2013

Polynomial knot and link invariants from the virtual biquandle

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Abstract

The Alexander biquandle of a virtual knot or link is a module over a 2-variable Laurent polynomial ring which is an invariant of virtual knots and links. The elementary ideals of this module are then invariants of virtual isotopy which determine both the generalized Alexander polynomial (also known as the Sawollek polynomial) for virtual knots and the classical Alexander polynomial for classical knots. For a fixed monomial ordering $<$, the Gröbner bases for these ideals are computable, comparable invariants which fully determine the elementary ideals and which generalize and unify the classical and generalized Alexander polynomials. We provide examples to illustrate the usefulness of these invariants and propose questions for future work.

KEYWORDS: virtual knot, generalized Alexander polynomial, virtual Alexander polynomial, Sawollek polynomial, biquandle, Alexander biquandle.

2010 MSC: 57M27, 57M25

1 Introduction

The Alexander biquandle of an oriented classical or virtual knot or link is a module over a ring of 2-variable Laurent polynomials which is invariant under virtual Reidemeister moves. In this paper we describe methods of obtaining computable invariants of classical and virtual knots and links from the Alexander biquandle. These families of invariants generalize and unify the classical Alexander polynomials for classical knots and the generalized Alexander polynomial for virtual knots and links (also known as the *virtual Alexander polynomial* [7, 9, 10] and the *Sawollek polynomial* [13]), $Z_D(x, y)$.

This work was inspired by the Remark following Theorem 3 in Sawollek's paper [13] which states, "For a connected sum $D_1 \# D_2$ of virtual link diagrams D_1 and D_2, a formula of the form

$$Z_{D_1 \# D_2}(x, y) = c Z_{D_1}(x, y) Z_{D_2}(x, y)$$

with a constant c does not hold in general..." This is not surprising since the connected sum of virtual knots and links is not well-defined, yet in certain cases the result nevertheless holds. Computing the virtual Alexander polynomial of various connected sums of virtual knots reveals that, as anticipated, these polynomials differed depending on where the connect sum is performed. Moreover, these polynomials gave no indication of where the connect sum was taken. It is natural, then, to ask under what circumstances does $Z_D(x, y)$ satisfy the above equation.

Since the classical Alexander polynomial generates the $k = 1$ elementary ideal of the Alexander matrix, we expect the factor relationship of the polynomials of connected summands to result in an

inclusion relationship at the level of ideals. To begin our study of such relationships, therefore, we turned our attention to describing these ideal-valued invariants in more detail as a first step. The result is the present paper; we anticipate future work taking these ideas further.

The paper is organized as follows. In Section 2 we recall the Alexander biquandle. In Section 3 we define invariants of oriented virtual and classical knots and links derived from the Alexander biquandle using principal ideals. In Section 4 we use reduced Gröbner bases to define further invariants from the Alexander biquandle. In Section 5 we collect some computations and examples. We conclude with some questions for future work in Section 6.

2 The Alexander biquandle

Recall from [7] that an *oriented virtual link* is an equivalence class of *oriented virtual link diagrams* under the equivalence relation defined by the *virtual Reidemeister moves*, obtained by considering all possible oriented versions of the moves pictured below:
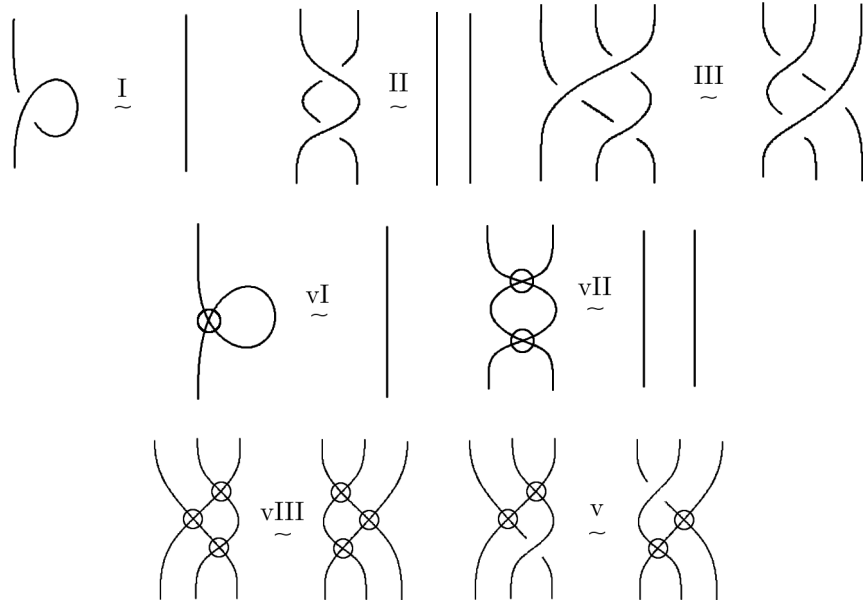
We now review the definitions of the virtual biquandle and the fundamental virtual biquandle. This second definition will allow us to see the relationship between biquandles and virtual knots.

