show that the map $\kappa\colon P_0\Omega G \times P_0\Omega G \to \mathrm{U}(1)$ defined by

$$\kappa(f,g) = \exp\left(2ik\int_0^{2\pi}\int_0^{2\pi}\langle f(t)^{-1}f'(t), g'(\theta)g(\theta)^{-1}\rangle\, d\theta\, dt\right) \tag{12}$$

is a group 2-cocycle. This 2-cocycle κ makes $P_0\Omega G \times \mathrm{U}(1)$ into a group with the following product:

$$(f_1,z_1)\cdot(f_2,z_2) = (f_1 f_2, z_1 z_2\, \kappa(f_1,f_2))\,.$$

Let N be the subset of $P_0\Omega G\times\mathrm{U}(1)$ consisting of pairs (γ, z) such that $\gamma\colon [0,2\pi] \to \Omega G$ is a loop based at $1 \in \Omega G$ and

$$z = \exp\left(-ik\int_{D_\gamma}\omega\right)$$

where D_γ is any disk in ΩG with γ as its boundary. It is easy to check that N is a normal subgroup of the group $P_0\Omega G \times \mathrm{U}(1)$ with the product defined as above. To construct $\widehat{\Omega_k G}$ we form the quotient group $(P_0\Omega G \times \mathrm{U}(1))/N$. In [17] it is shown that the resulting central extension is isomorphic to the central extension of ΩG at level k. So we have the commutative diagram

$$\begin{tikzcd} P_0\Omega G \times \mathrm{U}(1) \arrow[r] \arrow[d] & \widehat{\Omega_k G} \arrow[d] \\ P_0\Omega G \arrow[r, "\pi"] & \Omega G \end{tikzcd} \tag{13}$$

where the horizontal maps are quotient maps, the upper horizontal map corresponding to the normal subgroup N, and the lower horizontal map corresponding to the normal subgroup $\Omega^2 G$ of $P_0\Omega G$.

Notice that the group of based paths $P_0 G$ acts on ΩG by conjugation. The next proposition shows that this action lifts to an action on $\widehat{\Omega_k G}$:

Proposition 24. *The action of $P_0 G$ on ΩG by conjugation lifts to a smooth action α of $P_0 G$ on $\widehat{\Omega_k G}$, whose differential gives an action $d\alpha$ of the Lie algebra $P_0\mathfrak{g}$ on the Lie algebra $\widehat{\Omega_k \mathfrak{g}}$ with*

$$d\alpha(p)(\ell, c) = \big([p,\ell],\ 2k\int_0^{2\pi} \langle p(\theta), \ell'(\theta)\rangle\, d\theta\,\big).$$

for all $p \in P_0\mathfrak{g}$ and all $(\ell, c) \in \Omega\mathfrak{g} \oplus \mathbb{R} \cong \widehat{\Omega_k \mathfrak{g}}$.

Proof. To construct α it suffices to construct a smooth action of $P_0 G$ on $P_0\Omega G \times \mathrm{U}(1)$ that preserves the product on this group and also preserves the normal subgroup N. Let $p\colon [0, 2\pi] \to G$ be an element of $P_0 G$, so that $p(0) = 1$. Define the action of p on a point $(f, z) \in P_0\Omega G \times \mathrm{U}(1)$ to be

$$p \cdot (f, z) = \big(pfp^{-1},\ z\exp(ik\int_0^{2\pi} \beta_p(f(t)^{-1}f'(t))\, dt)\ \big)$$

where β_p is the left-invariant 1-form on ΩG corresponding to the linear map $\beta_p\colon \Omega\mathfrak{g} \to \mathbb{R}$ given by:

$$\beta_p(\xi) = -2\int_0^{2\pi} \langle \xi(\theta), p(\theta)^{-1}p'(\theta)\rangle\, d\theta.$$

for $\xi \in \Omega\mathfrak{g}$. To check that this action preserves the product on $P_0\Omega G \times \mathrm{U}(1)$, we have to show that

$$\big(pf_1p^{-1},\ z_1\exp(ik\int_0^{2\pi} \beta_p(f_1(t)^{-1}f_1'(t))\, dt)\,\big)\cdot\big(pf_2p^{-1},\ z_2\exp(ik\int_0^{2\pi} \beta_p(f_2(t)^{-1}f_2'(t))\, dt)\,\big)$$

$$= \big(pf_1f_2p^{-1},\ z_1z_2\kappa(f_1, f_2)\exp(ik\int_0^{2\pi} \beta_p((f_1f_2)(t)^{-1}(f_1f_2)'(t))\, dt)\,\big).$$

It therefore suffices to establish the identity

$$\kappa(pf_1p^{-1}, pf_2p^{-1}) = \kappa(f_1, f_2) \exp\Big(ik \int_0^{2\pi} \big(\beta_p((f_1f_2)(t)^{-1}(f_1f_2)'(t)) - \beta_p(f_1(t)^{-1}f_1'(t)) - \beta_p(f_2(t)^{-1}f_2'(t))\big)\ dt\Big).$$

This is a straightforward computation that can safely be left to the reader.

Next we check that the normal subgroup N is preserved by the action of P_0G. For this we must show that if $(f, z) \in N$ then

$$\big(pfp^{-1},\ z\exp(ik \int_0^{2\pi} \beta_p(f^{-1}f')\, dt)\big) \in N.$$

Recall that N consists of pairs (γ, z) such that $\gamma \in \Omega^2 G$ and $z = \exp(-ik \int_{D_\gamma} \omega)$ where D_γ is a disk in ΩG with boundary γ. Therefore we need to show that

$$\exp\left(ik \int_{D_{p^{-1}\gamma p}} \omega\right) = \exp\left(ik \int_{D_\gamma} \omega\right) \exp\left(-ik \int_0^{2\pi} \beta_p(\gamma^{-1}\gamma')dt\right).$$

This follows immediately from the identity

$$\mathrm{Ad}(p)^*\omega = \omega - d\beta_p,$$

which is easily established by direct computation.

Finally, we have to check the formula for $d\alpha$. On passing to Lie algebras, diagram (13) gives rise to the following commutative diagram of Lie algebras:

$$\begin{tikzcd}
P_0\Omega\mathfrak{g} \oplus \mathbb{R} \arrow[r, "\overline{\mathrm{ev}}"] \arrow[d] & \Omega\mathfrak{g} \oplus \mathbb{R} \arrow[d] \\
P_0\Omega\mathfrak{g} \arrow[r, "\mathrm{ev}"] & \Omega\mathfrak{g}
\end{tikzcd}$$

where $\overline{\mathrm{ev}}$ is the homomorphism $(f, c) \mapsto (f(2\pi), c)$ for $f \in P_0\Omega\mathfrak{g}$ and $c \in \mathbb{R}$. To calculate $d\alpha(p)(\ell, c)$ we compute $\overline{\mathrm{ev}}(d\alpha(p)(\tilde{\ell}, c))$ where $\tilde{\ell}$ satisfies $\mathrm{ev}(\tilde{\ell}) = \ell$ (take, for example, $\tilde{\ell}(t) = t\ell/2\pi$). It is then straightforward to compute that

$$\overline{\mathrm{ev}}(d\alpha(p)(\tilde{\ell}, c)) = \big([p, \ell], 2k \int_0^{2\pi} \langle p(\theta), \ell'(\theta)\rangle\, d\theta\big).$$

□

4 The Lie 2-Group $\mathcal{P}_kG$

Having completed our review of Lie 2-algebras and loop groups, we now study a Lie 2-group $\mathcal{P}_kG$ whose Lie 2-algebra $\mathcal{P}_k\mathfrak{g}$ is equivalent to $\mathfrak{g}_k$. We begin in

Section 4.1 by giving a construction of $\mathcal{P}_k G$ in terms of the central extension $\widehat{\Omega_k G}$ of the loop group of G. This yields a description of $\mathcal{P}_k \mathfrak{g}$ which we use later to prove that this Lie 2-algebra is equivalent to $\mathfrak{g}_k$.

