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Discrete Velocity Fields with Explicitly Computable Lagrangian Law

Curtis D. Bennett
Loyola Marymount University, cbennett@lmu.edu

Craig L. Zirbel
Bowling Green State University

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Curtis D. Bennett [1]
Craig L. Zirbel [2]

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Abstract. We introduce a class of random velocity fields on the periodic lattice and in discrete time having a certain hidden Markov structure. The generalized Lagrangian velocity (the velocity field as viewed from the location of a single moving particle) has similar hidden Markov structure, and its law is found explicitly. Its rate of convergence to equilibrium is studied in small numerical examples and in rigorous results giving absolute and relative bounds on the size of the second–largest eigenvalue modulus. The effect of molecular diffusion on the rate of convergence is also investigated; in some cases it slows convergence to equilibrium. After repeating the velocity field periodically throughout the integer lattice, it is shown that, with the usual diffusive rescaling, the single–particle motion converges to Brownian motion in both compressible and incompressible cases. An exact formula for the effective diffusivity is given and numerical examples are shown.

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1 Introduction

A fundamental and longstanding problem in statistical fluid mechanics is this: given a random velocity field U having a known probability law, determine the law of motion of a single particle moved by U, with or without the additional influence of molecular diffusion. This article introduces velocity fields in discrete space and time for which one may explicitly write down the probability law of the particle's velocity and thus obtain the probability law of the particle's motion. These models are well adapted to doing exact numerical calculations of Lagrangian statistics, rather than simulating, as is usually done.

We begin with the continuous space–time setting of the problem. Let $U = \{U(x,t),\ x \in \mathbb{R}^d,\ t \geq 0\}$ be a random vector field taking values in $\mathbb{R}^d$. Think of U as a velocity field and consider the motion of a particle whose position satisfies the trajectory equation

$$\frac{dX_t}{dt} = U(X_t, t), \qquad t > 0, \tag{1.1}$$

assuming sufficient smoothness of U. If molecular diffusion is desired, consider instead the stochastic differential equation

$$dX_t = U(X_t, t)dt + \sigma(X_t, t)dW_t, \qquad t > 0, \tag{1.2}$$

where σ is the molecular diffusivity and W is a Wiener process independent of U.

[1]Department of Mathematics; Loyola Marymount University; Los Angeles, CA 90045

[2]Corresponding author: Department of Mathematics and Statistics; Bowling Green State University; Bowling Green, OH 43403-0221; phone: (419) 372-7466; fax: (419) 372-6092; e-mail: zirbel@bgnet.bgsu.edu

The fundamental problem is to determine the law of the particle's trajectory $X_t,\ t \geq 0$, from knowledge of the law of the velocity field U and the molecular diffusivity σ. A slightly simpler goal is to determine the law of the *Lagrangian velocity* process $U(X_t, t),\ t \geq 0$, which is the particle's velocity under (1.1) or its drift under (1.2). (By contrast, U is called the *Eulerian velocity* field.) Note that even determining the appropriate law for U is nontrivial, however moving forward from a given law is an important part of the larger problem.

The most widely applicable way to study particle motion in a random velocity field is to repeatedly drop a particle into identically distributed realizations of the velocity field, track it, and analyze the data statistically. For example, Avellaneda *et al.* (1993), Elliott and Majda (1996), and Carmona and Cerou (1999) have examined particle motion in numerical simulations of model velocity fields with given laws. Similarly, Yeung and Pope (1989) and Gotoh *et al.* (1993) have studied particle motion in numerical simulations of forced Navier–Stokes turbulence. The same technique is used in physical flows, for example in oceanography, where drifting instruments have been tracked remotely to obtain approximate particle trajectories which can be used to estimate the laws of $X_t,\ t \geq 0$ and $U(X_t, t),\ t \geq 0$ in the ocean; see the review Davis (1991) and references therein.

Statistical methods may be used to estimate numerical parameters of the law of particle motion, but the estimates always have a margin of error which makes it difficult to tell how small changes in the Eulerian law will affect the Lagrangian law. Thus, for example, it is difficult to make or refute conjectures about the Lagrangian law.

There are several rigorous results concerning the asymptotic convergence of the rescaled processes $\varepsilon X_{t/\varepsilon^2},\ t \geq 0$ to Brownian motion as $\varepsilon \to 0$; among these are Bhattacharya (1985), Molchanov (1996), Carmona and Xu (1997), and Fannjiang and Komorowski (1997, 1999). These are known as homogenization results from the original connection with partial differential equations; indeed the mean concentration in these cases evolves according to an effective diffusion equation under a similar rescaling of space and time. Majda and Kramer (1999) give shear flow models for which the effective diffusivity can be computed exactly.

A key ingredient in many homogenization results is that U is homogeneous, stationary, and divergence free. Under these conditions, the Lagrangian velocity $U(X_t, t),\ t \geq 0$ is strictly stationary for both (1.1) and (1.2), as was shown by Lumley (1962) and Zirbel (2001). Moreover, the *generalized Lagrangian velocity* V defined by

$$V(x, t) = U(x + X_t, t), \qquad x \in \mathbb{R}^d,\ t \geq 0, \tag{1.3}$$

is strictly stationary, as shown by Osada (1982) for (1.2) and Zirbel (2001) for (1.1) and (1.2). The generalized Lagrangian velocity is the view of the whole velocity field at time t from the location of the particle at time t.

When the Eulerian velocity field U is homogeneous and Markov in time, the generalized Lagrangian velocity V is Markov, even when U is divergent, cf. Zirbel (2000). Carmona and Xu (1997) showed this in a case in which U is, in addition, stationary, Gaussian, and divergence free. They and Fannjiang and Komorowski (1999) have computed the generator of V and have obtained L^2 rate of convergence results for functionals of V in terms of the spectral gap of the Eulerian field, provided it is divergence free.

In the current paper, we introduce a large class of Eulerian velocity fields U on the periodic lattice and in discrete time for which the law of the generalized Lagrangian velocity V can be found explicitly in both incompressible and compressible cases. Exact numerical calculations of Lagrangian statistics may be performed instead of simulations. Thus, one may quickly check (or reject) conjectures numerically. The models are especially well suited to studying the phenomenology of particle motion due to simple fluid motifs such as localized vortices which move over time, as the examples will show. The fundamental difficulty in continuous space and time is the non–linear relationship between U and X. This is retained in the discrete setting with analogues of (1.1) and (1.2); see (2.1) and (8.1). While the discrete nature of the model might appear to be a limitation, we note that numerical approximations of particle motion in both random and Navier–Stokes velocity fields are necessarily discrete also, albeit with a very fine lattice.

When U is homogeneous and Markov in time, it can be described by two parameters, the velocity field type I and the location L. Together (I, L) forms a Markov chain. This observation forms the basis for the class of velocity fields we introduce (Section 3). By choosing the state space of I to be large enough, a very large class of Eulerian laws is allowed, including non–Markov models and numerical simulations of Navier–Stokes turbulence.