Definition 1 Let X be a set and define the *diagonal map* $\Delta : X \to X \times X$ by $\Delta(x) = (x, x)$. A *virtual biquandle* structure on X consists of two invertible maps $B, V : X \times X \to X \times X$ satisfying the axioms:

(0) $V^2 = \mathrm{Id} : X \times X \to X \times X$,

(1) There exist unique invertible maps $S, vS : X \times X \to X \times X$ (called the *sideways maps*) satisfying

$$S(B_1(x, y), x) = (B_2(x, y), y) \quad \text{and} \quad vS(V_1(x, y), x) = (V_2(x, y), y)$$

for all $x, y \in X$,

(2) $(S^{\pm 1} \circ \Delta)_j$ and $(vS^{\pm 1} \circ \Delta)_j$ are bijections for $j = 1, 2$ satisfying

$$(S \circ \Delta)_1 = (S \circ \Delta)_2 \quad \text{and} \quad (vS \circ \Delta)_1 = (vS \circ \Delta)_2, \text{ and}$$

(3) B and V satisfy the *set-theoretic Yang-Baxter equations*

$$\begin{array}{rcl}
(B \times I)(I \times B)(B \times I) & = & (B \times I)(B \times I)(I \times B) \\
(V \times I)(I \times V)(V \times I) & = & (V \times I)(V \times I)(I \times V) \\
(V \times I)(I \times B)(V \times I) & = & (V \times I)(B \times I)(I \times V)
\end{array}$$

We note that these axioms result from by applying the labeling condition pictured below to the virtual Reidemeister moves. For instance, the invertibility of B and V along with Axiom (1) represent Reidemeister 2 moves. Reidemeister 1 moves are represented by Axiom (2), and Reidemeister 3 moves are related to Axiom (3).

Suppose we are given a virtual knot or link L with fixed diagram. Let $X = \{x_1, \ldots, x_n\}$ be a set of generators that are in one-to-one correspondence with the set of *semiarcs* in L, i.e. the portions of the virtual knot or link between crossing points (whether virtual, classical over or classical under). The set of *virtual biquandle words in X*, denoted $W(X)$, is defined recursively by the rules:

- $x \in X$ implies $x \in W(X)$ and

- $x, y \in W(X)$ implies $B_j^{\pm 1}(x, y)$, $V_j^{\pm 1}(x, y)$, $S_j^{\pm 1}(x, y)$, and $vS_j^{\pm 1}(x, y) \in W(X)$ for $j = 1, 2$.

The *free virtual biquandle on X*, denoted $FV(X)$, is the set of equivalence classes of virtual biquandle words in X under the equivalence relation on $W(X)$ generated by the virtual biquandle axioms.

We remark that $FV(X)$ gives provides almost no information about our original knot or link L. In order to capture information contained in our fixed diagram of L, we mod out by relations suggested by the crossings to obtain the fundamental virtual biquandle.
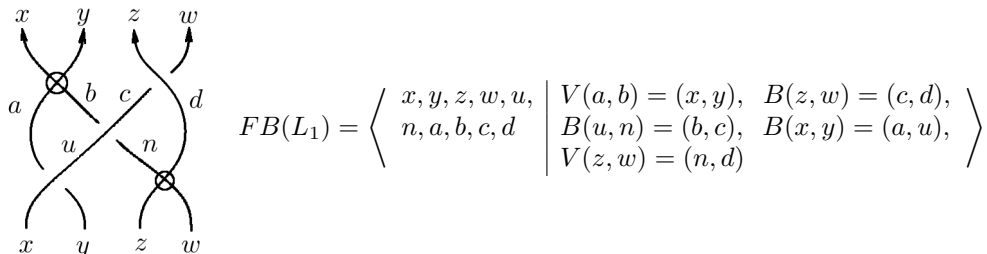
The *fundamental virtual biquandle of L*, denoted $FB(L)$, is the set of equivalence classes of $FV(X)$ under the equivalence relation generated by the *crossing relations* in L:

$$B(x, y) = (z, w) \qquad\qquad V(x, y) = (z, w)$$
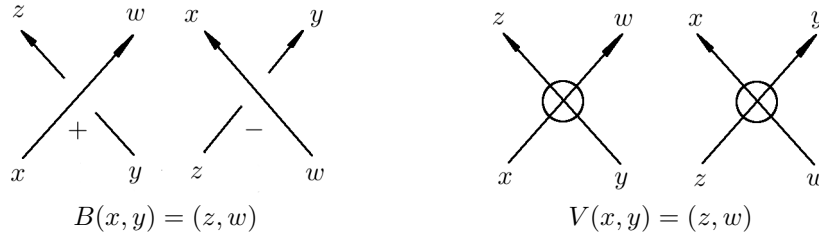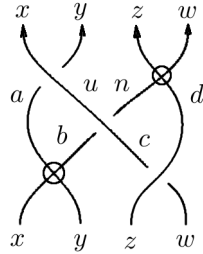
Alternatively, we can describe $FB(L)$ more directly as the set of equivalence classes in $W(X)$ under the equivalence relation generated by both the crossing relations and the virtual biquandle axioms.

Theorem 1 *Let L be a virtual link written as a closed virtual braid β, that is $L = \widehat{\beta}$. The fundamental virtual biquandle of a virtual link L is isomorphic to the fundamental virtual biquandle of the link L' obtained from L by reversing the direction of all strands in the closure of the inverse braid $\widehat{\beta^{-1}}$.*

The proof is effectively the same as the analogous result for classical biquandles of virtual links in [11]; we provide an illustration. Consider the virtual link $L_1 = \widehat{\beta}$ given by the closure of the braid

$$FB(L_1) = \left\langle \begin{array}{c|c} x, y, z, w, u, & V(a, b) = (x, y), \quad B(z, w) = (c, d), \\ n, a, b, c, d & B(u, n) = (b, c), \quad B(x, y) = (a, u), \\ & V(z, w) = (n, d) \end{array} \right\rangle$$

3

The reversed inverse braid closure $L_2 = \widehat{r\beta^{-1}}$ has isomorphic virtual biquandle (indeed, with identical presentation) to that of L_1
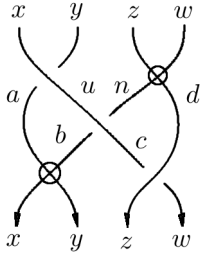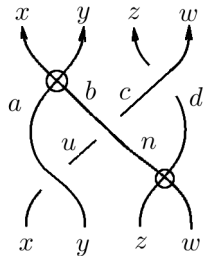
$$FB(L_2) = \left\langle \begin{array}{c} x,y,z,w,u, \\ n,a,b,c,d \end{array} \middle| \begin{array}{ll} V(a,b)=(x,y), & B(z,w)=(c,d), \\ B(u,n)=(b,c), & B(x,y)=(a,u), \\ V(z,w)=(n,d) \end{array} \right\rangle$$

while the unreversed inverse braid closure $L_3 = \widehat{\beta^{-1}}$ has a generally distinct virtual biquandle:

$$FB(L_3) = \left\langle \begin{array}{c} x,y,z,w,u, \\ n,a,b,c,d \end{array} \middle| \begin{array}{ll} B(u,a)=(y,x), & V(w,z)=(d,n), \\ B(n,u)=(c,b), & V(b,a)=(y,x), \\ B(w,z)=(d,c) \end{array} \right\rangle$$

Remark 1 The reversed inverse braid closure $\widehat{r\beta^{-1}}$ is called the *vertical mirror image* in [11] and is one of the 2^c possible orientations for a c-component link of what the Knot Atlas [4] calls the *horizontal mirror image.* Another "mirror image" of a virtual link can be obtained by switching every overcrossing to an undercrossing while fixing the diagram outside a neighborhood of the crossing. This operation is called the *mirror image* in [11] and is one orientation of the Knot Atlas' *vertical mirror image.* To avoid confusion we will refer to this operation as the *sign switch* of L, denoted sL; it is in general a distinct oriented virtual link from L:

$$FB(sL) = \left\langle \begin{array}{c} x,y,z,w,u, \\ n,a,b,c,d \end{array} \middle| \begin{array}{ll} B(a,u)=(x,y), & V(z,w)=(n,d), \\ B(b,c)=(u,n), & V(a,b)=(x,y), \\ B(c,d)=(z,w) \end{array} \right\rangle$$

An important example of a virtual biquandle, and one to which we will devote considerable attention, is the Alexander biquandle. We begin with some notation. Let $\Lambda = \mathbb{Z}[t^{\pm 1}, s^{\pm 1}]$ be the ring of Laurent polynomials in two variables t, s. We can think of Λ as a quotient ring of the four-variable polynomial ring $\tilde{\Lambda} = \mathbb{Z}[t, s, t^{-1}, s^{-1}]$ by the ideal $I = \langle tt^{-1} - 1, ss^{-1} - 1 \rangle$

