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ON LEVI-CIVITA'S ALTERNATING SYMBOL, SCHOUTEN'S ALTERNATING UNIT TENSORS, CPT, AND QUANTIZATION

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Abstract: The purpose of the present article is to demonstrate that by adopting a unifying differential geometric perspective on certain themes in physics one reaps remarkable new dividends in both microscopic and macroscopic domains. By replacing algebraic objects by tensor-transforming objects and introducing methods from the theory of differentiable manifolds at a very fundamental level we obtain a Kottler-Cartan metric-independent general invariance of the Maxwell field, which in turn makes for a global quantum superstructure for Gauss-Ampère and Aharonov-Bohm "quantum integrals." Beyond this, our approach shows that postulating a Riemannian metric at the quantum level is an unnecessary concept and our differential geometric, or more accurately topological yoga can substitute successfully for statistical mechanics.

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1. Aspects of the Theory of Differentiable Manifolds

1.1. Enantiomorphism

The existence of non-identical mirror pairs of quartz crystals constitutes an explicit example of the natural occurrence of orientation-based phenomena. Objects of this kind have long played a role in crystallography. Among the 32 crystal classes defining its external macroscopic forms, 11 classes are due to rotations and are said to be *enantiomorphic*; the remaining 21 classes are obtained by rotations followed by reflections or inversions and have no *enantiomorphic* companions.

Other examples of this type of pairing outside of the realm of crystallography are our hands and feet, and this underscores the fact that enantiomorphism does not involve size and even allows measures of deformations. So, there is something of a topological quality to enantiomorphism; characteristic features of the objects on which enantiomorphism act remain present under more general transformations than rigid motions.

Moreover, similar phenomena to enantiomorphism pairing occur even in the microscopic realm where the operative term is parity, P. In particle physics parity is a property or from a slightly different perspective, an operator, which, together with the charge conjugate C and time-reversal T contributes to a fundamental theorem titled the CPT invariance. Here PC and TC, as operators, engender a sign change in electric charge, while, in physical space PC also serves to identify charge as a pseudo-scalar; beyond this, taken together with TC, the operator PC provides space-time invariance with a positive determinant.

This said, and taking these observations as motivation, we now turn our attention to space-time, taking the position that it should be regarded as a Riemannian 4-manifold whose orientability, generally regarded as a metric-related affair, can in fact be discerned by examining enantiomorphic "objects" which are present in some abundance.

1.2. Space-Time

Of course, it is more proper to say Minkowski space-time, given the historical role played by Hermann Minkowski (1860-1909) in stipulating that space-time should be an orientable manifold equipped with a generally invariant indeterminate metric with Lorentz signature (+, -, -, -) since this makes the best framework for general relativity.

But we assert that orientability per se actually has an undeniable pre-metric aspect evidenced by time-reversal T above and this suggests that we should commence our discussion, or rather the development of our theme, by characterizing space-time as a manifold whose orientability, indeed, whose orientation, is gleaned from enantiomorphy. Any consideration of the Einstein metric can be left aside for the time being; and our subtext for this manoeuvre is that we will show below that in our chosen geometric-topological setup, working so to speak pre-metrically, nothing less than quantization can be achieved with no recourse to the statistical methods favored by the Copenhagen interpretation of quantum mechanics. The latter interpretation is evidently due to Max Born (1882-1970) of Göttingen, not Copenhagen, but we begin with Minkowski.

The sticking point is that outside of the discipline of particle physics, electrical charge is considered to be an absolute scalar and so it is that largely because of the prevalence of orientation preserving transformations, vector analysis tends to prevail. Thus, the matrix group of spatial rotations, *SR(3)*, takes central stage as the operative transformation group; it is this convention that we challenge in what follows, in the presence of enantiomorphy and parity considerations in particle physics.