The generalized Lagrangian velocity V has the same hidden Markov structure as U, except that its location parameter M evolves differently than L (Section 4). The transition matrix of (I, M) can be written in terms of that of (I, L) in an elegant way. We give bounds on the rate of convergence to equilibrium of (I, M) (and thus V) in Sections 5 – 7. The effect of adding a discrete analogue of molecular diffusion is studied in Section 8. Finally, homogenization of single particle motion in $\mathbb{Z}^d$ is studied by periodically repeating U throughout $\mathbb{Z}^d$ (Section 9). Under the usual diffusive scaling, this motion converges to Brownian motion in both incompressible and compressible cases. We give an exact, computable formula for the effective diffusivity and provide some numerical examples which show that the effective diffusivity is increased by compressibility.

2 Discrete velocity fields

Particle motion on the lattice has become very familiar with the various studies of Markov motion in a homogeneous random environment. Our situation differs primarily in that the environment (the velocity field) changes over time (it is not "quenched") and that it models particles carried by a fluid rather than the molecular motion of systems of particles. The recent papers on card shuffling (for example, Bayer and Diaconis (1992)) also bear some similarity to our situation, with each shuffle moving all particles (cards) incompressibly. The main difference is that successive shuffles are independent, whereas we seek models in which the velocity field exhibits strong dependence in time. Similar comments apply to random transposition models; see Diaconis (1988).

Let us now turn to describing our situation. The spatial domain will be $\mathbb{D} = \{0, 1, \ldots, n_1 - 1\} \times \cdots \times \{0, 1, \ldots, n_d - 1\}$, which has $n = n_1 n_2 \cdots n_d$ points. Addition of elements of $\mathbb{D}$ is done componentwise modulo the numbers $n_1, \ldots, n_d$, which we call addition modulo $\mathbb{D}$.

We think of $\mathbb{D}$ as a lattice with periodic boundary conditions.

A velocity field u on $\mathbb{D}$ is a mapping from $\mathbb{D}$ to $\mathbb{Z}^d$. Figure **??** illustrates two–dimensional discrete fields on successively finer grids which approximate a continuous velocity field. We denote by $\mathcal{U}$ the set of all velocity fields on $\mathbb{D}$. A random velocity field U is a stochastic process $U_t,\ t = 0, 1, \ldots$ taking values in $\mathcal{U}$. A velocity field u in $\mathcal{U}$ generates a mapping $\alpha : \mathbb{D} \to \mathbb{D}$ by $\alpha(x) = x + u(x), x \in \mathbb{D}$. If α is a permutation on $\mathbb{D}$, we say that u is *incompressible*. This is the discrete–space analogue of a divergence–free vector field. We denote by $\mathcal{U}_0$ the set of all incompressible vector fields. A random velocity field U taking values in $\mathcal{U}_0$ is said to be *incompressible*, otherwise it is called *compressible*.

The velocity field U is said to be *homogeneous* if, for all z in $\mathbb{D}$, the random velocity field $\widetilde{U}$ defined by $\widetilde{U}_t(x) = U_t(x+z), x \in \mathbb{D},\ t = 0, 1, \ldots$ has the same distribution as U. In other words, the law of U is invariant under spatial translation. The definition of stationarity is similar, but for temporal translations.

The trajectory equation in discrete time with no molecular diffusion is

$$X_{t+1} = X_t + U_t(X_t), \qquad t = 0, 1, \ldots, \tag{2.1}$$

with X_0 fixed and using addition modulo $\mathbb{D}$. Molecular diffusion is discussed in Section 8. The generalized Lagrangian velocity V is defined by

$$V_t(x) = U_t(x + X_t), \qquad x \in \mathbb{D},\ t = 0, 1, \ldots, \tag{2.2}$$

which is the same as (1.3). Note that $X_t = X_0 + \sum_{s=0}^{t-1} V_s(0)$ for $t = 0, 1, \ldots$, which is an additive functional of V. Also note that if U is homogeneous, stationary, and incompressible, then V is strictly stationary, as in the continuous case, cf. Zirbel (2001).

3 Eulerian velocity fields with hidden Markov structure

We now introduce the class of Eulerian velocity fields with hidden Markov structure that will be considered in the remainder of the paper. Fix $m \geq 1$ and let $\mathcal{I} = \{1, \ldots, m\}$. Let $u : \mathcal{I} \to \mathcal{U}$, so that each i in $\mathcal{I}$ is associated with a vector field $u(i, \cdot)$ in $\mathcal{U}$. We call $u(1, \cdot), \ldots, u(m, \cdot)$ *vortex types*. Let $I_t,\ t = 0, 1, \ldots$ be a Markov chain on $\mathcal{I}$ with transition matrix R. We assume that this *type process* is irreducible and aperiodic. The distribution of I_0 is immaterial at this point.

For example, we may have $d = 2$ and $m = 2$ with $u(1, \cdot)$ being a clockwise vortex and $u(2, \cdot)$ being an anticlockwise vortex as in Figure **??**. Then $u(I_t, x),\ t = 0, 1, \ldots, x \in \mathbb{D}$ is a random velocity field which alternates between clockwise and anticlockwise vortices over time.

Next, we allow the vortices to move in $\mathbb{D}$, each in their own characteristic way. For each i and j in $\mathcal{I}$, let $c_{ij} : \mathbb{D} \to [0, 1]$ be a function for which $\sum_{x \in \mathbb{D}} c_{ij}(x) = 1$. For all types i, j and times $t = 0, 1, \ldots$, let $A_t(i, j)$ be a random variable taking values in $\mathbb{D}$ with $\mathbb{P}(A_t(i, j) =$

$x) = c_{ij}(x)$, and, moreover, let the collection $A = \{A_t(i,j),\ i,j \in \mathcal{I},\quad t = 0,1,\ldots\}$ be mutually independent and independent of I. Let L_0 be a random variable taking values in $\mathbb{D}$ and independent of A and I. Define the *location process* $L_t,\ t = 0,1,\ldots$ recursively by

$$L_{t+1} = L_t + A_t(I_t, I_{t+1}), \qquad t = 0,1,\ldots, \tag{3.1}$$

with addition modulo $\mathbb{D}$. We must impose a mild condition on the c_{ij} to guarantee that for all t large enough, the distribution of L_t is supported on all of $\mathbb{D}$. We require that there be a sequence $i_1, i_2, \ldots, i_N$ of types for which $R_{i_1 i_2}, \ldots, R_{i_{N-1} i_N} > 0$ and the distribution of $A_1(i_1, i_2) + \cdots + A_{N-1}(i_{N-1}, i_N)$ is supported on all of $\mathbb{D}$, since by irreducibility of R, the type process I will make all of these transitions at some point.

It is clear that the paired process $(I_t, L_t),\ t = 0,1,\ldots$ is an aperiodic, irreducible Markov chain. Now define the velocity field U by

$$U_t(x) = u(I_t, x - L_t), \qquad t = 0,1,\ldots. \tag{3.2}$$

Note that U is not Markov in general, but will be Markov if knowledge of U_t allows one to uniquely determine I_t and L_t. The process $U_t,\ t = 0,1,\ldots$ will be homogeneous if the initial location L_0 is uniformly distributed on $\mathbb{D}$. It will be stationary if, in addition, I_0 has the invariant distribution for R. It can be shown that, if U is homogeneous and Markov, then U can be written as in (3.2). Thus, this construction generalizes the case of homogeneous Markov velocity fields.