Definition 2 Let L be an oriented virtual knot or link diagram and let $X = \{x_1, \ldots, x_n\}$ be a set of generators corresponding to the semiarcs of L. Then the *Alexander biquandle* of L, denoted $AB(L)$, is the Λ-module generated by X with the relations pictured below at positively $(+)$ and

negatively $(-)$ oriented classical crossings and at virtual crossings:

$$
\begin{aligned}
z &= ty + (1 - st)x \\
w &= sx
\end{aligned}
\qquad\qquad
\begin{aligned}
z &= y \\
w &= x
\end{aligned}
$$
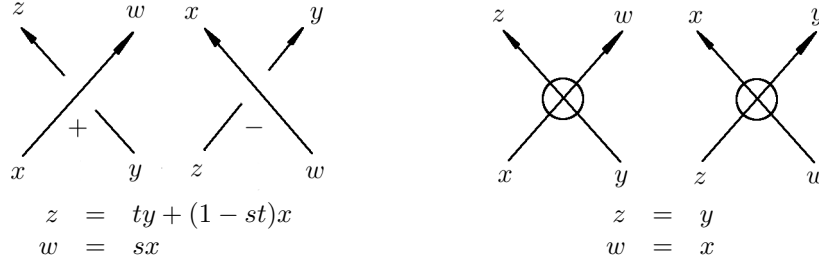
In particular, $AB(L)$ is obtained from the fundamental virtual biquandle of L by setting

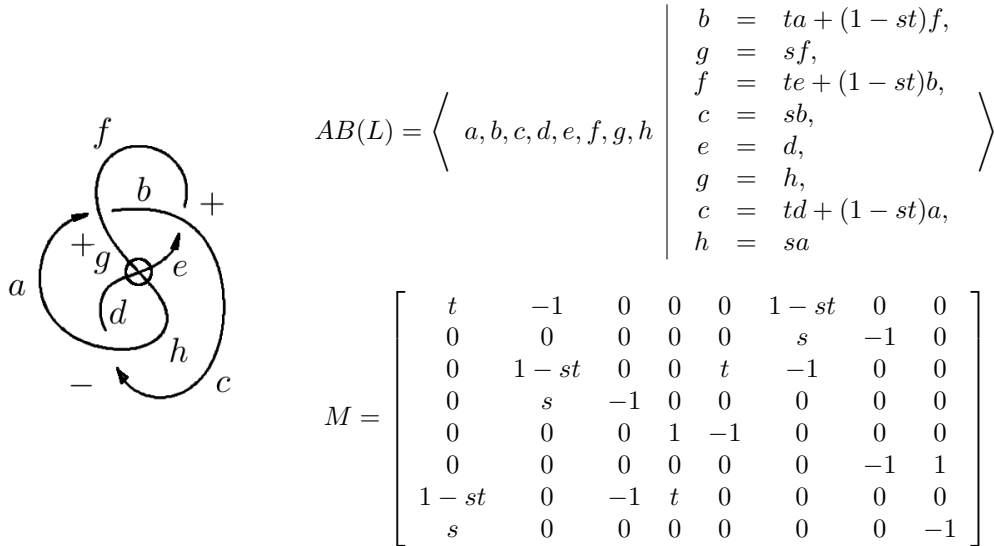$$B(x,y) = (ty + (1 - st)x, sx) \quad \text{and} \quad V(x,y) = (y, x).$$

A straightforward check verifies that the Alexander biquandle of a virtual knot is invariant under virtual Reidemeister moves; see [10] for details.

Remark 2 It is natural to consider, as we initially did, including a coefficient at the virtual crossings, e.g. set $V(x,y) = (vy, v^{-1}x)$. However, Theorem 7.1 in [1] implies that the resulting $\mathbb{Z}[t^{\pm 1}, s^{\pm 1}, v^{\pm 1}]$-module contains the same information as the simpler $\mathbb{Z}[t^{\pm 1}, s^{\pm 1}]$-module. The authors would like to thank Lou Kauffman for bringing this result to their attention.

We can represent the Alexander biquandle of a virtual knot or link with the coefficient matrix of the homogeneous system of equations determined by the crossings, known as a *presentation matrix*.

Example 3 The virtual knot below has Alexander biquandle with presentation and presentation matrix as listed.

$$
AB(L) = \left\langle\; a,b,c,d,e,f,g,h \;\middle|\;
\begin{aligned}
b &= ta + (1 - st)f, \\
g &= sf, \\
f &= te + (1 - st)b, \\
c &= sb, \\
e &= d, \\
g &= h, \\
c &= td + (1 - st)a, \\
h &= sa
\end{aligned}
\;\right\rangle
$$

$$
M = \begin{bmatrix}
t & -1 & 0 & 0 & 0 & 1-st & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & s & -1 & 0 \\
0 & 1-st & 0 & 0 & t & -1 & 0 & 0 \\
0 & s & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
1-st & 0 & -1 & t & 0 & 0 & 0 & 0 \\
s & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}
$$

For comparison, the Sawollek [13] matrix $M - P$ is:

$$
M - P = \begin{bmatrix}
1-x & -y & 0 & 0 & -1 & 0 \\
-x/y & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1-x & -y & 0 & -1 \\
0 & -1 & -x/y & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -y/x \\
-1 & 0 & 0 & 0 & -1/y & 1-1/x
\end{bmatrix}.
$$

We describe the relationship between these matrices in Example 5.