In point of fact, the hegemony of vector analytic methods in physics already found itself challenged early on with the work of Pierre Curie (1857-1906) in France and Woldemar Voigt (1850-1919) in Germany. Specifically, since piezoelectricity invokes directional derivatives in a linear manner, sign changes of the free electric charge density, ρ, no longer cancel and charge conjugation for particles subsequently implies that charge should be a pseudo-scalar. On the other hand a physical tradition older than crystal physics has long held that charge should be an absolute scalar because the existence of pseudo-scalars is not accounted for by *SR(3)*. Hence-forth, the physical representation was compromised by the limitations of vector analysis in $SR(3)$.

Enter Minkowski [6]. In his hands Maxwell's equations themselves were especially adapted to a space-time context with the major operations being generalized curl and divergence acting on antisymmetric species of covariant and contravariant tensorial entities. We submit that distinctions between covariance and contraviance reflect an essential duality between actual space-time and Einstein's construct of space as a collective of frames and references.

Indeed, Minkowski effected the space-time description of the Lorentz-invariant properties of the Maxwell equations as follows. Taking $\boldsymbol{E},\boldsymbol{B},\boldsymbol{D},\boldsymbol{H}$ to mean as usual, the electric field intensity, the magnetic field density, electric induction or flux density, and the intensity of the magnetic field, we have, identically,

$$\boldsymbol{E} = -\nabla\varphi - \frac{\partial \boldsymbol{A}}{\partial t}, \tag{1}$$

φ being the electric potential and $\boldsymbol{A}$ the magnetic vector potential;

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}; \tag{2}$$

$$\nabla \cdot \boldsymbol{D} = \rho, \tag{3}$$

the free charge density; and

$$\nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t} = \boldsymbol{J}, \tag{4}$$

the electric current density. But Minkowski joined these magnetic and electric properties of the magnetic field into the components of the anti-symmetric space-time tensors, with $\boldsymbol{E}, \boldsymbol{B}$ joined into F_{lk} and $\boldsymbol{H}$,$\boldsymbol{D}$ joined into $G^{lk}, l = 0, 1, 2, 3$, with 0 as time label, and $1, 2, 3$ as space labels. Under these circumstances the Maxwell equations take the following appearance:

$$\partial_s F_{lk} + \partial_k F_{sl} + \partial_l F_{ks} = 0, \tag{5}$$

being a local Faraday-Maxwell law (generalized curl);

$$\partial_l A_k - \partial_l A_l = \partial_l F_{lk}, \tag{6}$$

being the definition of the vector potential;

$$\partial_k G^{lk} = C^l, \tag{7}$$

being Ampère's law, displacement current and generalized divergence; with

$$\partial_k C^k = 0, \tag{8}$$

being the local charge conservation. The linear invariance present in equations (6) and (8) reduces to Lorentz invariance if, with c the speed of light in vacuum, the free-space constitutive equations

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} \tag{9}$$

and

$$\boldsymbol{B} = \mu_0 \boldsymbol{H}, \tag{10}$$

where

$$\varepsilon_0 \mu_0 c^2 = 1 \tag{11}$$

are obeyed. We note with emphasis that at this point there are as yet no P, T specifications.

Enter Einstein. In the wake of the general theory of relativity, appearing on the historical scene in 1914, Maxwell's equations underwent an evolution, so to speak, as a result of the pervasive manoeuvre of replacing partial differentiation by covariant differentiation. Using this recipe equations (1) and (2) remain unchanged because the Christoffel terms cancel; but things are different for relations (3) and (4): each of the latter two of Maxwell's equations gains an extra term involving Christoffel symbols. Thus, there is something of an asymmetry between the first pair of Maxwell's equations, comprising the so-called Faraday part, and the second pair, the Ampère part, all for transparent historical reasons. This asymmetry was explicated in 1922 by the Austrian physicist, Friedrich Kottler (1886-1965) [5] who realized that the contravariant tensors G^{lk} and C^k, or, in more proper notation $G^{\lambda\mu}$ and C^λ, should be regarded as tensor and vector densities of weight -1. With this stipulation in place, the rules for covariant differentiation are extended, taking density into account, and this makes for a restoration of the form or appearance of both parts of Maxwell's equations.