We now compute the transition matrix P of (I, L) for later use. First, note that

$$P((i,y);(j,z)) = \mathbb{P}(L_{t+1} - L_t = z - y | I_{t+1} = j, I_t = i)\mathbb{P}(I_{t+1} = j | I_t = i) = c_{ij}(z-y)R_{ij}.$$

For each i, j in $\mathcal{I}$, define a matrix C_{ij} by $C_{ij}(y,z) = c_{ij}(z-y),\ y,z \in \mathbb{D}$. Note that C_{ij} is doubly indexed by $\mathbb{D}$ and is doubly stochastic. When $\mathbb{D}$ is one–dimensional, C_{ij} is a circulant matrix. In higher–dimensions, C_{ij} is more awkward to write out as a conventional matrix, but it still has the analogue of the circulant property, which is closely related to homogeneity. Now we may write P in block form:

$$P = [R_{ij}C_{ij}] = \begin{bmatrix} R_{11}C_{11} & \cdots & R_{1m}C_{1m} \\ \vdots & & \vdots \\ R_{m1}C_{m1} & \cdots & R_{mm}C_{mm} \end{bmatrix}. \tag{3.3}$$

We think of the blocks as being indexed by type, with type transitions given by the R_{ij}. Once the type transition is made, the appropriate circulant matrix C_{ij} is used to determine the transition of the location.

4 The law of the generalized Lagrangian velocity

The generalized Lagrangian velocity V can be written in terms of the vortex type I_t, the vortex location L_t, and the particle position X_t as $V_t(x) = u(I_t, x - (L_t - X_t))$. Define the

Lagrangian location process M by $M_t = L_t - X_t,\ t = 0, 1, \ldots$. A short calculation shows that M evolves according to

$$M_{t+1} = \sigma(I_t, M_t) + A_t(I_t, I_{t+1}), \qquad t = 0, 1, \ldots, \tag{4.1}$$

where for each i in $\mathcal{I}$, we define $\sigma(i, \cdot) : \mathbb{D} \to \mathbb{D}$ by $\sigma(i, x) = x - u(i, -x),\ x \in \mathbb{D}$. Note that $\sigma(i, \cdot)$ is a permutation if and only if $u(i, \cdot)$ is incompressible.

It is clear that the paired process (I, M) is Markov. Thus, the Eulerian and generalized Lagrangian velocity processes have very similar structure: they are obtained in the same way from Markov chains (I, L) and (I, M), respectively, and the type process I is the same in both cases. The only difference is that the Lagrangian location M_t undergoes a "shuffle" $\sigma(I_t, \cdot)$ before it is shifted by A, to account for the change of perspective brought about by the motion of the particle, while the Eulerian location L_t is just shifted.

We now find the transition matrix Q of (I, M). First, M makes a deterministic transition $m \to \sigma(i, m)$ according to the current value of I_t. This may be represented by a block diagonal matrix Σ:

$$\Sigma = \begin{bmatrix} \Sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Sigma_m \end{bmatrix}, \tag{4.2}$$

where Σ_i is the transition matrix corresponding to $\sigma(i, \cdot)$. That is, $\Sigma_i(x, y)$ equals 1 if $y = \sigma(i, x)$ and is 0 otherwise. The off–diagonal blocks of Σ are zero because only the location changes at this step. The transition given by Σ is followed by a transition of I and the addition of $A_t(I_t, I_{t+1})$, which is accomplished by P. Thus, we have the very simple formula $Q = \Sigma P$ for the relation between the transition matrix P underlying the Eulerian velocity field and the transition matrix Q underlying the Lagrangian field.

When the Eulerian velocity field U is incompressible, the matrices Σ_i and Σ are permutation matrices. Thus, $Q = \Sigma P$ is obtained by permuting rows of P within each block according to the velocity field corresponding to that block. When U is compressible, at least one Σ_i is not a permutation matrix, and at least one row of P appears twice in Q. As we shall see, this has many consequences for the law of V and makes this case more difficult than the incompressible case.

Note that we have found the exact law of the generalized Lagrangian velocity V in terms of the law of the Eulerian velocity field U. Moreover, since the particle position X satisfies $X_t = X_0 + \sum_{s=0}^{t-1} u(I_s, -M_s)$, it is an additive functional of the Markov chain (I, M), and so, in principle, its law is known.

5 Eigenvectors of P and Q

Here we begin the study of the eigenvectors and eigenvalues of the transition matrices P and Q underlying U and V in order to understand their rates of convergence to equilibrium. The rows and columns of P, Σ, and Q are indexed by type $i \in \mathcal{I}$ and location $x \in \mathbb{D}$. They

are functions f from $\mathcal{I} \times \mathbb{D}$ to $\mathbb{C}$, which can be thought of as vectors in $\mathbb{C}^{mn}$. For b in $\mathbb{C}^m$ and c in $\mathbb{C}^{\mathbb{D}}$, we will write $f = b \otimes c$ for the Kronecker product $f(i,x) = b_i c(x)$, where $b_1, \ldots, b_m$ are the components of b. Note that $b \otimes c$ may denote either a row or a column vector; which it is will be clear from the context. We denote by $\mathbf{1}$ the vector in $\mathbb{C}^{\mathbb{D}}$ with all components equal to 1.

Let π denote the invariant distribution of the type transition matrix R. Because the C_{ij} are doubly stochastic, $(\pi \otimes \mathbf{1})P = (\pi \otimes \mathbf{1})$, so that under the invariant distribution $\frac{1}{n}(\pi \otimes \mathbf{1})$, the type I_t is independent of the Eulerian location L_t, which is uniformly distributed on $\mathbb{D}$. Moreover, in the incompressible case, the matrix Σ is a permutation matrix, and so $\pi \otimes \mathbf{1}$ is also the invariant distribution for Q, so the same comments apply to (I, M). In particular, when the Eulerian parameters (I, L) are started in the invariant distribution, both the Eulerian and generalized Lagrangian velocity are stationary with the same invariant distribution. The same is true in general, cf. Zirbel (2001). However, in the compressible case, $\pi \otimes \mathbf{1}$ fails to be invariant for Q because Σ is no longer doubly stochastic. The invariant type distribution is still π, but the Lagrangian location parameter M is now dependent on the type under the invariant distribution.

Vectors of the form $b \otimes \mathbf{1}$ may also be right eigenvectors, for

$$P(b \otimes \mathbf{1}) = Q(b \otimes \mathbf{1}) = (Rb) \otimes \mathbf{1}, \tag{5.1}$$

in both the incompressible and compressible cases. Thus, if b is an eigenvector of R, then $b \otimes \mathbf{1}$ is an eigenvector of both P and Q with the same eigenvalue as b has for R. Consider the set $\mathcal{G} = \{b \otimes \mathbf{1} : b \in \mathbb{C}^m\}$. The following result shows that in the incompressible case, the eigenvectors of Q split naturally over $\mathcal{G}$ and its orthogonal complement $\mathcal{H} = \{h \in \mathbb{C}^{mn} : g^* h = 0 \text{ for all } g \in \mathcal{G}\}$, where $*$ denotes conjugate transpose. It is useful to think of $\mathcal{G}$ as the set of vectors which are constant on blocks and of $\mathcal{H}$ as the set of vectors which sum to zero on each block.