Definition 3 Let L be a virtual link, $AB(L)$ the Alexander biquandle of L, and M an $m \times n$ presentation matrix for $AB(L)$. The ideal I_k of Λ generated by the $(m-k)$-minors of M is the *kth elementary ideal* of M.

It is well-known that:

Theorem 2 *The elementary ideals of a module over a commutative ring with identity do not depend on the choice of presentation matrix for the module.*

In particular, any two presentation matrices of the same module differ by a sequence of moves of the forms:

- reordering of rows or columns,

- adding or deleting a row of all zeroes,

- adding or deleting a row and column with a 1 in the intersection and all other entries 0,

- adding a scalar multiple of one row (or column, respectively) to another row (or column, respectively), or

- replacing a row or column by an invertible scalar multiple of itself.

One then checks that these moves do not change the ideal. For instance, switching the order of two rows will multiply the minors by -1, but this does not change the ideal since -1 is a unit. Similarly, multilinearity of the determinant ensures that adding a scalar multiple of one row to another does not change the minors, etc. See [2, 12] for more details.

It then follows that:

Corollary 3 *Let L be a virtual link and let M be a presentation matrix for $AB(L)$. Then the elementary ideals I_k of M are invariants of virtual isotopy.*

In order to compare ideals, we need some machinery. In the next sections we describe two methods for comparing the ideals I_k.

3   Principal Alexander Polynomials

In this section we employ the tried and true method of obtaining invariants from I_k using principal ideals, i.e. ideals generated by a single element of $\Lambda = \mathbb{Z}[t^{\pm 1}, s^{\pm 1}]$.

Definition 4 Let L be a virtual link. For any $k = 0, 1, 2, \ldots$, the *kth principal Alexander polynomial* of L, denoted $\Delta_k^p(L)$, is the generator of the smallest principal ideal $P_k \subset \Lambda$ containing the kth elementary ideal I_k of $AB(L)$.

Remark 4 Note that $\Delta_k^p(L)$ is only defined up to multiplication by units in Λ, so two values f and g of $\Delta_k^p(L)$ are equivalent if $f = \pm t^n s^m g$ for some $n, m \in \mathbb{Z}$.

The classical Alexander polynomials can be obtained by the analogous construction starting with a presentation matrix for the the Alexander quandle, which is the special case of the Alexander biquandle with

$$B(x, y) = (ty + (1 - t)x, y) \quad \text{and} \quad V(x, y) = (y, x)$$

or equivalently the result of specializing $s = 1$ in $\Delta_k^p(L)$. In the literature, Δ_0^p is known as the *generalized Alexander polynomial* or *virtual Alexander polynomial*, and, after a change of variables, the *Sawollek polynomial* [7, 9, 10]. In particular, specializing $s = 1$ in $\Delta_1^p(L)$ for classical links yields the classical Alexander polynomial $\Delta(L)$.

Example 5 The virtual knot in Example 3 has 0th principal polynomial

$$\Delta_0^p(K) = (1 - st)(1 - s)(1 - t)$$

Taking the determinant of the Sawollek matrix in Example 3, performing the change of variables $x = st$ and $y = -s$ and canceling units gives the generalized Alexander polynomial in the notation of [7]:

$$G_K(s, t) = (s - 1)(t - 1)(st - 1) = \Delta_0^p(K).$$

The classical Alexander polynomial of this virtual knot is $\Delta(K) = 1$.

Since a virtual knot always has $AB(L)$ presented by a square matrix, the top level $k = 0$ ideal is always principal, generated by the determinant of the presentation matrix. To compute $\Delta_k^p(L)$ for $k > 0$, we can find the $(m - k)$-minors of the presentation matrix, and after multiplying by units if necessary to get elements of the UFD (Unique Factorization Domain) $\mathbb{Z}[t, s]$, find the greatest common divisor.

4 Alexander-Gröbner Invariants

The principal polynomials Δ_k^p have the advantages of being fairly quick to compute and easy to compare; however, for any principal ideal $P \subset \Lambda$, there are potentially many distinct non-principal ideals contained in P, and thus passing from I_k to P_k represents a loss of information. To avoid this loss of information, we can employ the idea of a Gröbner basis.