Kottler's conclusions were presently confirmed in a purely differential geometric context by the French mathematician Elie Cartan (1869-1951). Starting with Maxwell's equations in this integral formulation, so that, in addition to Gauss's laws (for electricity and magnetism), we get *qua* Maxwell-Faraday

$$\int_{\partial S} \boldsymbol{E} \cdot d\vec{l} = -\frac{\partial \varphi_{\boldsymbol{B},\mathrm{S}}}{\partial t} \tag{12}$$

and *qua* Ampère

$$\int_{\partial S} \boldsymbol{B} \cdot d\vec{l} = \mu_0 I_S + \mu_0 \varepsilon_0 \frac{\partial \varphi_{\boldsymbol{E},\mathrm{S}}}{\partial t} \tag{13}$$

with $\varphi_{\boldsymbol{B},\mathrm{S}}$ and $\varphi_{\boldsymbol{E},\mathrm{S}}$, representing the magnetic and electric flux through the surface S, respectively, demonstrating that Kottler's line of reasoning is part and parcel of a duality property for differential manifolds. Thus, and most importantly, generalized curl and divergence emerge as dual operations independent of the metric structure on the ambient space, thus as an intrinsic part of space-time with no need to invoke metric properties at all.

1.3. Schouten's Tensor Calculus

We noted above that the presentation of Maxwell's equations in terms of a generalized curl and divergence originates with Minkowski; we went on to go from Minkowski to Kottler to argue that there is no need to impose a Riemannian metric on space-time at this stage of our discussion. This position can be strengthened by bringing to bear on these matters a transparent means of accounting for the "duality" (accruing to space-time as a manifold) effected by the Faraday versus Ampère asymmetry in Maxwell's equations, namely, the yoga of anti-symmetric unit tensors introduced by Jan Arnoldus Schouten (1883-1971)[10, 11].

The thrust of Schouten's simplifications is that these anti-symmetric unit tensors convert Levi-Civita's determinant-forming symbol into two tensor densities that can convert generalized curls into equivalent generalized contravariant divergences, and vice-versa. Indeed, if we now return briefly to the role these generalized operators play in the aforementioned evolved version of Maxwell's equations, we find ourselves in the position to prove, by means of Schouten's tensor calculus, that with the electrodynamics of space-time (as a manifold) entirely characterized by Maxwell's equations there is no need to introduce any metric, Lorentz or otherwise, at this early developmental stage.

However, we would first need some details about Schouten's tensor calculus. To wit, let Greek indices, either subscripted (covariant) or superscripted (contravariant), indicate general coordinate validity; however, orientation changes must be included in our notation; these are denoted by a tilde. The tensor densities that transform with the sign of the Jacobian, Δ, are then denoted as

$$\widetilde{U}^{\nu_0'\nu_1'\nu_2'\nu_3'} = \Delta^{-1} A^{\nu_0'}_{\nu_0} A^{\nu_1'}_{\nu_1} A^{\nu_2'}_{\nu_2} A^{\nu_3'}_{\nu_3} \widetilde{U}^{\nu_0\nu_1\nu_2\nu_3} \tag{14}$$

$$\widetilde{U}_{\nu_0'\nu_1'\nu_2'\nu_3'} = \Delta A^{\nu_0}_{\nu_0'} A^{\nu_1}_{\nu_1'} A^{\nu_2}_{\nu_2'} A^{\nu_3}_{\nu_3'} \widetilde{U}_{\nu_0\nu_1\nu_2\nu_3} \tag{15}$$

where the ν_j and $\nu_j', j = 0, 1, 2, 3$ refer to the initial and the new frames of reference, respectively, and $A^{\nu'}_{\nu} = \frac{\partial x^{\nu'}}{\partial x^{\nu}}$, $A^{\nu}_{\nu'} = \frac{\partial x^{\nu}}{\partial x^{\nu'}}$ (relative to local coordinates in the usual way); as always $\sum_{\nu'} A^{\nu'}_{\nu} A^{\sigma}_{\nu'}$ is the Kronecker unit tensor δ^{σ}_{ν}.