5.2 **Proposition.** *Suppose that Σ is a permutation matrix of the form (4.2) and that $Q = \Sigma P$ is diagonalizable. Then a basis of eigenvectors of Q can be chosen so that each is either an element of $\mathcal{H}$ or of the form $b \otimes \mathbf{1}$ where b is a right eigenvector of R. Moreover, Q can be written as the direct sum of its restrictions to $\mathcal{G}$ and $\mathcal{H}$.*

Proof: Equation (5.1) shows that $Q\mathcal{G} \subseteq \mathcal{G}$. We claim that $Q\mathcal{H} \subseteq \mathcal{H}$ as well. Let $h \in \mathcal{H}$ and $g = b \otimes \mathbf{1} \in \mathcal{G}$. We must show that $g^* Q h = 0$. But $g^* Q h = g^* \Sigma P h$, and $g^* \Sigma = g^*$ because Σ is a permutation matrix of the form (4.2). Moreover, $g^* P = ((b^* R)^* \otimes \mathbf{1})^*$, and so $g^* P h = 0$. Thus, $Q\mathcal{H} \subseteq \mathcal{H}$. This establishes the last claim of the proposition.

Now let $\{f^{(j)}\}$ be a basis for $\mathbb{C}^{mn}$ consisting of eigenvectors of Q with eigenvalues $\mu^{(j)}$. We may write $f^{(j)} = g^{(j)} + h^{(j)}$ where $g^{(j)} \in \mathcal{G}$ and $h^{(j)} \in \mathcal{H}$. Because $Q\mathcal{G} \subseteq \mathcal{G}$ and $Q\mathcal{H} \subseteq \mathcal{H}$, we have that $Qg^{(j)} = \mu^{(j)} g^{(j)}$ and $Qh^{(j)} = \mu^{(j)} h^{(j)}$. From among the $g^{(j)}$ we may choose a basis for $\mathcal{G}$, and from the $h^{(j)}$ a basis for $\mathcal{H}$, which gives the basis claimed.

Finally, each $g^{(j)}$ chosen may be written in the form $b \otimes \mathbf{1}$. The equation $Qg = \lambda g$ becomes $(Rb) \otimes \mathbf{1} = (\lambda b) \otimes \mathbf{1}$ by (5.1), and so b must be an eigenvector of R, as claimed. □

The eigenvectors of P are related to the eigenvectors of circulant matrices. Let $c : \mathbb{D} \to \mathbb{C}$ and define a matrix C doubly–indexed by $\mathbb{D}$ by $C(y,z) = c(z-y),\ y,z \in \mathbb{D}$. We say that C is *circulant* by extension of the case $d = 1$. For each k in $\mathbb{D}$ define a vector $\phi^{(k)}$ in $\mathbb{C}^{\mathbb{D}}$ by $\phi^{(k)}(x) = \exp(2\pi i k \cdot x)/\sqrt{n}, x \in \mathbb{D}$, where $n = n_1 n_2 \cdots n_d$ and the inner product is defined by $k \cdot x = k_1 x_1/n_1 + \cdots + k_d x_d/n_d$. The vectors $\phi^{(k)}$ are the standard Fourier basis. They are orthonormal, $\phi^{(k)*}\phi^{(\ell)} = \delta_{k\ell},\ k,\ell \in \mathbb{D}$, and for each k in $\mathbb{D}$, $\phi^{(k)}$ is an eigenvector of C with eigenvalue $\lambda^{(k)} = \sqrt{n} c^T \phi^{(k)}$.

The eigenvectors of P split over a large number of mutually orthogonal subspaces, as the next result will show. It is an extension of Proposition 5.2 for the case $\Sigma = I$. For each k in $\mathbb{D}$, let $\mathcal{G}^{(k)}$ be the space $\{b \otimes \phi^{(k)} : b \in \mathbb{C}^m\}$. These spaces are then mutually orthogonal and $\mathcal{G}^0 = \mathcal{G}$ from above. Let $\lambda_{ij}^{(k)}$ denote the eigenvalue of C_{ij} corresponding to $\phi^{(k)}$. For each k in $\mathbb{D}$, let $R^{(k)}$ be the matrix $R^{(k)} = [R_{ij}\lambda_{ij}^{(k)}]$, and note that $R^{(0)} = nR$.

5.3 **Proposition.** *Suppose that P is diagonalizable. Then the eigenvectors of P are $b^{(i,k)} \otimes \phi^{(k)}, i \in \mathcal{I},\ k \in \mathbb{D}$ where $b^{(i,k)}, i \in \mathcal{I}$ are the eigenvectors of $R^{(k)}$.*

Proof: A simple computation similar to (5.1) shows that $P(b \otimes \phi^{(k)}) = (R^{(k)}b) \otimes \phi^{(k)}$. Thus, for each k in $\mathbb{D}$, $P\mathcal{G}^{(k)} \subseteq \mathcal{G}^{(k)}$. We now proceed as in the proof of Proposition 5.2 to select a basis $b^{(i,k)} \otimes \phi^{(k)}, i \in \mathcal{I}$ of eigenvectors of P for each subspace $\mathcal{G}^{(k)}$ and recognize that the $b^{(i,k)}$ must be eigenvectors of $R^{(k)}$. □

6 Absolute bound on eigenvalues

The moduli of eigenvectors determine the rate of convergence to equilibrium for Markov chains. As shown in (5.1), the matrices P and Q share eigenvectors of the form $b \otimes \mathbf{1}$, where b is an eigenvector of R. These eigenvalues depend only on type transitions and not on the velocity fields encoded in Σ, so they are of relatively little interest. By Proposition 5.2, in the incompressible case, the remaining eigenvalues correspond to eigenvectors which lie entirely in $\mathcal{H}$. As our techniques work only for such eigenvectors, we restrict attention to the incompressible case. If M is a diagonalizable matrix, we denote by $\mathrm{eig}_1(M)$ the largest of the moduli of the eigenvalues of M and the second largest by $\mathrm{eig}_2(M)$. Similarly, we write $\mathrm{eig}_1(M, \mathcal{L})$ for the largest eigenvalue modulus among eigenvectors in the subspace $\mathcal{L}$.

6.1 **Theorem.** *Suppose that Σ is a permutation matrix and that $Q = \Sigma P$ is diagonalizable. Then $\mathrm{eig}_1(Q, \mathcal{H}) \le \mathrm{eig}_1(T)$, where $T = [R_{ij}\mathrm{eig}_2(C_{ij})]$ is an $m \times m$ matrix.*

Proof: Let $h \in \mathcal{H}$ be an eigenvector of Q with eigenvalue μ. Writing h_i for the blocks of h, the equation $Qh = \mu h$ becomes $\sum_{j=1}^m R_{ij}\Sigma_i C_{ij} h_j = \mu h_i,\ i = 1, \ldots, m$. Fix i. Noting that the $\mathbb{C}^{\mathbb{D}}$ vector norm is invariant under the permutation matrix Σ_i, we obtain

$$|\mu|\|h_i\| = \left\| \sum_{j=1}^m R_{ij} C_{ij} h_j \right\| \le \sum_{j=1}^m R_{ij} \|C_{ij} h_j\|, \tag{6.2}$$

by the triangle inequality. Now h is in $\mathcal{H}$ and so, in particular, h is orthogonal to $e_j \otimes \mathbf{1}$ where e_j is the jth standard basis vector for $\mathbb{C}^m$. Thus, $\mathbf{1}^T h_j = 0$, and so h_j is orthogonal to $\phi^{(0)}$. Writing h_j as a linear combination of the $\phi^{(k)}$, $k \neq 0$, we see that

$$\|C_{ij} h_j\|^2 = \sum_{k \neq 0} |\phi^{(k)*} h_j|^2 |\lambda_{ij}^{(k)}|^2 \leq \mathrm{eig}_2(C_{ij})^2 \|h_j\|^2. \tag{6.3}$$

