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VALUE MONOIDS OF ZERO-DIMENSIONAL VALUATIONS OF RANK ONE

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Abstract. Classically, Gröbner bases are computed by first prescribing a set monomial order. Moss Sweedler suggested an alternative and developed a framework to perform such computations by using valuation rings in place of monomial orders. We build on these ideas by providing a class of valuations on $k(x, y)$ that are suitable for this framework. For these valuations, we compute $\nu(k[x, y]^*)$ and use this to perform computations concerning ideals in the polynomial ring $k[x, y]$. Interestingly, for these valuations, some ideals have a finite Gröbner basis with respect to the valuation that is not a Gröbner basis with respect to any monomial order, whereas other ideals only have Gröbner bases that are infinite with respect to the valuation.

1. Introduction

Unless stated otherwise, k will denote an arbitrary field, and $\mathbb{N}$ will denote the set of nonnegative integers. Whenever R is a ring or monoid, we denote by R^* the nonzero elements of R.

One of the fundamental ideas of the theory of Gröbner bases is that monomial orders are well-orderings on the set of monomials, which leads us to a natural reduction process using multivariate polynomial division. In this section, we provide a brief account of a generalized theory of Gröbner bases that uses valuations in place of monomial orders, which will yield a more general reduction process. The development of this theory can be found in the unpublished manuscript [Sw] of Sweedler, and it is briefly discussed in this section solely for the sake of completeness. In that manuscript, Sweedler develops the theory in terms of valuation rings. Here we present the same results in terms of valuations rather than valuation rings. Proofs are omitted since they can all be found in [Sw].

Suppose k is a subfield of a field F. A **valuation on** F is a homomorphism ν from the additive group of nonzero elements of F to an ordered group (called the **value group**) such that for $f, g \in F^*$ where $f + g \neq 0$, $\nu(f + g) \leq \max\{\nu(f), \nu(g)\}$. Note that the triangle inequality was chosen to be opposite of the most common definition, which is so that our results most closely coincide with those concerning monomial orders. For more details, see [MoSw1], [MoSw2], and [M]. A **valuation on** F **over** k is a valuation on F such that its restriction to k^* is the zero map. For our purposes, we restrict our attention to valuations on rational function fields. In this setting, we require that our valuations have the additional properties given in the following definition.

Definition 1.1. We say that a valuation ν on $k(\mathbf{x})$ over k is **suitable relative to** $k[\mathbf{x}]$ if satisfies the following three properties.

(i) For all $f \in k[\mathbf{x}]$, $\nu(f) = 0$ iff $f \in k$.

(ii) If $\nu(f) = \nu(g)$ where $f, g \in k(\mathbf{x})^*$, then $\exists! \lambda \in k^*$ such that $f = \lambda g$ or $\nu(f - \lambda g) < \nu(f)$.

(iii) $\nu(k[\mathbf{x}]^*)$ is a well-ordered monoid.

When using monomial orders, one must determine divisibility among monomials. The analogue for valuations uses arithmetic in the monoid $\nu(k[\mathbf{x}]^*)$.

Definition 1.2. Let ν be a valuation on $k(\mathbf{x})$. Given $f, g \in k[\mathbf{x}]$, we say that $\nu(g)$ **divides** $\nu(f)$, denoted $\nu(g) \mid \nu(f)$, if there exists $h \in k[\mathbf{x}]$ such that $\nu(f) = \nu(gh)$. We say that h is an **approximate quotient** of f by g (relative to ν), if $f = gh$, or if $f \neq gh$ and $\nu(f - gh) < \nu(f)$.

The following simple proposition follows from the definition above.

Proposition 1.3. *Let ν be a valuation on $k(\mathbf{x})$ over k that is suitable relative to $k[\mathbf{x}]$. Let $f, g \in k[\mathbf{x}]$. Then $\nu(g)$ divides $\nu(f)$ if and only if there exists an approximate quotient h of f by g.*

The following is a generalized form of the standard polynomial reduction algorithm that makes use of valuations.

Algorithm 1.4. Let ν be a valuation on $k(\mathbf{x})$ over k that is suitable relative to $k[\mathbf{x}]$. Let ν be a valuation on $k(\mathbf{x})$ over k. Let I be an ideal in $k[\mathbf{x}]$ and G be a generating set for I. The following algorithm computes a reduction of a polynomial $a \in k[\mathbf{x}]$ over G relative to ν.

- Set $i = 0$ and $f_0 = f$.
- While $f_i \neq 0$ and $\nu(g) \mid \nu(f_i)$ for some $g \in G$ do:
  Choose $g_i \in G$ such that $\nu(g_i) \mid \nu(f_i)$. Let h_i be an an approximate quotient of f_i by g_i. Set $f_{i+1} = f_i - g_i h_i$. Increment i by 1.

We say that f_n is the nth reductum of f over G. We say that f reduces to b if b is a reductum of f. It can be shown that if ν is suitable with respect to $k[\mathbf{x}]$, then reduction of any element of $k[\mathbf{x}]$ over G terminates after a finite number of steps. We will call a subset $G \subset I^*$ a **Gröbner basis for I with respect to ν** if it satisfies the equivalent conditions of the following proposition.