Lemma 1.3.1. *The transformations $\widetilde{U}^{\nu_0\nu_1\nu_2\nu_3}$ and $\widetilde{U}_{\nu_0\nu_1\nu_2\nu_3}$ are unit tensor densities.*

Proof. It suffices to show that such a transformation, for example $\widetilde{U}^{\nu_0'\nu_1'\nu_2'\nu_3'}$, replicates an alternating symmetric tensor. This is seen by assigning values of 1, -1 to the "untransformed" components in, say, the first equation and summing according to the permutation order of $\nu_0, \nu_1, \nu_2, \nu_3$, reproducing the same parity; In other words, the same assignment of 1, -1 on the left side of the equation produces a Jacobian factor, Δ, canceling the given Δ^{-1}. The same argument gives the desired result for equation (15), with the roles of Δ^{-1} and Δ interchanged. $\square$

Furthermore, since the unit tensors are constants they commute with differentiation and integration and accordingly remain valid under all transformations rising in the general theory.

Now, in the light of the foregoing, consider the following option of expressing the operator of current density C^{λ} in terms of $\widetilde{U}^{\nu_0\nu_1\nu_2\nu_3}$ and $\widetilde{U}_{\nu_0\nu_1\nu_2\nu_3}$; introducing the covariant entities $C_{\nu_1\nu_2\nu_3}, \widetilde{C}_{\nu_1\nu_2\nu_3}$ in the process, stipulate that $C_{\nu_1\nu_2\nu_3} = \widetilde{C}^{\lambda}\widetilde{U}_{\lambda\nu_1\nu_2\nu_3}$ and $C^{\lambda}\widetilde{U}_{\lambda\nu_1\nu_2\nu_3} = \widetilde{C}_{\nu_1\nu_2\nu_3}$, and reciprocally obtain

$$\frac{1}{6}\widetilde{U}^{\lambda\nu_1\nu_2\nu_3}C_{\nu_1\nu_2\nu_3} = \widetilde{C}^{\lambda} \tag{16}$$

$$\frac{1}{6}\widetilde{U}^{\lambda\nu_1\nu_2\nu_3}\widetilde{C}_{\nu_1\nu_2\nu_3} = C^{\lambda} \tag{17}$$

with bold faced quantities being densities transforming with some powers of Δ and the tilde signifying quantities transforming with the sign of Δ. It behooves us to note that it was apparently Georges de Rham (1903-1990) who noted that the preceding covariant tri-vector defines the coefficient of a pair differential 3-form in equation (14) and the coefficients of an impair 3-form in equation (15).

All this having been said, however, the first point to be taken is the following observation:

Proposition 1.3.1. *Generalized curl and divergence are dual generally invariant operators in the holonomic frame, regardless of whether there is a metric in the game; in other words,* $\partial_{\lambda}A_{\nu}^{\kappa'} - \partial_{\nu}A_{\nu}^{\kappa'} = 0$.

Proof. This follows easily from the fact that the Schouten unit tensors are invariant under differentiation and integration even in the realm of general coordinates. □

This fact obviously undergirds our discussion in Section 1.2.

The second point to take note of is that with de Rham's interpretation available we can use Maxwell's equations, reformulated as above, as a point d'appui for a critique of certain strongly held positions vis à vis the framework of quantum mechanics(**QM**), by the Copenhagen school and its interpretation of **QM**.

2. Differential Forms

2.1. More on Maxwell's Equations

We can make the following initial stipulations in view of the discussions in Sections 1.2 and 1.3. The Faraday-Maxwell law, involving $F = F(\mathbf{E}, \mathbf{B})$ and $\boldsymbol{A}$ i.e.