Using this in (6.2) yields $|\mu| \|h_i\| \leq \sum_{j=1}^m T_{ij} \|h_j\|$. Now T has non–negative entries, so by a corollary of the Perron–Frobenius Theorem (cf. Theorem 15.5.1 of Lancaster and Tismenetsky (1985)), T has a left eigenvector a with non–negative entries and a positive real eigenvalue equal to $\mathrm{eig}_1(T)$. Multiplying $|\mu| \|h_i\|$ by a_i and summing over i yields

$$|\mu| \sum_{i=1}^m a_i \|h_i\| \leq \sum_{j=1}^m \sum_{i=1}^m a_i T_{ij} \|h_j\| = \sum_{j=1}^m \mathrm{eig}_1(T) a_j \|h_j\|, \tag{6.4}$$

from which we conclude $|\mu| \leq \mathrm{eig}_1(T)$. □

7 Relative bounds on eigenvalues

For non–divergent random velocity fields in continuous space and time, Fannjiang and Komorowski (1999) have shown that the spectral gap of the generalized Lagrangian velocity exceeds that of the Eulerian velocity, so that the Lagrangian velocity converges to equilibrium at least as quickly as the Eulerian. Surprisingly, in discrete space and time, one can find counterexamples to this behavior. In this section we give a simple counterexample and then identify two conditions on the Eulerian velocity field which are sufficient to guarantee that $\mathrm{eig}_1(Q, \mathcal{H}) \leq \mathrm{eig}_1(P, \mathcal{H})$, meaning that the generalized Lagrangian velocity V converges to equilibrium at least as fast as U.

7.1 **Example.** Consider the one–dimensional velocity fields $u(1, \cdot) = [1 \;\; -2 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1]$ and $u(2, \cdot) = [-1 \;\; -1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 2]$. The type I makes transitions according to the transition matrix $R = \left[\begin{smallmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{smallmatrix}\right]$. Given the type transition, the vortex location L makes transitions according to circulant matrices C_{ij} whose first rows are given by

$$\begin{aligned} c_{11} &= [0.2 \;\; 0.75 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0.05] \\ c_{12} = c_{21} &= [1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0] \\ c_{22} &= [0.2 \;\; 0.05 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0.75] \end{aligned}$$

Then we find $\mathrm{eig}_1(P, \mathcal{H}) = 0.6813$ and yet $\mathrm{eig}_1(Q, \mathcal{H}) = 0.8455$. However, if we interchange c_{11} and c_{22}, then $\mathrm{eig}_1(P, \mathcal{H}) = 0.6813$ and $\mathrm{eig}_1(Q, \mathcal{H}) = 0.6546$. □

7.2 **Proposition.** *Suppose that U is incompressible, $C_{ij} = C$ for all i, j in $\mathcal{I}$ and that R is diagonalizable. Then* $\mathrm{eig}_1(Q, \mathcal{H}) \leq \mathrm{eig}_1(P, \mathcal{H})$.

Proof: Let $b^{(i)}, i = 1, \ldots, m$ denote the eigenvectors of R and $\rho^{(i)}$ the corresponding eigenvalues. Recalling the definition of $R^{(k)}$, we have $R^{(k)} = [R_{ij}\lambda^{(k)}] = \lambda^{(k)}R$, and so the $b^{(i)}$ are also the eigenvectors of $R^{(k)}$. The eigenvectors of P are thus of the form $b^{(i)} \otimes \phi^{(k)}$ and the corresponding eigenvalues are $\rho^{(i)}\lambda^{(k)}$. The eigenvalues over $\mathcal{H}$ correspond to $k \neq 0$, and the largest of these occurs when $\rho^{(i)} = 1$ and $|\lambda^{(k)}| = \mathrm{eig}_2(C)$. Thus $\mathrm{eig}_1(P, \mathcal{H}) = \mathrm{eig}_2(C)$. In the current case, $T = [R_{ij}\mathrm{eig}_2(C)] = \mathrm{eig}_2(C)R$, and so $\mathrm{eig}_1(T) = \mathrm{eig}_2(C)$. Theorem 6.1 gives $\mathrm{eig}_1(Q, \mathcal{H}) \leq \mathrm{eig}_1(T) = \mathrm{eig}_2(C) = \mathrm{eig}_1(P, \mathcal{H})$, which is the desired result. □

7.3 **Remark.** Equations (3.1) and (4.1) become, in the present case, $L_{t+1} = L_t + A_t$ and $M_{t+1} = \sigma(I_t, M_t) + A_t$. We have written A_t in place of $A_t(I_t, I_{t+1})$ because the law of $A_t(i, j)$ does not depend on i and j when $C_{ij} = C$. It is clear that the processes I and L evolve independently. These equations can be thought of as describing the motion of a particle L undergoing diffusion and a particle M subject to diffusion and incompressible advection. The advecting velocity field $W_t(x) = \sigma(I_t, x) - x$ need not be homogeneous, and can be made to have virtually any law by suitably enlarging the state space of I.

Now L is Markov but M alone is not. For a proper comparison of their rates of convergence to equilibrium we compare the Markov chains (I, L) and (I, M). The eigenvalues over $\mathcal{H}$ are germane to the rate of convergence of the distributions of L and M to the uniform distribution, and Proposition 7.2 concludes that $\mathrm{eig}_1(Q, \mathcal{H}) \leq \mathrm{eig}_1(P, \mathcal{H})$, which means that diffusion plus incompressible advection makes particle location converge to uniform more quickly than diffusion alone. The analogue for motion on $\mathbb{Z}^d$ or $\mathbb{R}^d$ is that the effective diffusivity exceeds the molecular diffusivity; cf. Isichenko (1992), Equation (4.16). □

Next, we will assume that the Eulerian velocity field is incompressible and reversible and show that $\mathrm{eig}_1(Q, \mathcal{H}) \leq \mathrm{eig}_1(P, \mathcal{H})$. The inspiration for this case is Section 2 of Carmona and Xu (1997), where the Eulerian velocity field is incompressible, Markov, and reversible.

As in Section 5, let π be the invariant distribution of the type process I, so that $\frac{1}{n}(\pi \otimes \mathbf{1})$ is the invariant distribution of (I, L). Note that the components of π are non–zero. Set $\Pi = \mathrm{diag}(\pi \otimes \mathbf{1})$. The Markov chain $(I_t, L_t),\ t = 0, 1, \ldots$ is reversible if ΠP is symmetric; then the law of U is invariant under time reversal.