Section 4.2 gives another viewpoint on $\mathcal{P}_k G$, which goes a long way toward explaining the significance of this 2-group. For this, we study the topological group $|\mathcal{P}_k G|$ formed by taking the geometric realization of the nerve of $\mathcal{P}_k G$.

4.1 Constructing $\mathcal{P}_k G$

In Proposition 24 we saw that the action of the path group $P_0 G$ on the loop group ΩG by conjugation lifts to an action α of $P_0 G$ on the central extension $\widehat{\Omega_k G}$. This allows us to define a Fréchet Lie group $P_0 G \ltimes \widehat{\Omega_k G}$ in which multiplication is given by:

$$(p_1, \hat{\ell}_1) \cdot (p_2, \hat{\ell}_2) = \big(p_1 p_2,\ \hat{\ell}_1 \alpha(p_1)(\hat{\ell}_2) \big).$$

This, in turn, allows us to construct the 2-group $\mathcal{P}_k G$ which plays the starring role in this paper:

Proposition 25. *Suppose G is a simply-connected compact simple Lie group and $k \in \mathbb{Z}$. Then there is a Fréchet Lie 2-group $\mathcal{P}_k G$ for which:*

- *The Fréchet Lie group of objects $\mathrm{Ob}(\mathcal{P}_k G)$ is $P_0 G$.*
- *The Fréchet Lie group of morphisms $\mathrm{Mor}(\mathcal{P}_k G)$ is $P_0 G \ltimes \widehat{\Omega_k G}$.*
- *The source and target maps $s, t \colon \mathrm{Mor}(\mathcal{P}_k G) \to \mathrm{Ob}(\mathcal{P}_k G)$ are given by:*

$$\begin{array}{rcl} s(p, \hat{\ell}) & = & p \\ t(p, \hat{\ell}) & = & \partial(\hat{\ell}) p \end{array}$$

  where $p \in P_0 G$, $\hat{\ell} \in \widehat{\Omega_k G}$, and $\partial \colon \widehat{\Omega_k G} \to P_0 G$ is the composite:

$$\widehat{\Omega_k G} \to \Omega G \hookrightarrow P_0 G\,.$$

- *The identity-assigning map $i \colon \mathrm{Ob}(\mathcal{P}_k G) \to \mathrm{Mor}(\mathcal{P}_k G)$ is given by:*

$$i(p) = (p, 1).$$

- *The composition map $\circ \colon \mathrm{Mor}(\mathcal{P}_k G) \times_{\mathrm{Ob}(\mathcal{P}_k G)} \mathrm{Mor}(\mathcal{P}_k G) \to \mathrm{Mor}(\mathcal{P}_k G)$ is given by:*

$$(p_1, \hat{\ell}_1) \circ (p_2, \hat{\ell}_2) = (p_2, \hat{\ell}_1 \hat{\ell}_2)$$

  whenever $(p_1, \hat{\ell}_1), (p_2, \hat{\ell}_2)$ are composable morphisms in $\mathcal{P}_k G$.

Proof. One can check directly that $s, t, i, \circ$ are Fréchet Lie group homomorphisms and that these operations make $\mathcal{P}_k G$ into a category. Alternatively, one can check that $(P_0 G, \widehat{\Omega_k G}, \alpha, \partial)$ is a crossed module in the category of Fréchet manifolds. This merely requires checking that

$$\partial(\alpha(p)(\hat{\ell})) = p\,\partial(\hat{\ell})\,p^{-1} \tag{14}$$

and

$$\alpha(\partial(\hat{\ell}_1))(\hat{\ell}_2) = \hat{\ell}_1 \hat{\ell}_2 \hat{\ell}_1^{-1}. \tag{15}$$

Then one can use the fact that crossed modules in the category of Fréchet manifolds are the same as Fréchet Lie 2-groups (see for example HDA6). $\square$

We denote the Lie 2-algebra of $\mathcal{P}_k G$ by $\mathcal{P}_k \mathfrak{g}$. To prove this Lie 2-algebra is equivalent to $\mathfrak{g}_k$ in Section 5, we will use an explicit description of its corresponding L_∞-algebra:

Proposition 26. *The 2-term L_∞-algebra V corresponding to the Lie 2-algebra $\mathcal{P}_k \mathfrak{g}$ has:*

- $V_0 = P_0 \mathfrak{g}$ *and* $V_1 = \widehat{\Omega_k \mathfrak{g}} \cong \Omega \mathfrak{g} \oplus \mathbb{R}$,
- $d \colon V_1 \to V_0$ *equal to the composite*

$$\widehat{\Omega_k \mathfrak{g}} \to \Omega \mathfrak{g} \hookrightarrow P_0 \mathfrak{g}\,,$$

- $l_2 \colon V_0 \times V_0 \to V_1$ *given by the bracket in* $P_0 \mathfrak{g}$:

$$l_2(p_1, p_2) = [p_1, p_2],$$

and $l_2 \colon V_0 \times V_1 \to V_1$ *given by the action* $d\alpha$ *of* $P_0 \mathfrak{g}$ *on* $\widehat{\Omega_k \mathfrak{g}}$*, or explicitly:*

$$l_2(p, (\ell, c)) = \big([p, \ell],\ 2k \int_0^{2\pi} \langle p(\theta), \ell'(\theta) \rangle \, d\theta\ \big)$$

for all $p \in P_0 \mathfrak{g}$, $\ell \in \Omega G$ *and* $c \in \mathbb{R}$.

- $l_3 \colon V_0 \times V_0 \times V_0 \to V_1$ *equal to zero.*

Proof. This is a straightforward application of the correspondence described in Section 2.2. The formula for $l_2 \colon V_0 \times V_1 \to V_1$ comes from Proposition 24, while $\ell_3 = 0$ because the Lie 2-algebra $\mathcal{P}_k \mathfrak{g}$ is strict. $\square$

4.2 The Topology of $|\mathcal{P}_k G|$

In this section we construct an exact sequence of Fréchet Lie 2-groups:

$$1 \to \mathcal{L}_k G \xrightarrow{\iota} \mathcal{P}_k G \xrightarrow{\pi} G \to 1\,,$$

where G is considered as a Fréchet Lie 2-group with only identity morphisms. Applying a certain procedure for turning topological 2-groups into topological groups, described below, we obtain this exact sequence of topological groups:

$$1 \to |\mathcal{L}_k G| \xrightarrow{|\iota|} |\mathcal{P}_k G| \xrightarrow{|\pi|} G \to 1\,.$$

Note that $|G| = G$. We then show that the topological group $|\mathcal{L}_k G|$ has the homotopy type of the Eilenberg–Mac Lane space $K(\mathbb{Z}, 2)$. Since $K(\mathbb{Z}, 2)$ is also the classifying space $B\mathrm{U}(1)$, the above exact sequence is a topological analogue of the exact sequence of Lie 2-algebras describing how $\mathfrak{g}_k$ is built from $\mathfrak{g}$ and $\mathfrak{u}(1)$:

$$0 \to \mathrm{b}\mathfrak{u}(1) \to \mathfrak{g}_k \to \mathfrak{g} \to 0\,,$$

where $\mathrm{b}\mathfrak{u}(1)$ is the Lie 2-algebra with a 0-dimensional space of objects and $\mathfrak{u}(1)$ as its space of morphisms.

The above exact sequence of topological groups exhibits $|\mathcal{P}_k G|$ as the total space of a principal $K(\mathbb{Z}, 2)$ bundle over G. Bundles of this sort are classified by their 'Dixmier–Douady class', which is an element of the integral third cohomology group of the base space. In the case at hand, this cohomology group is $H^3(G) \cong \mathbb{Z}$, generated by the element we called $[\nu/2\pi]$ in the Introduction. We shall show that the Dixmier–Douady class of the bundle $|\mathcal{P}_k G| \to G$ equals $k[\nu/2\pi]$. Using this, we show that for $k = \pm 1$, $|\mathcal{P}_k G|$ is a version of $\hat{G}$ — the topological group obtained from G by killing its third homotopy group.