Briefly, in a multivariable polynomial ring $R[x_1, \ldots, x_n]$ over a PID (Principal Ideal Domain) R, a *term ordering* is a well-ordering on the set of monomials; such an ordering then gives each polynomial a well-defined *leading term.* Standard examples of term orderings include:

- *lexicographical ordering.* Starting with an ordering on the variables, e.g. $x_1 < x_2 < \cdots < x_n$, one compares terms by comparing powers on the variables in order, with ties in one variable resolved by comparing the next. For example, if $x < y < z$ then we have

$$xy^2z < xy^2z^2 < xy^3 < x^2.$$

- *graded lexicographical ordering.* Here we start by comparing total degree, with ties broken lexicographically. Thus for the previous example in graded lexicographical ordering we have

$$x^2 < xy^2 < xy^3 < xy^2z^2.$$

A generating set G for an ideal I in a polynomial ring is a *Gröbner basis* for I with respect to a choice of term ordering $<$ if the leading term of every element of I is divisible by the leading term of some element of G. Gröbner basis have many useful properties, such as:

- reducing a polynomial $f \in I$ by G using the multivariable division algorithm (i.e. repeatedly subtracting from f multiples of elements of G to cancel leading terms) always results in 0 (this is not true for arbitrary generating sets), and

- the remainder of a polynomial after multivariable division by G is unique, enabling computations in the quotient ring Λ/I.

Moreover, a Gröbner basis is *reduced* if redundant elements have been removed, i.e. if every leading term has coefficient 1 and no monomial in any element $g \in G$ is in the ideal generated by the leading terms of $G - \{g\}$. Importantly for us, for a given choice of term ordering, reduced Gröbner bases of ideals are unique.

Thus, we would like to compare the ideals I_k by finding and comparing Gröbner bases. One slight problem is that $\Lambda = \mathbb{Z}[t^{\pm 1}, s^{\pm 1}]$ is a ring of *Laurent* polynomials, not polynomials. To address this, we will pull back from the ring Λ to the four-variable polynomial ring $\tilde{\Lambda} = \mathbb{Z}[t, s, t^{-1}, s^{-1}]$ in which t^{-1} and s^{-1} are considered new variables as opposed to powers of t and s.

Definition 5 Let L be a virtual link and M a presentation matrix for $AB(L)$. For each elementary ideal I_k, let $\tilde{I}_k$ be the preimage of I_k in $\tilde{\Lambda}$. For a choice of term ordering $<$, the kth *Alexander-Gröbner Invariant* $\Delta_k^<(L)$ is the Gröbner basis of $\tilde{\Lambda}$ with respect to the term ordering $<$.

To compute $\Delta_k^<(L)$, one uses *Buchberger's algorithm* [3] starting by setting G equal to the set of $(m-k)$-minors together with the generators $1 - tt^{-1}$ and $1 - ss^{-1}$. For each pair of elements $g, g' \in G$, we find a difference $S = pg - qg'$ using the least common multiple of the leading terms of g and g' such that that the leading terms of pg and qg' cancel. After reducing S mod G, if the remainder is nonzero, add it to G and start over. When no more polynomials are added, we have a Gröbner basis. We then remove elements whose monomials lie in the ideal generated by the leading terms of the other elements of G to obtain the reduced Gröbner basis, unique up to choice of term ordering $<$. See [3] for more.

Since the resulting sets of polynomials can be quite large, we may employ various strategies to obtain more conveniently comparable invariants at the cost of losing some information. These include but are certainly not limited to:

- *Alexander-Gröbner cardinality.* The number of elements in the Gröbner basis of $\tilde{I}_k$, $\Delta_k^{<,\mathbb{Z}}(L) = |\Delta_k^<(L)|$;

- *Alexander-Gröbner sum.* The sum of the elements of $\Delta_k^<(L)$,

$$\Delta_k^{<,\Sigma}(L) = \sum_{g \in \Delta_k^<} g;$$

  and

- *Alexander-Gröbner maximum polynomial.* The maximal element of $\Delta_k^<(L)$ with respect to the term ordering $<$, $\Delta_k^{<,M}(L)$.

We note that the last two invariants coincide with the principal polynomials when the ideals I_k are principal; while the first invariant is 1 iff the ideal I_k is prinicpal.

5 Computations and Examples

We provide several examples that justify our proposed generalizations of the Sawollek and other Alexander-type polynomials. In each of the examples below we use the graded lexicographical ordering on $\tilde{\Lambda}$. Our custom `python` code is available at `www.esotericka.org`.