Reversibility puts rather severe restrictions on U. The ij block of ΠP is $\pi_i R_{ij} C_{ij}$. Symmetry of ΠP requires the matrix equation $\pi_i R_{ij} C_{ij} = \pi_j R_{ji} C_{ji}^T$ to be satisfied. Since C_{ij} (and hence C_{ji}) is doubly stochastic, summing across the first row yields $\pi_i R_{ij} = \pi_j R_{ji}$. But then we must also have $C_{ij} = C_{ji}^T$. Thus, when (I, L) is reversible, the type process I must be reversible and the motion of the velocity fields must satisfy $C_{ij} = C_{ji}^T$. In particular, when type i is followed by type i, the location must make a transition according to a *symmetric* circulant matrix C_{ii}. This prevents preferential drift in any direction for type i.

7.4 **Proposition.** *Suppose U is incompressible and that (I, L) is reversible. Then* $\mathrm{eig}_1(Q, \mathcal{H}) \leq \mathrm{eig}_1(P, \mathcal{H})$.

Proof: For vectors f and g in $\mathbb{C}^{mn}$, define an inner product by $\langle f, g \rangle_\Pi = f^* \Pi g$, where $*$ denotes conjugate transpose. The inner product induces a norm by $\|f\|_\Pi^2 = \langle f, f \rangle_\Pi =$

$\sum_{i=1}^{m} \pi_i \|f(i,\cdot)\|^2$. Because Σ is a permutation matrix with block diagonal form (4.2), it does not change the norms of the blocks $f(i,\cdot)$, and so $\|\Sigma f\|_\Pi = \|f\|_\Pi$ for all f in $\mathbb{C}^{mn}$.

Let $h \in \mathcal{H}$ be an eigenvector of Q with eigenvalue μ. Then

$$|\mu| \|h\|_\Pi = \|\mu h\|_\Pi = \|Qh\|_\Pi = \|\Sigma P h\|_\Pi = \|Ph\|_\Pi \tag{7.5}$$

by the preceding paragraph. An easy computation shows that P is self–adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_\Pi$. Also, by the proof of Proposition 5.2, P preserves the subspace $\mathcal{H}$, and so its restriction to $\mathcal{H}$ is self–adjoint. Thus, by the spectral theorem, there exists a basis for $\mathcal{H}$ consisting of eigenvectors of P, and these eigenvectors are orthonormal. A standard argument (cf. (6.3)) shows that $\|Ph\|_\Pi \le \mathrm{eig}_1(P, \mathcal{H}) \|h\|_\Pi$. Combining this with (7.5) yields $|\mu| \le \mathrm{eig}_1(P, \mathcal{H})$, which was to be shown. □

8 Molecular diffusion

In this section we see that molecular diffusivity directly reduces the bounds found in the previous sections. The analogue of motion with diffusion is

$$X_{t+1} = X_t + U_t(X_t) + \Delta_t, \qquad t = 0, 1, \ldots, \tag{8.1}$$

where $\Delta_0, \Delta_1, \ldots$ are independent, identically distributed random variables taking values in $\mathbb{Z}^d$, and independent of U. The common distribution of Δ_t is given by a function d_0 on $\mathbb{D}$. The strength of the diffusion can be controlled by concentrating d_0 around 0 or not.

Note that (8.1) is identical to the numerical implementation of the Euler scheme for simulating the solution of a stochastic differential equation with Brownian noise, except that we are working on a coarse lattice rather than a lattice of machine–precision numbers.

The diffusion does not affect the Eulerian velocity field U, but it does affect the generalized Lagrangian velocity: Equation (4.1) for the Lagrangian location parameter becomes

$$M_{t+1} = \sigma(I_t, M_t) + \Delta_t + A_t(I_t, I_{t+1}), \qquad t = 0, 1, \ldots. \tag{8.2}$$

It is easy to see that (I, M) is still Markov. Its transition matrix Q_D can be written as

$$Q_D = \Sigma D P, \tag{8.3}$$

where D is a block diagonal matrix with each diagonal block equal to the circulant matrix $D_0(y, z) = d_0(z - y)$. Note that in the degenerate case in which $\Delta_t = 0$ with probability 1, D reduces to the identity matrix and (8.3) reduces to $Q = \Sigma P$ as in Section 4.

The presence of diffusion modifies the absolute bound on eigenvalues for incompressible velocity fields in the following way. The product DP is a block matrix of the form $[R_{ij} D_0 C_{ij}]$, and so in Theorem 6.1 we may replace C_{ij} by $D_0 C_{ij}$. As D_0 and C_{ij} are both circulant, they have the same eigenvectors. Denoting the eigenvalues of D_0 and C_{ij} by $\delta^{(k)}$ and $\lambda_{ij}^{(k)}$, the eigenvalues of $D_0 C_{ij}$ are equal to $\delta^{(k)} \cdot \lambda_{ij}^{(k)}$, $k \in \mathbb{D}$. Then T can be

replaced by $(T_D)_{ij} = R_{ij} \max(|\delta^{(k)} \cdot \lambda_{ij}^{(k)}|,\ k \neq 0)$, and the bound of Theorem 6.1 reads $\mathrm{eig}_1(Q_D, \mathcal{H}) \leq \mathrm{eig}_1(T_D)$. In particular, we have $(T_D)_{ij} \leq R_{ij}\mathrm{eig}_2(D_0)\mathrm{eig}_2(C_{ij})$, from which,

$$\mathrm{eig}_1(Q_D, \mathcal{H}) \leq \mathrm{eig}_1(T_D) \leq \mathrm{eig}_2(D_0)\mathrm{eig}_1(T) \leq \mathrm{eig}_2(D_0). \tag{8.4}$$

The second inequality shows a reduction in the absolute bound on eigenvalues due to diffusion, and the third shows an absolute bound which depends *solely* on the molecular diffusion.

While the presence of diffusion lowers the eigenvalue bound $\mathrm{eig}_1(T_D)$, in specific instances it can be shown to *increase* $\mathrm{eig}_1(Q_D, \mathcal{H})$. That is, in some cases we may have $\mathrm{eig}_1(\Sigma P, \mathcal{H}) < \mathrm{eig}_1(\Sigma DP, \mathcal{H})$. In such cases, the addition of diffusion slows the convergence of the Lagrangian location parameter M to its equilibrium distribution.

8.5 **Example.** *Diffusion slows convergence.* Consider again the velocity fields of Example 7.1. Let I make transitions according to $R = \left[\begin{smallmatrix} 1/2 & 1/2 \\ 2/3 & 1/3 \end{smallmatrix}\right]$. Let the first rows of C_{ij} be

$$\begin{aligned} c_{11} &= [0.262\ \ 0.143\ \ 0.037\ \ 0\ \ 0.098\ \ 0.162\ \ 0.298] \\ c_{12} &= [0.275\ \ 0.184\ \ 0.042\ \ 0\ \ 0.035\ \ 0.189\ \ 0.275] \\ c_{21} &= [0.099\ \ 0.211\ \ 0.104\ \ 0\ \ 0.106\ \ 0.132\ \ 0.348] \\ c_{22} &= [0.166\ \ 0.237\ \ 0.113\ \ 0\ \ 0.147\ \ 0.158\ \ 0.179], \end{aligned}$$

and let the distribution of diffusion be given by $d_0 = [0.5\ \ 0.25\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0.25]$. Then we find $\mathrm{eig}_1(\Sigma P, \mathcal{H}) = 0.1623$ and yet $\mathrm{eig}_1(\Sigma DP, \mathcal{H}) = 0.2201$, so that the addition of diffusion slows convergence to equilibrium. □

Finally, let us consider the results of Section 7 in the presence of diffusion. The case $C_{ij} = C$ is already closely connected with diffusion; we simply replace C by D_0C. The only change is that now Proposition 7.2 may conclude with the stronger inequality $\mathrm{eig}_1(Q_D, \mathcal{H}) \leq \mathrm{eig}_2(D_0)\mathrm{eig}_1(P, \mathcal{H})$ using (8.4). In the reversible case, the argument of Proposition 7.4 carries through once we note that $\|DPh\|_\Pi^2 = \sum_{i=1}^m \pi_i \|D_0(Ph)_i\|^2 \leq \mathrm{eig}_2(D_0)^2 \|Ph\|_\Pi^2$, using an argument similar to (6.3). Again we conclude that $\mathrm{eig}_1(Q_D, \mathcal{H}) \leq \mathrm{eig}_2(D_0)\mathrm{eig}_1(P, \mathcal{H})$.