electric field intensity and magnetic field density joined (in F) into the components of an anti-symmetric space-time tensor and magnetic vector potential ($\boldsymbol{A}$), precipitates pair (or even) differential forms related to an absolute scalar flux source with quantum step size $\frac{\widetilde{h}}{2\widetilde{e}}$. Dually (as it were) the Ampère-Gauss formalism engendered by the behavior of $\widetilde{C}$ and $\widetilde{G}(\boldsymbol{H},\boldsymbol{D})$, i.e. magnetic field intensity and flux density joined (in $\widetilde{G}$) into a similar tensor, precipitates impair (or odd) differential forms related to the pseudo-scalar paired charge $\pm\widetilde{e}$, recalling that the "tilde" notation designates, by definition, orientation changes, as per Section 1.3. Of course, the scalars $\frac{\widetilde{h}}{2\widetilde{e}}$ and $\pm\widetilde{e}$ require comment. We will provide the required rationale at greater lengths presently, but simply note, for now, that these scalars experimentally arise from the analysis of the fractional Quantum Hall Effect along the lines of Aharonov-Bohm.

Training our focus for the moment on the indicated Ampère-Gauss formalism in Maxwell's laws, i.e. the stipulated relations between $\widetilde{C}$ and $\widetilde{G}$, it behooves us to cast it in the language of differential forms and path-integrals:

$$\iint_{\partial V} \widetilde{G}_{\lambda\nu} dx_1^\lambda dx_2^\nu = \iiint_V \widetilde{C}_{\lambda\nu_1\nu_2} dx_1^\lambda dx_2^\nu dx_3^\kappa \tag{18}$$

with $\widetilde{C} = d\widetilde{G}$, d being the exterior derivative characterized by $d^2 = d \circ d = 0$ and $dx \wedge dx = 0$, bringing the wedge product into play; of course, the latter stipulation is easily rendered equivalently as $dx \wedge dy = -dy \wedge dx$.

The point to be taken, again, is that with this approach to Maxwell's equations via the calculus of differential forms (the exterior calculus) we are still operating in a pre-metric or even metric-free environment, where any differential manifold will do. To wit, Maxwell's equations are pre-metric statements about the fields associated with $F, \widetilde{G} \in \Omega^2(\mathcal{M})$ as 2-forms, $\boldsymbol{A} \in \Omega^1(\mathcal{M})$ as a 1-form, and $\widetilde{C} \in \Omega^3(\mathcal{M})$ as a 3-form on the space-time manifold $\mathcal{M}$.

Additionally, qua orientation we reiterate that (with de Rham, really) fields and their associated forms are expressible as alternating form aggregates that are either pair (even) or impair (odd), with the latter transforming in accordance with sign changes determined by the sign of the Jacobian factor, as discussed earlier.

2.2. The Pair vs Impair Distinction and the Electric Charge

We are now in a position to propose something bold though mathematically quite prosaic.

Lemma 2.2.1. *$F(\boldsymbol{E}, \boldsymbol{B})$ pair $\Longrightarrow$ $\widetilde{G}(\boldsymbol{D}, \boldsymbol{H})$ impair.*

Proof. Consider the Gaussian integral $q = \iint_S \boldsymbol{D} \cdot dS$, indicating that the total flux out of the surface S is the net charge q within the surface. The surface vector on S does not change sign under inversion, but, in order to coexist with $\boldsymbol{E}$, the dielectric displacement, $\boldsymbol{D}$, does. This infers that upon admitting space inversion P into the system, charge becomes a pseudo-scalar. The charge in rotation $q \mapsto \tilde{q}$ (following Schouten's earlier protocols), and the fact that P is not a symmetry implies that $P\tilde{q} = -\tilde{q}$. So, since $P \circ P = \boldsymbol{E}$ the identity (the unit operation), $\tilde{q} + P\tilde{q}$ is obviously parity invariant. $\square$

Thus, the enantiomorphic nature of the electric charge, to coin a phrase, suggests, in view of the preceding lemma, that the prevailing connections holding that the charge should be an absolute scalar are safely jettisoned. For a global rendering of electricity and magnetism we can go to the language of differential forms where the pair versus impair distinction suffices.

2.3. The Theorem of de Rham

In preparation for the third and final part of this article we now briefly expound the preceding theme of pair versus impair differential forms by developing what is usually referred to as de Rham's theorem, but in its older form. In its present form it conveys the fact or the assertion that the natural cohomology attached to differential forms in the presence of the exterior derivative, i.e. de Rham cohomology, as a graded ring or vector space, is isomorphic to the singular cohomology of the underlying manifold.