We start by defining a map $\pi\colon \mathcal{P}_k G \to G$ as follows. We define π on objects $p \in \mathcal{P}_k G$ as follows:

$$\pi(p) = p(2\pi) \in G.$$

In other words, π applied to a based path in G gives the endpoint of this path. We define π on morphisms in the only way possible, sending any morphism $(p, \hat{\ell})\colon p \to \partial(\hat{\ell})p$ to the identity morphism on $\pi(p)$. It is easy to see that π is a **strict homomorphism** of Fréchet Lie 2-groups: in other words, a map that strictly preserves all the Fréchet Lie 2-group structure. Moreover, it is easy to see that π is onto both for objects and morphisms.

Next, we define the Fréchet Lie 2-group $\mathcal{L}_k G$ to be the **strict kernel** of π. In other words, the objects of $\mathcal{L}_k G$ are objects of $\mathcal{P}_k G$ that are mapped to 1 by π, and similarly for the morphisms of $\mathcal{L}_k G$, while the source, target, identity-assigning and composition maps for $\mathcal{L}_k G$ are just restrictions of those for $\mathcal{P}_k G$. So:

- the Fréchet Lie group of objects $\mathrm{Ob}(\mathcal{L}_k G)$ is ΩG,
- the Fréchet Lie group of morphisms $\mathrm{Mor}(\mathcal{L}_k G)$ is $\Omega G \ltimes \widehat{\Omega_k G}$,

where the semidirect product is formed using the action α restricted to ΩG. Moreover, the formulas for $s, t, i, \circ$ are just as in Proposition 25, but with loops replacing paths.

It is easy to see that the inclusion $\iota\colon \mathcal{L}_k G \to \mathcal{P}_k G$ is a strict homomorphism of Fréchet Lie 2-groups. We thus obtain:

Proposition 27. *The sequence of strict Fréchet 2-group homomorphisms*

$$1 \to \mathcal{L}_k G \xrightarrow{\iota} \mathcal{P}_k G \xrightarrow{\pi} G \to 1$$

is **strictly exact***, meaning that the image of each arrow is equal to the kernel of the next, both on objects and on morphisms.*

Any Fréchet Lie 2-group C is, among other things, a **topological category**: a category where the sets $\mathrm{Ob}(C)$ and $\mathrm{Mor}(C)$ are topological spaces and the source, target, identity-assigning and composition maps are continuous. Homotopy theorists have a standard procedure for taking the 'nerve' of a topological category and obtaining a simplicial space. They also know how to take the 'geometric realization' of any simplicial space, obtaining a topological space. We use $|C|$ to denote the geometric realization of the nerve of a topological category C. If C is in fact a topological 2-group — for example a Fréchet Lie 2-group — then $|C|$ naturally becomes a topological group.

For readers unfamiliar with these constructions, let us give a more hands-on description of how to build $|C|$. First for any $n \in \mathbb{N}$ we construct a space $|C|_n$. A point in $|C|_n$ consists of a string of n composable morphisms in C:

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} x_{n-1} \xrightarrow{f_n} x_n$$

together with a point in the standard n-simplex:

$$a \in \Delta_n = \{(a_0, \ldots, a_n) \in [0,1]\colon\ a_0 + \cdots + a_n = 1\}.$$

Since $|C|_n$ is a subset of $\mathrm{Mor}(C)^n \times \Delta_n$, we give it the subspace topology. There are well-known face maps $d_i\colon \Delta_n \to \Delta_{n+1}$ and degeneracies $s_i\colon \Delta_n \to \Delta_{n-1}$. We use these to build $|C|$ by gluing together all the spaces $|C|_n$ via the following identifications:

$$\left(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n, a\right) \sim \left(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_i} x_i \xrightarrow{1} x_i \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_n} x_n, d_i(a)\right)$$

for $0 \le i \le n$, and

$$\left(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n, a\right) \sim \left(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i-2}} x_{i-1} \xrightarrow{f_i f_{i+1}} x_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} x_n, s_i(a)\right)$$

for $0 < i < n$, together with

$$\left(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n, a\right) \sim \left(x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} x_n, s_0(a)\right)$$

and

$$\left(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n, a\right) \sim \left(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-2}} x_{n-2} \xrightarrow{f_{n-1}} x_{n-1}, s_n(a)\right)$$

This defines $|C|$ as a topological space, but when C is a topological 2-group the multiplication in C makes $|C|$ into a topological group. Moreover, if G is a topological group viewed as a topological 2-group with only identity morphisms, we have $|G| \cong G$.

Applying the functor $|\cdot|$ to the exact sequence in Proposition 27, we obtain this result, which implies Theorem 3:

Theorem 28. *The sequence of topological groups*

$$1 \to |\mathcal{L}_k G| \xrightarrow{|\iota|} |\mathcal{P}_k G| \xrightarrow{|\pi|} G \to 1$$

is exact, and $|\mathcal{L}_k G|$ has the homotopy type of $K(\mathbb{Z}, 2)$. Thus, $|\mathcal{P}_k G|$ is the total space of a $K(\mathbb{Z}, 2)$ bundle over G. The Dixmier–Douady class of this bundle is $k[\nu/2\pi] \in H^3(G)$. Moreover, $\mathcal{P}_k G$ is $\hat{G}$ when $k = \pm 1$.

Proof. It is easy to see directly that the functor $|\cdot|$ carries strictly exact sequences of topological 2-groups to exact sequences of topological groups. To show that $|\mathcal{L}_k G|$ is a $K(\mathbb{Z}, 2)$, we prove there is a strictly exact sequence of Fréchet Lie 2-groups

$$1 \to \mathrm{U}(1) \to \mathcal{E}\widehat{\Omega_k G} \to \mathcal{L}_k G \to 1 \,. \tag{16}$$

Here U(1) is regarded as a Fréchet Lie 2-group with only identity morphisms, while $\mathcal{E}\widehat{\Omega_k G}$ is the Fréchet Lie 2-group with $\widehat{\Omega_k G}$ as its Fréchet Lie group of objects and precisely one morphism from any object to any other. In general:

Lemma 29. *For any Fréchet Lie group $\mathcal{G}$, there is a Fréchet Lie 2-group $\mathcal{E}\mathcal{G}$ with:*

- *$\mathcal{G}$ as its Fréchet Lie group of objects,*
- *$\mathcal{G} \ltimes \mathcal{G}$ as its Fréchet Lie group of morphisms, where the semidirect product is defined using the conjugation action of $\mathcal{G}$ on itself,*

and:

- *source and target maps given by $s(g, g') = g$, $t(g, g') = gg'$,*
- *identity-assigning map given by $i(g) = (g, 1)$,*
- *composition map given by $(g_1, g_1') \circ (g_2, g_2') = (g_2, g_1' g_2')$ whenever (g_1, g_1'), (g_2, g_2') are composable morphisms in $\mathcal{E}\mathcal{G}$.*

Proof. It is straightforward to check that this gives a Fréchet Lie 2-group. Note that $\mathcal{E}\mathcal{G}$ has $\mathcal{G}$ as objects and one morphism from any object to any other. $\square$

In fact, Segal [20] has already introduced $\mathcal{E}\mathcal{G}$ under the name $\overline{\mathcal{G}}$, treating it as a topological category. He proved that $|\mathcal{E}\mathcal{G}|$ is contractible. In fact, he exhibited $|\mathcal{E}\mathcal{G}|$ as a model of $E\mathcal{G}$, the total space of the universal bundle over the classifying space $B\mathcal{G}$ of $\mathcal{G}$. Therefore, applying the functor $|\cdot|$ to the exact sequence (16), we obtain this short exact sequence of topological groups:

$$1 \to \mathrm{U}(1) \to E\widehat{\Omega_k G} \to |\mathcal{L}_k G| \to 1 \,.$$

Since $E\widehat{\Omega_k G}$ is contractible, it follows that $|\mathcal{L}_k G| \cong E\widehat{\Omega G}/\mathrm{U}(1)$ has the homotopy type of $B\mathrm{U}(1) \simeq K(\mathbb{Z}, 2)$.