9 Homogenization and effective diffusivity

Repeat the velocity field U periodically throughout $\mathbb{Z}^d$ and let Y move in this velocity field according to $Y_{t+1} = Y_t + U_t(Y_t \bmod \mathbb{D}) + \Delta_t,\ t = 0, 1, \ldots$, where the addition is no longer modulo $\mathbb{D}$. We will see that Y converges to Brownian motion upon rescaling space and time in the usual way, and we will compute the limiting diffusion coefficient exactly in terms of the law of the Eulerian velocity field and the distribution of the diffusion. The result holds for both compressible and incompressible cases.

If we set $Y_0 = 0$ and $X_t = Y_t \bmod \mathbb{D}$, then X and the type–location process (I, M) evolve as in Section 8. Then Y evolves according to,

$$Y_{t+1} = Y_t + u(I_t, -M_t) + \Delta_t, \qquad t = 0, 1, \ldots. \tag{9.1}$$

The process (I, M, Δ) is Markov and Y is an additive functional of it. As such, we expect that Y will converge to Brownian motion when properly scaled. More importantly, the limiting diffusivity can be expressed in terms of the transition matrix Q_D of (I, M) and the covariance of Δ, rather than using the much larger transition matrix of (I, M, Δ).

Let π be the invariant distribution corresponding to the transition matrix Q_D of (8.3). Let $\pi u = \sum_{i,m} \pi(i,m)u(i,-m)$ be the mean drift due to U and $\mathbb{E}\Delta = \sum_{a\in\mathbb{Z}^d} d_0(a)a$ be the mean drift due to the diffusion. For $t = 0, 1, \ldots$, let $Z_t = Y_t - (\pi u + \mathbb{E}\Delta)t$ and for non–integer t define Z_t by linear interpolation. Finally, define an inner product for functions from $\mathcal{I}\times\mathbb{D}$ to $\mathbb{R}$ by $\langle f, g\rangle_\pi = \sum_{i,m} \pi(i,m)f(i,m)g(i,m)$.

9.2 **Theorem.** Suppose that Q_D is irreducible and aperiodic. Then as $\varepsilon \to 0$, the processes $\varepsilon Z_{t/\varepsilon^2},\ t \ge 0$ converge in distribution to d–dimensional Brownian motion with zero drift and covariance matrix β given by

$$\beta^{k\ell} = -\langle f^k, f^\ell\rangle_\pi - \langle f^k, g^\ell\rangle_\pi - \langle g^k, f^\ell\rangle_\pi + \mathrm{Cov}(\Delta^k, \Delta^\ell) - \langle 1, \Sigma D_k P g^\ell + \Sigma D_\ell P g^k\rangle_\pi, \tag{9.3}$$

where $f : \mathcal{I}\times\mathbb{D} \to \mathbb{R}^d$ is given by $f(i,m) = u(i,-m) - \pi u$, g^k is the solution of

$$(I - Q_D)g^k = -f^k, \qquad k = 1, \ldots, d \tag{9.4}$$

and D_k is the matrix $D_k = \sum_{a\in\mathbb{Z}^d} d_0(a)(a^k - \mathbb{E}\Delta^k)\Sigma_a$. For each a in $\mathbb{Z}^d$, Σ_a is the transition matrix on $\mathcal{I}\times\mathbb{D}$ corresponding to the addition of a to the second component modulo $\mathbb{D}$.

9.5 **Remark.** The first three terms in (9.3) reflect the limiting covariance of $\sum_{s=0}^{t-1} u(I_s, -M_s)$, the fourth comes straight from molecular diffusion, and the last term reflects an interaction between diffusion and advection. In the absence of molecular diffusion, $\beta^{k\ell} = -\langle f^k, f^\ell\rangle_\pi - \langle f^k, g^\ell\rangle_\pi - \langle g^k, f^\ell\rangle_\pi$ and g^k satisfies $(I - Q)g^k = -f^k$, cf. (9.7). □

Proof: By (9.1), at integer times, $Z_t = \sum_{s=0}^{t-1} \hat{f}(I_s, M_s, \Delta_s)$, where $\hat{f}$ is defined by $\hat{f}(i,m,a) = f(i,m) + \mu(a)$, with $\mu(a) = a - \mathbb{E}\Delta$. The process (I, M, Δ) is a Markov chain with transition matrix $\hat{Q}$ given by

$$\hat{Q}(i,m,a;\ j,n,b) = R_{ij}\cdot C_{ij}(n - \sigma(i,m) - a)\cdot d_0(b). \tag{9.6}$$

This chain has state space $\mathcal{I}\times\mathbb{D}\times\mathrm{support}(d_0)$ and is irreducible aperiodic because Q_D is irreducible aperiodic and the value of Δ is chosen independently at each step. The invariant distribution $\hat{\pi}$ of (I, M, Δ) satisfies $\hat{\pi}(i,m,a) = \pi(i,m)d_0(a)$.

Note that the mean of $\hat{f}$ under $\hat{\pi}$ is zero. Thus, for each $k = 1, \ldots, d$, the equation $(I - \hat{Q})\hat{g}^k = -\hat{f}^k$ has a solution, as will be explained below. By Theorem VIII.3.74 of Jacod and Shiryaev (1987), the processes $\varepsilon Z_{t/\varepsilon^2},\ t \ge 0$ converge in distribution as $\varepsilon \to 0$ to a Brownian motion with zero drift and covariance matrix β given by

$$\beta^{k\ell} = -\langle \hat{f}^k, \hat{f}^\ell\rangle_{\hat{\pi}} - \langle \hat{f}^k, \hat{g}^\ell\rangle_{\hat{\pi}} - \langle \hat{g}^k, \hat{f}^\ell\rangle_{\hat{\pi}}, \qquad k, \ell = 1, \ldots, d. \tag{9.7}$$

The inner product here is analogous to $\langle\cdot,\cdot\rangle_\pi$ defined before the theorem. To be precise, Jacod and Shiryaev (1987) establish (9.7) for $k = \ell$, but it may be checked for $k \ne \ell$ by consideration of the additive functional based on $\hat{f}^k + \hat{f}^\ell$ and polarization.