Example 6 In [9], the authors give two examples of knots that are not detected by the generalized Alexander polynomial. One knot is the Kishino knot. The other is the following Kishino-like knot K:
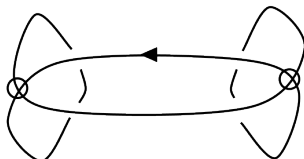
The (standard) Kishino knot has trivial values for Δ_0^p, $\Delta_0^<$ and $\Delta_1^<$, but this modified Kishino-like knot, K, has the following non-trivial value of $\Delta_1^<$:

$$\Delta_1^<(K) = \{1 - t^{-1} + t^{-2}, \ -1 + t^{-1} + t, \ -1 + s, \ -1 + s^{-1}\}$$

Example 7 Two more knots that are of interest to us are Slavik's knot and Miyazawa's knot. Slavik's knot is not detected by the arrow polynomial and Miyazawa's knot is not detected by the Miyazawa polynomial [8]. While Slavik's knot isn't detected by Δ_0^p, it is detected by $\Delta_1^<$. On the other hand, $\Delta_1^<$ is trivial for Miyazawa's knot, but Δ_0^p and $\Delta_0^<$ are both non-trivial.
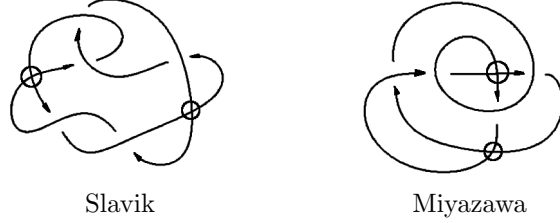
Slavik   Miyazawa

$$\Delta_1^<(\text{Slavik}) = \{3t^{-1}s - s^2 - 2t^{-2} + s^{-1}t^{-3}, \ 3 - st - 2s^{-1}t^{-1} + s^{-2}t^{-2},$$
$$3s^{-1}t - t^2 - 2s^{-2} + t^{-1}s^{-3}, \ -3s^{-1}t^2 + 2ts^{-2} + t^3 - s^{-3},$$
$$-3t + 2s^{-1} + st^{-2} - t^{-1}s^{-2}, \ -3s + 2t^{-1} + ts^{-2} - s^{-1}t^{-2},$$
$$-3t^{-1}s^2 + 2st^{-2} + s^3 - t^{-3}, \ -1 + tt^{-1}, \ -1 + ss^{-1}\}$$

$$\Delta_0^p(\text{Miyazawa}) = (st - 1)(s - 1)(t - 1)$$
$$\Delta_0^<(\text{Miyazawa}) = \{-(1 - s^{-1})(1 - t)(s^{-1} - t), \ (1 - t^{-1})(1 - s)(s - t^{-1}),$$
$$(1 - t^{-1})(1 + t^{-1} - s - s^{-1}t^{-1}), \ (1 - s^{-1})(1 + s^{-1} - t - s^{-1}t^{-1})$$
$$s^{-1} + t^{-1} - s - t + st - s^{-1}t^{-1}, \ -1 + tt^{-1}, \ -1 + ss^{-1}\}.$$

Example 8 Our next example considers connected sums of virtual knot diagrams. For classical knots, the connected sum operation is well-defined. For virtual knots, however, the knot type of the diagram obtained by taking a connected sum depends on *where* the connected sum is taken. We give two connected sums, $K_1^\#$ and $K_2^\#$, of the virtual trefoil knot, 2.1, with itself:
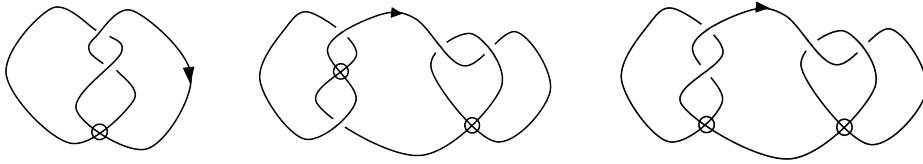
We see from the sets below that the invariants Δ_0^p and $\Delta_1^<$ distinguish these two connected sums. Note that $\Delta_1^<$ is trivial on both the virtual trefoil itself and on $K_1^\#$.

$$\Delta_0^p(2.1) = (1 - s)(1 - t)(1 - st)$$
$$\Delta_0^p(K_1^\#) = (1 - s)(1 - t)(1 - st)(1 - t + st^2 + s^2t^2)$$
$$\Delta_0^p(K_2^\#) = (1 - s)(1 - t)(1 - st)(1 + s - t + st^2 + s^2t^2 - ts^2)$$

$$\Delta_1^<(2.1) = \{1\}$$
$$\Delta_1^<(K_1^\#) = \{1\}$$
$$\Delta_1^<(K_2^\#) = \{1 - t^{-1} + t^{-2}, \ -1 + t^{-1} + t, \ -1 + s, \ -1 + s^{-1}\}$$