One can check that $|\pi|\colon |\mathcal{P}_k G| \to G$ is a locally trivial fiber bundle, so it defines a principal $K(\mathbb{Z},2)$ bundle over G. Like any such bundle, this is the pullback of the universal principal $K(\mathbb{Z},2)$ bundle $p\colon EK(\mathbb{Z},2) \to BK(\mathbb{Z},2)$ along some map $f\colon G \to BK(\mathbb{Z},2)$, giving a commutative diagram of spaces:

$$
\begin{tikzcd}
|\mathcal{L}_k G| \arrow[r, "|\iota|"] \arrow[d, "\sim"'] & |\mathcal{P}_k G| \arrow[r, "|\pi|"] \arrow[d, "p^* f"] & G \arrow[d, "f"] \\
K(\mathbb{Z},2) \arrow[r, "i"] & EK(\mathbb{Z},2) \arrow[r, "p"] & BK(\mathbb{Z},2)
\end{tikzcd}
$$

Indeed, such bundles are classified up to isomorphism by the homotopy class of f. Since $BK(\mathbb{Z},2) \simeq K(\mathbb{Z},3)$, this homotopy class is determined by the Dixmier–Douady class $f^*\kappa$, where κ is the generator of $H^3(K(\mathbb{Z},3)) \cong \mathbb{Z}$. The next order of business is to show that $f^*\kappa = k[\nu/2\pi]$.

For this, it suffices to show that f maps the generator of $\pi_3(G) \cong \mathbb{Z}$ to k times the generator of $\pi_3(K(\mathbb{Z},3)) \cong \mathbb{Z}$. Consider this bit of the long exact sequences of homotopy groups coming from the above diagram:

$$
\begin{tikzcd}
\pi_3(G) \arrow[r, "\partial"] \arrow[d, "\pi_3(f)"'] & \pi_2(|\mathcal{L}_k G|) \arrow[d, "\cong"] \\
\pi_3(K(\mathbb{Z},3)) \arrow[r, "\partial'"] & \pi_2(K(\mathbb{Z},2))
\end{tikzcd}
$$

Since the connecting homomorphism ∂' and the map from $\pi_2(|\mathcal{L}_k G|)$ to $\pi_2(K(\mathbb{Z},2))$ are isomorphisms, we can treat these as the identity by a suitable choice of generators. Thus, to show that $\pi_3(f)$ is multiplication by k it suffices to show this for the connecting homomorphism ∂.

To do so, consider this commuting diagram of Frechét Lie 2-groups:

$$
\begin{tikzcd}
\Omega G \arrow[r, "\iota"] \arrow[d, "i"'] & P_0 G \arrow[r, "\pi"] \arrow[d, "i'"'] & G \arrow[d, "1"] \\
\mathcal{L}_k G \arrow[r, "\iota"] & \mathcal{P}_k G \arrow[r, "\pi"] & G
\end{tikzcd}
$$

Here we regard the groups on top as 2-groups with only identity morphisms; the downwards-pointing arrows include these in the 2-groups on the bottom row. Applying the functor $|\cdot|$, we obtain a diagram where each row is a principal bundle:

$$\begin{tikzcd}
\Omega G \arrow[r, "|\iota|"] \arrow[d, "|i|"'] & P_0 G \arrow[r, "|\pi|"] \arrow[d, "|i'|"'] & G \arrow[d, "1"] \\
|\mathcal{L}_k G| \arrow[r, "|\iota|"] & |\mathcal{P}_k G| \arrow[r, "|\pi|"] & G
\end{tikzcd}$$

Taking long exact sequences of homotopy groups, this gives:

$$\begin{tikzcd}
\pi_3(G) \arrow[r, "1"] \arrow[d, "1"'] & \pi_2(\Omega G) \arrow[d, "\pi_2(|i|)"] \\
\pi_3(G) \arrow[r, "\partial"] & \pi_2(|\mathcal{L}_k G|)
\end{tikzcd}$$

Thus, to show that ∂ is multiplication by k it suffices to show this for $\pi_2(|i|)$.

For this, we consider yet another commuting diagram of Fréchet Lie 2-groups:

$$\begin{tikzcd}
\mathrm{U}(1) \arrow[r] \arrow[d] & \widehat{\Omega_k G} \arrow[r] \arrow[d] & \Omega G \arrow[d, "i"'] \\
\mathrm{U}(1) \arrow[r] & \mathcal{E}\widehat{\Omega_k G} \arrow[r] & \mathcal{L}_k G
\end{tikzcd}$$

Applying $|\cdot|$, we obtain a diagram where each row is a principal $\mathrm{U}(1)$ bundle:

$$\begin{tikzcd}
\mathrm{U}(1) \arrow[r] \arrow[d] & \widehat{\Omega_k G} \arrow[r] \arrow[d] & \Omega G \arrow[d, "|i|"] \\
\mathrm{U}(1) \arrow[r] & |\mathcal{E}\widehat{\Omega_k G}| \arrow[r] & |\mathcal{L}_k G| \simeq K(\mathbb{Z},2)
\end{tikzcd}$$

Recall that the bottom row is the universal principal $\mathrm{U}(1)$ bundle. The arrow $|i|$ is the classifying map for the $\mathrm{U}(1)$ bundle $\widehat{\Omega_k G} \to \Omega G$. By Theorem 23, the Chern class of this bundle is k times the generator of $H^2(\Omega G)$, so $\pi_2(|i|)$ must map the generator of $\pi_2(\Omega G)$ to k times the generator of $\pi_2(K(\mathbb{Z},2))$.

Finally, let us show that $|\mathcal{P}_k G|$ is $\hat{G}$ when $k = \pm 1$. For this, it suffices to show that when $k = \pm 1$, the map $|\pi| \colon |\mathcal{P}_k G| \to G$ induces isomorphisms on all homotopy groups except the third, and that $\pi_3(|\mathcal{P}_k G|) = 0$. For this we examine the long exact sequence:

$$\cdots \longrightarrow \pi_n(|\mathcal{L}_k G|) \longrightarrow \pi_n(|\mathcal{P}_k G|) \longrightarrow \pi_n(G) \xrightarrow{\partial} \pi_{n-1}(|\mathcal{L}_k G|) \longrightarrow \cdots .$$

Since $|\mathcal{L}_k G| \simeq K(\mathbb{Z}, 2)$, its homotopy groups vanish except for $\pi_2(|\mathcal{L}_k G|) \cong \mathbb{Z}$, so $|\pi|$ induces an isomorphism on π_n except possibly for $n = 2, 3$. In this portion of the long exact sequence we have

$$0 \longrightarrow \pi_3(|\mathcal{P}_k G|) \longrightarrow \mathbb{Z} \xrightarrow{k} \mathbb{Z} \longrightarrow \pi_2(|\mathcal{P}_k G|) \longrightarrow 0$$

so $\pi_3(|\mathcal{P}_k G|) \cong 0$ unless $k = 0$, and $\pi_2(|\mathcal{P}_k G|) \cong \mathbb{Z}/k\mathbb{Z}$, so $\pi_2(|\mathcal{P}_k G|) \cong \pi_2(G) \cong 0$ when $k = \pm 1$. $\square$

5 The Equivalence Between $\mathcal{P}_k\mathfrak{g}$ and $\mathfrak{g}_k$

In this section we prove our main result, which implies Theorem 2:

Theorem 30. *There is a strictly exact sequence of Lie 2-algebra homomorphisms*

$$0 \to \mathcal{E}\Omega\mathfrak{g} \xrightarrow{\lambda} \mathcal{P}_k\mathfrak{g} \xrightarrow{\phi} \mathfrak{g}_k \to 0$$

where $\mathcal{E}\Omega\mathfrak{g}$ is equivalent to the trivial Lie 2-algebra and ϕ is an equivalence of Lie 2-algebras.