We now compute $\hat{g}$ and β in terms of the smaller matrices P, Σ, Q_D, and Σ_a. The solution of $(I - \hat{Q})\hat{g}^k = -\hat{f}^k$ is $\hat{g}^k = -\sum_{n=0}^{\infty} \hat{Q}^n \hat{f}^k$, which converges because $\hat{\pi}\hat{f} = 0$ and $\hat{Q}$ is irreducible aperiodic. Now $\hat{Q}^n \mu = 0$ for $n = 1, 2, \ldots$, so we have $\hat{g}^k = -\mu^k - \sum_{n=0}^{\infty} \hat{Q}^n f^k$, where we think of f as a function of i, m, and a, although it does not really depend on a. The infinite sum defining $\hat{g}^k$ may be written

$$\sum_{n=0}^{\infty} (\hat{Q}^n f^k)(i,m,a) = f(i,m) + \sum_{n=1}^{\infty} \mathbb{E}[f^k(I_n, M_n) | I_0 = i, M_0 = m, \Delta_0 = a].$$

The influence of Δ_0 does not last long. It only affects the value of M_1, since Δ_1 is independent of (I_0, M_0, Δ_0). After time 1, (I, M) evolves exactly as in Section 8, with transition matrix Q_D, and we may ignore the value of Δ. For the first step, note that from (8.2), $M_1 = \sigma(i,m) + a + A_0(i, I_1)$, so that the first transition of (I, M) is according to the matrix $\Sigma\Sigma_a P$, where Σ_a corresponds to the deterministic addition of a modulo $\mathbb{D}$. Thus,

$$\mathbb{E}[f^k(I_n, M_n) | I_0 = i, M_0 = m, \Delta_0 = a] = (\Sigma\Sigma_a P Q_D^{n-1} f^k)(i,m),$$

where we have returned to regarding f as a function of i and m alone. By changing the index of summation, we obtain

$$\hat{g}^k(\cdot,\cdot,a) = -\mu^k - f^k - \Sigma\Sigma_a P \sum_{n=0}^{\infty} Q_D^n f^k = -\mu^k - f^k + \Sigma\Sigma_a P g^k, \tag{9.8}$$

where g^k satisfies (9.4). Finally, we simplify terms in (9.7). Because $\sum_{a \in \mathbb{Z}^d} d_0(a)\mu(a) = 0$,

$$\langle \hat{f}^k, \hat{f}^\ell \rangle_{\hat{\pi}} = \langle f^k, f^\ell \rangle_\pi + \mathrm{Cov}(\Delta^k, \Delta^l). \tag{9.9}$$

Next, from (9.8), $\langle \hat{f}^k, \hat{g}^\ell \rangle_{\hat{\pi}}$ equals

$$\begin{aligned}
&\sum_{i,m,a} \pi(i,m) d_0(a) (f^k(i,m) + \mu^k(a))(-\mu^\ell(a) - f^\ell(i,m) + (\Sigma\Sigma_a P g^\ell)(i,m)) \\
&\qquad = -\langle f^k, f^\ell \rangle_\pi + \sum_{i,m} \pi(i,m) f^k(i,m) (Q_D g^\ell)(i,m) \\
&\qquad\quad -\mathrm{Cov}(\Delta^k, \Delta^l) + \sum_{i,m} \pi(i,m) (\Sigma D_k P g^\ell)(i,m) \\
&\qquad = \langle f^k, g^\ell \rangle_\pi - \mathrm{Cov}(\Delta^k, \Delta^l) + \langle 1, \Sigma D_k P g^\ell \rangle_\pi,
\end{aligned}$$

using $Q_D g^\ell = f^\ell + g^\ell$ from (9.4). Combining this and (9.9) into (9.7) yields (9.3). □

9.10 **Example.** *Numerical examples of effective diffusivity.* Consider a velocity field having two vortex types, $u(1,\cdot)$ and $u(2,\cdot)$. Switch types according to $R = \left[\begin{smallmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{smallmatrix}\right]$, so that typically each vortex type is used for a few steps, then the other, and back again. Define the matrix C_{11} so that vortex 1 moves up and to the right, specifically, with probabilities 0.3 (right one), 0.3 (up one), 0.2 (up one and right one), 0.2 (do not move). Define the matrix C_{22} so that vortex 2 moves down and to the right, specifically, with probabilities 0.3 (right one), 0.3 (down one), 0.2 (down one and right one), 0.2 (do not move).

The second largest eigenvalue modulus of the Eulerian velocity does not depend on the vortex types used. The value is 0.9051 for this Eulerian velocity field.

To generate several examples, $u(1,\cdot)$ and $u(2,\cdot)$ will be chosen from among the four vortex types shown in Figure **??**. The first letter stands for Incompressible or Compressible, the second for Clockwise or Anticlockwise.

Letting $u(1,\cdot) = \texttt{IC}$ and $u(2,\cdot) = \texttt{IA}$, the Lagrangian velocity has second largest eigenvalue modulus 0.8532, indicating that it converges to equilibrium more quickly than the Eulerian velocity field. This will be the case for all the examples here. The Lagrangian drift πu is zero, while the effective diffusivity matrix is $\beta = \begin{bmatrix} 0.1470 & 0.0000 \\ 0.0000 & 0.1550 \end{bmatrix}$. This is the smallest effective diffusivity among these five examples.

Letting $u(1,\cdot) = \texttt{IC}$ and $u(2,\cdot) = \texttt{IC}$, the Lagrangian velocity has second largest eigenvalue modulus 0.8197. The Lagrangian drift πu is zero, while the effective diffusivity matrix is $\beta = \begin{bmatrix} 0.3038 & 0.0000 \\ 0.0000 & 0.2081 \end{bmatrix}$. Here, the vortex has momentum either up and to the right, or down and to the left, because it continues several steps in one direction before switching to the other direction. This roughly doubles the diffusivity.

Letting $u(1,\cdot) = \texttt{IC}$ and $u(2,\cdot) = \texttt{CC}$, so one of the vortex types is compressible, the Lagrangian velocity has second largest eigenvalue modulus 0.8197. The Lagrangian drift πu is $[0.0769 \quad -0.0006]$, while the effective diffusivity matrix is $\beta = \begin{bmatrix} 0.4586 & -0.1455 \\ -0.1455 & 0.3581 \end{bmatrix}$. Compressible fields may have non–zero drift. Note that the diffusivity has increased compared to the incompressible examples.

Letting $u(1,\cdot) = \texttt{CC}$ and $u(2,\cdot) = \texttt{CA}$, the Lagrangian velocity has second largest eigenvalue modulus 0.8524. The Lagrangian drift πu is $[0.1880 \quad 0.0000]$, while the effective diffusivity matrix is $\beta = \begin{bmatrix} 0.6818 & 0.0000 \\ 0.0000 & 0.6067 \end{bmatrix}$. Letting $u(1,\cdot) = \texttt{CA}$ and $u(2,\cdot) = \texttt{CA}$, the Lagrangian velocity has second largest eigenvalue modulus 0.8558. The Lagrangian drift πu is $[0.2109 \quad -0.0169]$, while the effective diffusivity matrix is $\beta = \begin{bmatrix} 0.6871 & 0.0595 \\ 0.0595 & 0.5622 \end{bmatrix}$. Using two compressible vortex types gives the largest effective diffusivity. □

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