9

For our final example, we consider the case of *based virtual knots* or equivalently *long virtual knots*; these are oriented virtual knots with a base point which cannot move through a classical crossing. In terms of diagrams, this means that strands may not move classically over or under or virtually detour past the base point. Interpreting the base point as the point at infinity yields the long knot interpretation.
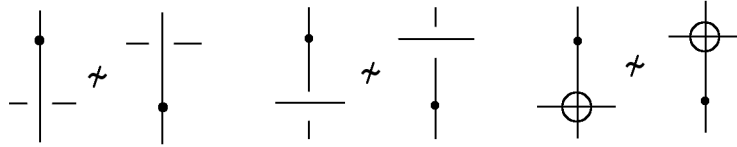
Our motivation for considering based virtual knots is the observation that while the connected sum operation is not well-defined for virtual knots, it *is* well-defined for based virtual knots provided we only permit the summing operation at the base point. Indeed, based virtual knots form a noncommutative monoid under based connected sum, since switching the order of summands requires forbidden moves.

The virtual biquandle of a based virtual knot with n crossings has $2n + 1$ generators since *two* distinct generators are assigned to the semiarc containing the base point while each remaining semiarc (of which there are $2n$) is assigned a single generator. Meanwhile, the based virtual biquandle has $2n$ relations: two coming from each of the n crossings, as usual. Because the presentation matrix is no longer square, its 0-level ideal I_0 need not be prinicipal. Therefore, it not surprising that the Gröbner invariants are stronger than the principal invariants in the based case, as our final example shows.

Example 9 The two pictured oriented based virtual trefoils both have the trivial value $\Delta_0^p = 1$, but are distinguished by $\Delta_0^<$:
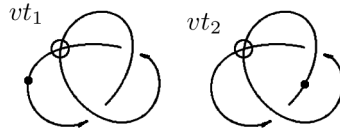
$$
\begin{aligned}
\Delta_0^p(vt_1) &= 1 \\
\Delta_0^p(vt_2) &= 1 \\
\Delta_0^<(vt_1) &= \{1\} \\
\Delta_0^<(vt_2) &= \{1 - t^{-1} + t^{-2},\ -1 + t^{-1} + t,\ -1 + s,\ -1 + s^{-1}\}
\end{aligned}
$$

Taking based connected sums, we have

$$
\begin{aligned}
\Delta_0^p(vt_2 \# vt_1) &= \Delta_0^<(vt_2) \\
\Delta_0^p(vt_2 \# vt_2) &= \{-2 + t + 3t^{-1} - 2t^{-2} + t^{-3},\ -(1 - s^{-1})(1 - t^{-1} + t^{-2}), \\
&\qquad 3 - 2t^{-1} - 2t + t^{-2} + t^2,\ -1 + t^{-1}t, \\
&\qquad (1 - s^{-1})(1 - t^{-1} - t),\ (1 - s^{-1})^2,\ \ s^{-1} - 2 + s\}
\end{aligned}
$$

6   Questions

Many interesting questions and promising avenues of exploration involving the Δ_k^p and $\Delta_k^<$ invariants await further work. These include but are not limited to:

- *Twisted Δ_k invariants.* In the classical case, a matrix representation of D_p coupled with a p-coloring of a knot diagram K lets us replace t in the Alexander matrix with a matrix depending on the p-colors at each crossing; the principal ideal invariants of the resulting matrix are known as the *twisted Alexander polynomials.* Matrix representations of labeling biquandles should be usable in an analogous way with $AB(K)$ to define twisted Δ_k^p invariants.

- *Multivariable Δ_k invariants.* Another variant of the Alexander polynomial for links involves multiple t variables; what is the analogous construction for Δ_k invariants?

- *Categorification of Δ_k^p.* We can apply a Khovanov-style construction to the state-sum expansion of Δ_0^p considered as a determinant. How does the result compare to Knot Floer homology?

- *Skein relations.* The classical Alexander polynomial satisfies the well-known Conway skein relation, and the Sawollek polynomial satisfies a similar skein relation. What skein relations, if any, are satisfied by Δ_k^p?

- *Connected sum behavior.* We return to our original question: what conditions are necessary and sufficient for a connected sum of virtual knot or link diagrams to behave like classical knots under connected sum with respect to the Δ_k^p invariants? A related question: what is the center of the monoid of based virtual knots?

- *Virtual links with boundary.* Based virtual knots are a special case of virtual knots with boundary, where the supporting surface of the virtual knot has a fixed boundary and knots or links on the surface may have endpoints in the boundary. Gluing such surfaces along boundary components such that endpoints of knots match up generalizes our based connected sum operation. What is the algebraic structure of such knots? The authors are grateful to Charlie Frohman for this observation.

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