Recall that by 'strictly exact' we mean that both on the vector spaces of objects and the vector spaces of morphisms, the image of each map is the kernel of the next.

We prove this result in a series of lemmas. We begin by describing $\mathcal{E}\Omega\mathfrak{g}$ and showing that it is equivalent to the trivial Lie 2-algebra. Recall that in Lemma 29 we constructed for any Fréchet Lie group $\mathcal{G}$ a Fréchet Lie 2-group $\mathcal{E}\mathcal{G}$ with $\mathcal{G}$ as its group of objects and precisely one morphism from any object to any other. We saw that the space $|\mathcal{E}\mathcal{G}|$ is contractible; this is a topological reflection of the fact that $\mathcal{E}\mathcal{G}$ is equivalent to the trivial Lie 2-group. Now we need the Lie algebra analogue of this construction:

Lemma 31. *Given a Lie algebra L, there is a 2-term L_∞-algebra V for which:*

- $V_0 = L$ *and* $V_1 = L$,
- $d\colon V_1 \to V_0$ *is the identity,*
- $l_2\colon V_0 \times V_0 \to V_1$ *and* $l_2\colon V_0 \times V_1 \to V_1$ *are given by the bracket in* L,
- $l_3\colon V_0 \times V_0 \times V_0 \to V_1$ *is equal to zero.*

We call the corresponding strict Lie 2-algebra $\mathcal{E}L$.

Proof. Straightforward. $\square$

Lemma 32. *For any Lie algebra L, the Lie 2-algebra $\mathcal{E}L$ is equivalent to the trivial Lie 2-algebra. That is, $\mathcal{E}L \simeq 0$.*

Proof. There is a unique homomorphism $\beta\colon \mathcal{E}L \to 0$ and a unique homomorphism $\gamma\colon 0 \to \mathcal{E}L$. Clearly $\beta \circ \gamma$ equals the identity. The composite $\gamma \circ \beta$ has:

$$\begin{array}{lccc} (\gamma\circ\beta)_0: & x & \mapsto & 0 \\ (\gamma\circ\beta)_1: & x & \mapsto & 0 \\ (\gamma\circ\beta)_2: & (x_1,x_2) & \mapsto & 0\,, \end{array}$$

while the identity homomorphism from $\mathcal{E}L$ to itself has:

$$\begin{array}{lccc} \mathrm{id}_0: & x & \mapsto & x \\ \mathrm{id}_1: & x & \mapsto & x \\ \mathrm{id}_2: & (x_1,x_2) & \mapsto & 0\,. \end{array}$$

There is a 2-isomorphism

$$\tau\colon \gamma\circ\beta \stackrel{\sim}{\Longrightarrow} \mathrm{id}$$

given by

$$\tau(x) = x\,,$$

where the x on the left is in V_0 and that on the right in V_1, but of course $V_0 = V_1$ here. $\Box$

We continue by defining the Lie 2-algebra homomorphism $\mathcal{P}_k\mathfrak{g} \xrightarrow{\phi} \mathfrak{g}_k$.

Lemma 33. *There exists a Lie 2-algebra homomorphism*

$$\phi\colon \mathcal{P}_k\mathfrak{g} \to \mathfrak{g}_k$$

which we describe in terms of its corresponding L_∞-homomorphism:

$$\begin{array}{lcl} \phi_0(p) & = & p(2\pi) \\ \\ \phi_1(\ell,c) & = & c \\ \\ \phi_2(p_1,p_2) & = & k\displaystyle\int_0^{2\pi} \left(\langle p_1,p_2'\rangle - \langle p_1',p_2\rangle\right)\,d\theta \end{array}$$

where $p,p_1,p_2 \in P_0\mathfrak{g}$ and $(\ell,c)\in\Omega\mathfrak{g}\oplus\mathbb{R}\cong\widehat{\Omega_k\mathfrak{g}}$.

Before beginning, note that the quantity

$$\int_0^{2\pi} \left(\langle p_1,p_2'\rangle - \langle p_1',p_2\rangle\right)\,d\theta = 2\int_0^{2\pi}\langle p_1,p_2'\rangle\,d\theta \;-\; \langle p_1(2\pi),p_2(2\pi)\rangle$$

is skew-symmetric, but not in general equal to

$$2\int_0^{2\pi}\langle p_1,p_2'\rangle\,d\theta$$

due to the boundary term. However, these quantities are equal when either p_1 or p_2 is a loop.

Proof. We must check that ϕ satisfies the conditions of Definition 10. First we show that ϕ is a chain map. That is, we show that ϕ_0 and ϕ_1 preserve the differentials:

$$\begin{tikzcd} \widehat{\Omega_k \mathfrak{g}} \arrow[r, "d"] \arrow[d, "\phi_1"'] & P_0\mathfrak{g} \arrow[d, "\phi_0"] \\ \mathbb{R} \arrow[r, "d'"] & \mathfrak{g} \end{tikzcd}$$

where d is the composite given in Proposition 26, and $d' = 0$ since $\mathfrak{g}_k$ is skeletal. This square commutes since ϕ_0 is also zero.

We continue by verifying conditions (3) - (5) of Definition 10. The bracket on objects is preserved on the nose, which implies that the right-hand side of (3) is zero. This is consistent with the fact that the differential in the L_∞-algebra for $\mathfrak{g}_k$ is zero, which implies that the left-hand side of (3) is also zero.

The right-hand side of (4) is given by:

$$\begin{aligned} \phi_1(l_2(p,(\ell,c)) - l_2(\phi_0(p),\phi_1(\ell,c)) &= \phi_1\Big([p,\ell], 2k\int \langle p,\ell'\rangle\, d\theta\Big) - \underbrace{l_2(p(2\pi),c)}_{=0} \\ &= 2k\int \langle p,\ell'\rangle\, d\theta. \end{aligned}$$

This matches the left-hand side of (4), namely:

$$\begin{aligned} \phi_2(p,d(\ell,c)) &= \phi_2(p,\ell) \\ &= k\int (\langle p,\ell'\rangle - \langle p',\ell\rangle)\, d\theta \\ &= 2k\int \langle p,\ell'\rangle\, d\theta \end{aligned}$$

Note that no boundary term appears here since one of the arguments is a loop.