Thus, if F is a p-form on our manifold $\mathcal{M}$ i.e. $F \in \Omega^p(\mathcal{M})$, we say that F is closed if $dF = 0$. More precisely, we can consider F and a submanifold $\mathcal{D} \subset \mathcal{M}$ such that F is closed on $\mathcal{D}$, meaning simply that throughout $\mathcal{D}$, $dF = 0$. We emphatically do not pre-suppose that the topology on $\mathcal{D}$ is such that it can not happen that $dF \neq 0$ in all sets $\mathcal{E}$ enclosed by $\mathcal{D}$. We also emphatically do not pre-suppose that the indicated fundamental group $\pi_1(\mathcal{D})$, should be trivial since $\mathcal{D}$ can have more than one connected component; $\mathcal{D}$ can be non-trivially linked, and so on. Thus it is feasible that there should be p-cycles c residing entirely in $\mathcal{D}$ (so that $dF = 0$ all along c) which enclose singularities or singular regions for a (finite) number $d \geq 1$ of domains where $dF \neq 0$. Evidently, for such $n^{th}(1 \leq n \leq d)$ singular region we can stipulate a cycle c_n, interior to c, such that $dF = 0$ all along c_n. Now we are in a position to state the following generalization to the residue theorem of complex analysis:

$$\int_c F = \sum_{n=1}^{d} \int_{c_n} F = \sum_{n=1}^{d} r_n =: r. \tag{19}$$

Of course, this relation delineates the behavior of some of the major players in Hodge theory, namely the interplay between p-forms and p-cycles. And the thrust of de Rham's theorem in this setting is that given pre-assigned periods r_n there is a closed p-form $F \in ker(d) \subset \Omega^p(\mathcal{D})$ on $\mathcal{D}$ such that its periods, $\int_{c_n} F$, coincide with the indicated r_n. Thus in brief we obtain the following

Theorem 2.3.1. *There exists a closed p-form on $\mathcal{M}$ having any pre-assigned set of values on its periods.*

3. Gauss-Ampère and Aharonov-Bohm Formalism and Pre-Statistical QM

3.1. The Indicated Formalisms

After Michael Faraday (1791-1867) established existence of electric charge e by his electrolytic experiments, which we may regard as a proto-quantum law, it might be said that the next such discovery would be that of Gauss-Ampère. Indeed, experimental verification of the existence of a smallest unit of flux $\frac{\tilde{h}}{2\tilde{e}}$, providing evidence for the discrete nature of quantization, was provided in 1961 by R. Doll and M. Näbauer [3] (and separately, by B.S. Deaver and W.M. Fairbank [2]). Prior to this, in 1959, Y. Aharonov and D.J. Bohm [1] had originally found a value of $\frac{\tilde{h}}{\tilde{e}}$ for the minimum unit, thereby engendering the Aharonov-Bohm law; they based this conclusion on electron beam interference experiments.

Now, first, the Gauss-Ampère law can be phrased in terms of the differential forms by stipulating that global flux conservation be rendered as $\iint_{c_2} F = 0$ for all 2-cycles c_2 with F an exact pair 2-form; thus its potential field can be rendered as a pair 1-form, $\boldsymbol{A}$, such that in those regions of space-time where $d\boldsymbol{A} = 0$ we get, as per Schouten and his tilde,

$$\int_{c_1} \boldsymbol{A} = n\frac{\tilde{h}}{2\tilde{e}}, \tag{20}$$

with the net number of $\pm$ linked elementary flux units linked by a 1-cycle c_1. Manifestly, this result agrees with the results of the Aharonov-Bohm experiment.

Second, as far as the global charge conservation is concerned, we begin with the statement that the impair 3-form $\widetilde{C}$ is exact i.e. $\iiint_{c_3} \widetilde{C} = 0$ for all 3-cycles c_3. But its potential field is just the impair 2-form $\widetilde{G}$, characterized by the fact that for all 2-cycles c_2 residing where $d\widetilde{G} = 0$ we get

$$\iint_{c_2} \widetilde{G} = s\tilde{e}, \tag{21}$$

where s is the sum of elementary charges of either polarity enclosed by c_2.