Finally, we check condition (5). Four terms in this equation vanish because $l_3 = 0$ in $\mathcal{P}_k\mathfrak{g}$ and $l_2 = 0$ in $\mathfrak{g}_k$. We are left needing to show

$$l_3(\phi_0(p_1),\phi_0(p_2),\phi_0(p_3)) = \phi_2(p_1,l_2(p_2,p_3)) + \phi_2(p_2,l_2(p_3,p_1)) + \phi_2(p_3,l_2(p_1,p_2))\,.$$

The left-hand side here equals $k\langle p_1(2\pi),[p_2(2\pi),p_3(2\pi)]\rangle$. The right-hand side equals:

$$\begin{aligned} & \phi_2(p_1,l_2(p_2,p_3)) \quad + \quad \text{cyclic permutations} \\ = \quad & k\int (\langle p_1,[p_2,p_3]'\rangle - \langle p_1',[p_2,p_3]\rangle)\, d\theta \quad + \text{ cyclic perms.} \\ = \quad & k\int (\langle p_1,[p_2',p_3]\rangle + \langle p_1,[p_2,p_3']\rangle - \langle p_1',[p_2,p_3]\rangle)\, d\theta \quad + \quad \text{cyclic perms.} \end{aligned}$$

Using the antisymmetry of $\langle \cdot, [\cdot, \cdot]\rangle$, this becomes:

$$k \int \left(\langle p_2', [p_3, p_1]\rangle + \langle p_3', [p_1, p_2]\rangle - \langle p_1', [p_2, p_3]\rangle \right) d\theta \quad + \quad \text{cyclic perms.}$$

The last two terms cancel when we add all their cyclic permutations, so we are left with all three cyclic permutations of the first term:

$$k \int \left(\langle p_1', [p_2, p_3]\rangle + \langle p_2', [p_3, p_1]\rangle + \langle p_3', [p_1, p_2]\rangle \right) d\theta \,.$$

If we apply integration by parts to the first term, we get:

$$k \int \left(-\langle p_1, [p_2', p_3]\rangle - \langle p_1, [p_2, p_3']\rangle + \langle p_2', [p_3, p_1]\rangle + \langle p_3', [p_1, p_2]\rangle \right) \, d\theta \, +$$

$$k\langle p_1(2\pi), [p_2(2\pi), p_3(2\pi)]\rangle \,.$$

By the antisymmetry of $\langle \cdot, [\cdot, \cdot]\rangle$, the four terms in the integral cancel, leaving just $k\langle p_1(2\pi), [p_2(2\pi), p_3(2\pi)]\rangle$, as desired. $\Box$

Next we show that the strict kernel of $\phi\colon \mathcal{P}_k\mathfrak{g} \to \mathfrak{g}_k$ is $\mathcal{E}\Omega\mathfrak{g}$:

Lemma 34. *There is a Lie 2-algebra homomorphism*

$$\lambda\colon \mathcal{E}\Omega\mathfrak{g} \to \mathcal{P}_k\mathfrak{g},$$

that is one-to-one both on objects and on morphisms, and whose range is precisely the kernel of $\phi\colon \mathcal{P}_k\mathfrak{g} \to \mathfrak{g}_k$, both on objects and on morphisms.

Proof. Glancing at the formula for ϕ in Lemma 33, we see that the kernel of ϕ_0 and the kernel of ϕ_1 are both $\Omega\mathfrak{g}$. We see from Lemma 31 that these are precisely the spaces V_0 and V_1 in the 2-term L_∞-algebra V corresponding to $\mathcal{E}\Omega\mathfrak{g}$. The differential $d\colon \ker(\phi_1) \to \ker(\phi_0)$ inherited from $\mathcal{E}\Omega\mathfrak{g}$ also matches that in V: it is the identity map on $\Omega\mathfrak{g}$.

Thus, we obtain an inclusion of 2-vector spaces $\lambda\colon \mathcal{E}\Omega\mathfrak{g} \to \mathcal{P}_k\mathfrak{g}$. This uniquely extends to a Lie 2-algebra homomorphism, which we describe in terms of its corresponding L_∞-homomorphism:

$$\begin{aligned}
\lambda_0(\ell) &= \ell \\
\lambda_1(\ell) &= (\ell, 0) \\
\lambda_2(\ell_1, \ell_2) &= \left(0, -2k \int_0^{2\pi} \langle \ell_1, \ell_2'\rangle \, d\theta\right)
\end{aligned}$$

where $\ell, \ell_1, \ell_2 \in \Omega\mathfrak{g}$, and the zero in the last line denotes the zero loop.

To prove this, we must show that the conditions of Definition 10 are satisfied. We first check that λ is a chain map, i.e., this square commutes:

$$\begin{tikzcd}
\Omega\mathfrak{g} \arrow[r, "d"] \arrow[d, "\lambda_1"'] & \Omega\mathfrak{g} \arrow[d, "\lambda_0"] \\
\widehat{\Omega_k\mathfrak{g}} \arrow[r, "d'"] & P_0\mathfrak{g}
\end{tikzcd}$$

where d is the identity and d' is the composite given in Proposition 26. To see this, note that $d'(\lambda_1(\ell)) = d'(\ell, 0) = \ell$ and $\lambda_0(d(\ell)) = \lambda_0(\ell) = \ell$.

We continue by verifying conditions (3) - (5) of Definition 10. The bracket on the space V_0 is strictly preserved by λ_0, which implies that the right-hand side of (3) is zero. It remains to show that the left-hand side, $d'(\lambda_2(\ell_1, \ell_2))$, is also zero. Indeed, we have:

$$d'(\lambda_2(\ell_1, \ell_2)) = d'\left(0, -2k \int \langle \ell_1, \ell_2' \rangle \, d\theta\right) = 0\,.$$

Next we check property (4). On the right-hand side, we have:

$$\begin{aligned}
\lambda_1(l_2(\ell_1, \ell_2)) - l_2(\lambda_0(\ell_1), \lambda_1(\ell_2)) &= ([\ell_1, \ell_2], 0) - \big([\ell_1, \ell_2], 2k \int \langle \ell_1, \ell_2' \rangle \, d\theta\big) \\
&= \big(0, -2k \int \langle \ell_1, \ell_2' \rangle \, d\theta\big)\,.
\end{aligned}$$

On the left-hand side, we have:

$$\lambda_2(\ell_1, d(\ell_2)) = \lambda_2(\ell_1, \ell_2) = \big(0, -2k \int \langle \ell_1, \ell_2' \rangle \, d\theta\big)$$

Note that this also shows that given the chain map defined by λ_0 and λ_1, the function λ_2 that extends this chain map to an L_∞-homomorphisms is unquely fixed by condition (4).

Finally, we show that λ_2 satisfies condition (5). The two terms involving l_3 vanish since λ is a map between two strict Lie 2-algebras. The three terms of the form $l_2(\lambda_0(\cdot), \lambda_2(\cdot, \cdot))$ vanish because the image of λ_2 lies in the center of $\widehat{\Omega_k\mathfrak{g}}$. It thus remains to show that

$$\lambda_2(\ell_1, l_2(\ell_2, \ell_3)) + \lambda_2(\ell_2, l_2(\ell_3, \ell_1)) + \lambda_2(\ell_3, l_2(\ell_1, \ell_2)) = 0.$$

This is just the cocycle property of the Kac–Moody cocycle, Equation (10). □

Next we check the exactness of the sequence

$$0 \to \mathcal{E}\Omega\mathfrak{g} \xrightarrow{\lambda} \mathcal{P}_k\mathfrak{g} \xrightarrow{\phi} \mathfrak{g}_k \to 0$$

at the middle point. Before doing so, we recall the formulas for the L_∞-homomorphisms corresponding to λ and ϕ. The L_∞-homomorphism corresponding to $\lambda\colon \mathcal{E}\Omega\mathfrak{g} \to \mathcal{P}_k\mathfrak{g}$ is given by

$$\begin{array}{rcl} \lambda_0(\ell) & = & \ell \\ \lambda_1(\ell) & = & (\ell, 0) \\ \lambda_2(\ell_1, \ell_2) & = & \big(0,\, -2k \displaystyle\int_0^{2\pi} \langle \ell_1, \ell_2' \rangle \, d\theta\big) \end{array}$$

where $\ell, \ell_1, \ell_2 \in \Omega\mathfrak{g}$, and that corresponding to $\phi\colon \mathcal{P}_k\mathfrak{g} \to \mathfrak{g}_k$ is given by:

$$\begin{array}{rcl} \phi_0(p) & = & p(2\pi) \\ \phi_1(\ell, c) & = & c \\ \phi_2(p_1, p_2) & = & k \displaystyle\int_0^{2\pi} \big(\langle p_1, p_2' \rangle - \langle p_1', p_2 \rangle\big) \, d\theta \end{array}$$

where $p, p_1, p_2 \in P_0\mathfrak{g}$, $\ell \in \Omega\mathfrak{g}$, and $c \in \mathbb{R}$.