It should be noted that both relations $\int_{c_1} \boldsymbol{A} = n\frac{\tilde{h}}{2\tilde{e}}$, and $\iint_{c_2} \widetilde{G} = s\tilde{e}$, introduce discreteness as expressed in terms of closed differential forms. There is no need whatsoever to bring statistical quantum mechanical considerations into play: it is ultimately all a matter of invoking de Rham's result in the given setting of closed potential fields.

3.2. Regarding Quantum Mechanics

Thus, the quantum theoretic laws discussed in the preceding sections in point of fact stand in silent conflict with the dogma of the Copenhagen interpretation of **QM** that would insist on statistical/probabilistic methods to be employed in all circumstances, including those involving single quantum mechanical systems (as though an average must accrue faithfully to each member of a varying collection). Taking into account differences between pair and impair differential forms, as well as orientation-changing characteristics (so that we might account for the pseudo-scalar nature of $\tilde{h}$ and $\tilde{e}$, the quanta of action and charge, respectively), the discussion of Section 3.1 succeeds in presenting the fundamental laws concerning $\tilde{h}$ and $\tilde{e}$ in terms of de Rham's (and Schouten's) language of differential forms.

More generally, the method of period integrals sketched here provides metric independent stipulations, given that they are fundamentally topological in nature rather than (Riemannian) geometric. Accordingly, questions covering general invariance at the core of **QM**'s philosophical underpinnings are more clearly addressed on a single system basis in this way, obviating the conceptual damage done by statistical methods. In fact, statistical states seem alien to notions of covariance in the micro domain, and the single system approach we propose is consonant with no less than Einstein's principle of general covariance.

Perhaps a few historical observations are in order, so as to bolster our contention that what we propose should indeed be regarded as a critique, and even a corrective measure of the Copenhagen interpretation of **QM**. To wit, Schrödinger's wave mechanics formalism, centered on the wave equation, was in due course corrected, or expanded, by P.A. M. Dirac into a formalism taking special relativity into account. This brilliant success, the centerpiece of which is the Dirac equation, proved to be something of the end of the line qua desirable generalizations because it proved intractible to get general relativistic invariants of these equations. Accordingly, in due course, something of a backlash developed for relativity, and, in particular, the important theme of finding ways to refine the principle of general relativistic invariance, initiated by Kottler and Cartan, as discussed in Section 1.2, fell by the wayside; on the other hand **QM**, under the watchful eye of the Copenhagen establishment, proceeded to assume more and more freedom in developing formalisms of its own. However, there were dissenters on the scene (with Erwin Schrödinger and, to some extent, also Dirac, among them) and in this vein a specific attack on Copenhagen's single system interpretation of the probabilistic entity Ψ was launched in the 1930's by Karl Popper (1901- 1993), his goal being to champion our ensemble view of this matter. But Copenhagen's non-classical statistics remained on the scene, essentially untouched by criticism. Another major dissident to the Copenhagen dogma was the aforementioned David Bohm (cf Section 3.1), whose heterodoxy contributed to his falling out with his once mentor, J Robert Oppenheimer (1904-1967) who was of course a pupil of Max Born, the originator of the statistical approach. Now, with Bohm, and specifically, with his work of the late 1960's together with Y. Aharonov, as discussed above, we encounter a potent and, we claim, undeniable, line of argument favoring a return to a pre-Schrödinger and anti-Copenhagen perspective, namely, that quantization for a single system can be evinced by global quantization of field integrals of the vector potential. This is part and parcel of our discussion in the proceeding sections; it remains for us to note, for historical accuracy and to give credit where it is due, that it was Robert M. Kiehn [4] who, some two decades after the Aharonov-Bohm experiment, suggested a complete set of period integral quantizers of flux, charge, and action, thereby setting the stage for what we now ultimately propose vis à vis de Rham cohomology.

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