Lemma 35. *The composite*

$$\mathcal{E}\Omega\mathfrak{g} \xrightarrow{\lambda} \mathcal{P}_k\mathfrak{g} \xrightarrow{\phi} \mathfrak{g}_k$$

is the zero homomorphism, and the kernel of ϕ is precisely the image of λ, both on objects and on morphisms.

Proof. The composites $(\phi \circ \lambda)_0$ and $(\phi \circ \lambda)_1$ clearly vanish. Moreover $(\phi \circ \lambda)_2$ vanishes since:

$$\begin{array}{rcll} (\phi \circ \lambda)_2(\ell_1, \ell_2) & = & \phi_2(\lambda_0(\ell_1), \lambda_0(\ell_2)) + \phi_1(\lambda_2(\ell_1, \ell_2)) & \text{by (6)} \\ & = & \phi_2(\ell_1, \ell_2) + \phi_1\big(0, -2k \displaystyle\int \langle \ell_1, \ell_2' \rangle \, d\theta\big) & \\ & = & k \displaystyle\int (\langle \ell_1, \ell_2' \rangle - \langle \ell_1', \ell_2 \rangle) \, d\theta - 2k \displaystyle\int \langle \ell_1, \ell_2' \rangle \, d\theta & \\ & = & 0 & \end{array}$$

with the help of integration by parts. The image of λ is precisely the kernel of ϕ by construction. $\square$

Note that ϕ is obviously onto, both for objects and morphisms, so we have an exact sequence

$$0 \to \mathcal{E}\Omega\mathfrak{g} \xrightarrow{\lambda} \mathcal{P}_k\mathfrak{g} \xrightarrow{\phi} \mathfrak{g}_k \to 0\,.$$

Next we construct a family of splittings $\psi\colon \mathfrak{g}_k \to \mathcal{P}_k\mathfrak{g}$ for this exact sequence:

Lemma 36. *Suppose*

$$f\colon [0, 2\pi] \to \mathbb{R}$$

is a smooth function with $f(0) = 0$ and $f(2\pi) = 1$. Then there is a Lie 2-algebra homomorphism

$$\psi\colon \mathfrak{g}_k \to \mathcal{P}_k\mathfrak{g}$$

whose corresponding L_∞-homomorphism is given by:

$$\begin{array}{rcl} \psi_0(x) & = & xf \\ \psi_1(c) & = & (0,c) \\ \psi_2(x_1,x_2) & = & ([x_1,x_2](f-f^2),0) \end{array}$$

where $x, x_1, x_2 \in \mathfrak{g}$ and $c \in \mathbb{R}$.

Proof. We show that ψ satisfies the conditions of Definition 10. We begin by showing that ψ is a chain map, meaning that the following square commutes:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "d"] \arrow[d, "\psi_1"'] & \mathfrak{g} \arrow[d, "\psi_0"] \\ \widehat{\Omega_k \mathfrak{g}} \arrow[r, "d'"] & P_0\mathfrak{g} \end{tikzcd}$$

where $d = 0$ since $\mathfrak{g}_k$ is skeletal and d' is the composite given in Proposition 26. This square commutes because $\psi_0(d(c)) = \psi_0(0) = 0$ and $d'(\psi_1(c)) = d'(0,c) = 0$.

We continue by verifying conditions (3) - (5) of Definition 10. The right-hand side of (3) equals:

$$\psi_0(l_2(x_1,x_2)) - l_2(\psi_0(x_1),\psi_0(x_2)) = [x_1,x_2](f-f^2)\,.$$

This equals the left-hand side $d'(\psi_2(x_1,x_2))$ by construction.

The right-hand side of (4) equals:

$$\psi_1(l_2(x,c)) - l_2(\psi_0(x),\psi_1(c)) \ = \ \psi_1(0) - l_2(xf,(0,c)) \ = \ 0$$

since both terms vanish separately. Since the left-hand side is $\psi_2(x,dc) = \psi_2(x,0) = 0$, this shows that ψ satisfies condition (4).

Finally we verify condition (5). The term $l_3(\psi_0(\cdot),\psi_0(\cdot),\psi_0(\cdot))$ vanishes because $\mathcal{P}_k\mathfrak{g}$ is strict. The sum of three other terms vanishes thanks to the Jacobi identity in $\mathfrak{g}$:

$$\begin{array}{rl} & \psi_2(x_1,l_2(x_2,x_3)) + \psi_2(x_2,l_2(x_3,x_1)) + \psi_2(x_3,l_2(x_1,x_2)) \\ = & \Big(([x_1,[x_2,x_3]] + [x_2,[x_3,x_1]] + [x_3,[x_1,x_2]])\,(f-f^2),\, 0\Big) \\ = & (0,0)\,. \end{array}$$

Thus, it remains to show that:

$$-\psi_1(l_3(x_1,x_2,x_3)) =$$

$$l_2(\psi_0(x_1),\psi_2(x_2,x_3)) + l_2(\psi_0(x_2),\psi_2(x_3,x_1)) + l_2(\psi_0(x_3),\psi_2(x_1,x_2))\,.$$

This goes as follows:

$$\begin{array}{rl} & l_2(\psi_0(x_1),\psi_2(x_2,x_3)) + l_2(\psi_0(x_2),\psi_2(x_3,x_1)) + l_2(\psi_0(x_3),\psi_2(x_1,x_2)) \\ = & \left(0, 3\cdot 2k \displaystyle\int_0^{2\pi} \langle x_1,[x_2,x_3]\rangle\, f(f-f^2)'\, d\theta\right) \\ = & (0,-k\langle x_1,[x_2,x_3]\rangle) \qquad \text{by the calculation below} \\ = & -\psi_1(l_3(x_1,x_2,x_3))\,. \end{array}$$

The value of the integral here is *universal*, independent of the choice of f:

$$\begin{aligned} \int_0^{2\pi} f(f-f^2)'\, d\theta &= \int_0^{2\pi} \big(f(\theta)f'(\theta) - 2f^2(\theta)f'(\theta)\big)\, d\theta \\ &= \frac{1}{2} - \frac{2}{3} = -\frac{1}{6}\,. \end{aligned}$$

□

The final step in proving Theorem 30 is to show that $\phi\circ\psi$ is the identity on $\mathfrak{g}_k$, while $\psi\circ\phi$ is isomorphic to the identity on $\mathcal{P}_k\mathfrak{g}$. For convenience, we recall the definitions first: $\phi\colon \mathcal{P}_k\mathfrak{g}\to\mathfrak{g}_k$ is given by:

$$\begin{aligned} \phi_0(p) &= p(2\pi) \\ \phi_1(\ell,c) &= c \\ \phi_2(p_1,p_2) &= k\int_0^{2\pi} \left(\langle p_1,p_2'\rangle - \langle p_1',p_2\rangle\right)\, d\theta \end{aligned}$$

where $p,p_1,p_2\in P_0\mathfrak{g}$, $\ell\in\widehat{\Omega_k\mathfrak{g}}$, and $c\in\mathbb{R}$, while $\psi\colon\mathfrak{g}_k\to\mathcal{P}_k\mathfrak{g}$ is given by:

$$\begin{aligned} \psi_0(x) &= xf \\ \psi_1(c) &= (0,c) \\ \psi_2(x_1,x_2) &= ([x_1,x_2](f-f^2),0) \end{aligned}$$

where $x,x_1,x_2\in\mathfrak{g}$, $c\in\mathbb{R}$, and $f\colon[0,2\pi]\to\mathbb{R}$ satisfies the conditions of Lemma 36.

Lemma 37. *With the above definitions we have:*

- *$\phi\circ\psi$ is the identity Lie 2-algebra homomorphism on $\mathfrak{g}_k$;*
- *$\psi\circ\phi$ is isomorphic, as a Lie 2-algebra homomorphism, to the identity on $\mathcal{P}_k\mathfrak{g}$.*

Proof. We begin by demonstrating that $\phi\circ\psi$ is the identity on $\mathfrak{g}_k$. First,

$$(\phi\circ\psi)_0(x) = \phi_0(\psi_0(x)) = \phi_0(xf) = xf(2\pi) = x,$$

since $f(2\pi)=1$ by the definition of f in Lemma 36. Second,

$$(\phi\circ\psi)_1(c) = \phi_1(\psi_1(c)) = \phi_1((0,c)) = c$$

Finally,

$$\begin{aligned}
(\phi\circ\psi)_2(x_1,x_2) &= \phi_2(\psi_0(x_1),\psi_0(x_2)) + \phi_1(\psi_2(x_1,x_2)) \qquad \text{by (6)}\\
&= \phi_2(x_1f, x_2f) \;+\; \phi_1([x_1,x_2](f-f^2),0)\\
&= k\int (\langle x_1f, x_2f'\rangle - \langle x_1f', x_2f\rangle)\,d\theta \;+\; 0\\
&= k\langle x_1,x_2\rangle \int (ff' - f'f)\,d\theta\\
&= 0\,.
\end{aligned}$$

Next we consider the composite

$$\psi\circ\phi\colon \mathcal{P}_k\mathfrak{g} \to \mathcal{P}_k\mathfrak{g}\,.$$

The corresponding L_∞-algebra homomorphism is given by:

$$\begin{aligned}
(\psi\circ\phi)_0(p) &= p(2\pi)f\\
(\psi\circ\phi)_1(\ell,c) &= (0,c)\\
(\psi\circ\phi)_2(p_1,p_2) &= \left([p_1(2\pi),p_2(2\pi)](f-f^2),\; k\int (\langle p_1,p_2'\rangle - \langle p_1',p_2\rangle)\;d\theta\right)
\end{aligned}$$

where again we use equation (6) to obtain the formula for $(\psi\circ\phi)_2$.

For this to be isomorphic to the identity there must exist a Lie 2-algebra 2-isomorphism

$$\tau\colon \psi\circ\phi \Rightarrow \mathrm{id}$$

where id is the identity on $\mathcal{P}_k\mathfrak{g}$. We define this in terms of its corresponding L_∞-2-homomorphism by setting:

$$\tau(p) = (p - p(2\pi)f, 0)\,.$$

Thus, τ turns a path p into the loop $p - p(2\pi)f$.

We must show that τ is a chain homotopy satisfying condition (7) of Definition 11. We begin by showing that τ is a chain homotopy. We have

$$\begin{aligned}
d(\tau(p)) \;=\; d(p-p(2\pi)f,0) &= p - p(2\pi)f\\
&= \mathrm{id}_0(p) - (\psi\circ\phi)_0(p)
\end{aligned}$$

and

$$\begin{aligned}
\tau(d(\ell,c)) \;=\; \tau(\ell) &= (\ell,0)\\
&= \mathrm{id}_1(\ell,c) - (\psi\circ\phi)_1(\ell,c)
\end{aligned}$$

so τ is indeed a chain homotopy.

We conclude by showing that τ satisfies condition (7):

$$(\psi\circ\phi)_2(p_1,p_2) = l_2((\psi\circ\phi)_0(p_1),\tau(p_2)) + l_2(\tau(p_1),p_2) - \tau(l_2(p_1,p_2))$$

In order to verify this equation, we write out the right-hand side more explicitly by inserting the formulas for $(\psi\circ\phi)_2$ and for τ, obtaining:

$$l_2\big(p_1(2\pi)f,\ (p_2-p_2(2\pi)f,0)\big)+l_2\big((p_1-p_1(2\pi)f,0),\ p_2\big)-([p_1,p_2]-[p_1(2\pi),p_2(2\pi)]f,\ 0)$$

This is an ordered pair consisting of a loop in $\mathfrak{g}$ and a real number. By collecting summands, the loop itself turns out to be:

$$[p_1(2\pi),p_2(2\pi)](f-f^2)\,.$$

Similarly, after some integration by parts the real number is found to be:

$$k\int_0^{2\pi}\left(\langle p_1,p_2'\rangle-\langle p_1',p_2\rangle\right)\,d\theta\,.$$

Comparing these results with the value of $(\psi\circ\phi)_2(p_1,p_2)$ given above, one sees that τ indeed satisfies (7). $\square$

6 Conclusions

We have seen that the Lie 2-algebra $\mathfrak{g}_k$ is equivalent to an infinite-dimensional Lie 2-algebra $\mathcal{P}_k\mathfrak{g}$, and that when k is an integer, $\mathcal{P}_k\mathfrak{g}$ comes from an infinite-dimensional Lie 2-group $\mathcal{P}_kG$. Just as the Lie 2-algebra $\mathfrak{g}_k$ is built from the simple Lie algebra $\mathfrak{g}$ and a shifted version of $\mathfrak{u}(1)$:

$$0\longrightarrow \mathrm{b}\mathfrak{u}(1)\longrightarrow\mathfrak{g}_k\longrightarrow\mathfrak{g}\longrightarrow 0\,,$$

the Lie 2-group $\mathcal{P}_kG$ is built from G and another Lie 2-group:

$$1\longrightarrow\mathcal{L}_kG\longrightarrow\mathcal{P}_kG\longrightarrow G\longrightarrow 1$$

whose geometric realization is a shifted version of U(1):

$$1\longrightarrow B\mathrm{U}(1)\longrightarrow|\mathcal{P}_kG|\longrightarrow G\longrightarrow 1\,.$$

None of these exact sequences split; in every case an interesting cocycle plays a role in defining the middle term. In the first case, the Jacobiator of $\mathfrak{g}_k$ is $k\nu\colon\Lambda^3\mathfrak{g}\to\mathbb{R}$. In the second case, composition of morphisms is defined using multiplication in the level-k Kac–Moody central extension of ΩG, which relies on the Kac–Moody cocycle $k\omega\colon\Lambda^2\Omega\mathfrak{g}\to\mathbb{R}$. In the third case, $|\mathcal{P}_kG|$ is the total space of a twisted $B\mathrm{U}(1)$-bundle over G whose Dixmier–Douady class is $k[\nu/2\pi]\in H^3(G)$. Of course, all these cocycles are different manifestations of the fact that every simply-connected compact simple Lie algebra has $H^3(G)=\mathbb{Z}$.

We conclude with some remarks of a more speculative nature. There is a theory of '2-bundles' in which a Lie 2-group plays the role of structure group [3, 4]. Connections on 2-bundles describe parallel transport of 1-dimensional extended objects, e.g. strings. Given the importance of the Kac–Moody extensions of loop groups in string theory, it is natural to guess that connections on 2-bundles with structure group $\mathcal{P}_kG$ will play a role in this theory.

The case when $G = \mathrm{Spin}(n)$ and $k = 1$ is particularly interesting, since then $|\mathcal{P}_k G| = \mathrm{String}(n)$. In this case we suspect that 2-bundles on a spin manifold M with structure 2-group $\mathcal{P}_k G$ can be thought as substitutes for principal $\mathrm{String}(n)$-bundles on M. It is interesting to think about 'string structures' [17] on M from this perspective: given a principal G-bundle P on M (thought of as a 2-bundle with only identity morphisms) one can consider the obstruction problem of trying to lift the structure 2-group from G to $\mathcal{P}_k G$. There should be a single topological obstruction in $H^4(M;\mathbb{Z})$ to finding a lift, namely the characteristic class $p_1/2$. When this characteristic class vanishes, every principal G-bundle on M should have a lift to a 2-bundle $\mathcal{P}$ on M with structure 2-group $\mathcal{P}_k G$. It is tempting to conjecture that the geometry of these 2-bundles is closely related to the enriched elliptic objects of Stolz and Teichner [21].

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