Proposition 1.5. *Let ν be a valuation on $k(\mathbf{x})$ over k that is suitable relative to $k[\mathbf{x}]$. Let I be an ideal in $k[\mathbf{x}]$ and $G \subseteq I^*$. The following are equivalent:*

(i) *Every nonzero element of I has a first reductum over G.*

(ii) *Every element of I reduces to 0 over G.*

(iii) *Given $f \in k[\mathbf{x}]$, $f \in I$ if and only if f reduces to 0 over G.*

We can use Gröbner bases in the generalized setting to solve the ideal membership problem in much the same way that we do in the case of monomial orders. Just as in the classical case, it can be shown that a Gröbner basis with respect to a valuation necessarily generates the given ideal. To compute Gröbner bases, we must work with ideals of $\nu(k[\mathbf{x}]^*)$, where an ideal J of a commutative monoid M is a subset $J \subset M$ such that for any $m \in M, j \in j$, $j + m \in J$. The smallest ideal containing $m_1, \ldots, m_\ell$ will be denoted $\langle m_1, \ldots, m_\ell \rangle$ and is called the ideal generated by $m_1, \ldots, m_\ell$

Definition 1.6. Let ν be a valuation on $k(\mathbf{x})$ over k that is suitable relative to $k[\mathbf{x}]$. We say that $T \subseteq \nu(k[\mathbf{x}]^*)$ is **an ideal generating set for f and g with respect to ν** if T generates the ideal $\langle \nu(f)\rangle \cap \langle \nu(g)\rangle$ in $\nu(k[\mathbf{x}]^*)$. It can be shown that for each $t \in T$ there are $a, b \in k[\mathbf{x}]^*$ such that $\nu(af) = \nu(bg) = t$ and $af = bg$ or $af \neq bg$ and $\nu(af - bg) < t$. This gives a map $T \to k[\mathbf{x}]$, $t \mapsto af - bg$. The image of this map is a **syzygy family for f and g indexed by T**. We say that $af - bg$ is the element of the family corresponding to t.

This definition shows one of the main differences between the generalized theory using valuations and the classical theory using monomial orders, namely, that each pair of polynomials may have many minimal syzygies. Sweedler constructs an example in [Sw] where this family consists of multiple elements. Using syzygy families, the algorithm below provides a method for constructing a Gröbner basis for an nonzero ideal I with generating set G.

Algorithm 1.7 (Gröbner Basis Construction Algorithm)**.** Let ν be a valuation on $k(\mathbf{x})$ over k that is suitable relative to $k[\mathbf{x}]$, and $G \subseteq I^*$ is a generating set for a nonzero ideal I.

(i) Set $G_0 = G$.
(ii) For each pair of distinct elements $g, h \in G$, find a monoid generating set $T^0_{g,h}$ for g, h and a syzygy family $S^0_{g,h}$ for g, h indexed by $T^0_{g,h}$. Define $U = \bigcup_{g \neq h \in G} S^0_{g,h}$.
(iii) Determine the set H_i of nonzero final reductums that occur from reducing the lements of U_i over G_i.
(iv) If H_i is empty, stop.
(v) Define $G_{i+1} = G_i \cup H_i$.
(vi) For each pair of distinct element $g \in G_{i+1}, h \in H_i$, find a monoid generating set $T^{i+1}_{g,h}$ for g, h and a syzygy family $S^{i+1}_{g,h}$ for g, h indexed by $T^{i+1}_{g,h}$. Define $U = \bigcup_{g \neq h \in G} S^{i+1}_{g,h}$.
(vii) Increment i by 1 and go to step (iii).

Sweedler shows that if G is finite and $\nu(I^*)$ is Noetherian (i.e., every ascending chain of ideals stabilizes), then the construction algorithm can be completed so that it terminates with a finite Gröbner basis. However, even if $\nu(I^*)$ isn't Noetherian, the set $\cup_{n=1}^{\infty} G_n$ is a Gröbner basis.

These algorithms will allow us to compute Gröbner bases using a class of valuations on $k(x, y)$ originally studied by Zariski in [Z]. In Section 2, we develop the background necessary to work with a valuation ν of this type, and we state one of the main results of the paper, which is an explicit formula for $\nu(k[x, y]^*)$. In Section 3, we prove some intermediate results concerning sequences associated with the valuations developed in Section 2. In particular, recursive formulas are given for a generating set of $\nu(k[x, y]^*)$. In Section 4, we build on these ideas to show that certain elements of $\nu(k[x, y]^*)$ have unique representations, which leads to a complete description of $\nu(k[x, y]^*)$ in Section 5. Finally, in Section 6, we use this description to make the algorithms developed by Sweedler constructive. With the exception of Section 4, all of the proofs herein are fairly elementary.

2. Value Groups and Monoids from Power Series

In this section, we examine a class of valuations of $k(x, y)$ studied by Zariski in [Z]. The value groups of these valuations were explicitly constructed by MacLane and Schilling in [MacSch]. In this section, we state one of our main results, which is an explicit construction of the restriction of such valuations to the underlying polynomial ring $k[x, y]$. Since the valuations of interest are constructed using generalized power series, we begin with a review of the relevant concepts.

We say that a set $T \subset \mathbb{Q}$ is **Noetherian** if every subset of T has a largest element. Given a function $z : \mathbb{Q} \to k$, the **support** of z is defined by $\mathrm{Supp}(z) = \{q \in \mathbb{Q} \mid z(q) \neq 0\}$. The collection of **Noetherian power series**, denoted by $k\langle\langle t^{\mathbb{Q}}\rangle\rangle$, consists of all functions from $\mathbb{Q}$ to k with Noetherian support. More commonly in the literature, generalized power series are defined as functions with well-ordered support, and we will freely use the analogues of these results for Noetherian power series. We choose the supports of our series to be opposite of the usual definition so that our results more closely fit with the theory of monomial orders and Gröbner bases.

As demonstrated in [H], the collection of Noetherian power series forms a field in which addition is defined pointwise and multiplication is defined via convolution; i.e., if $z_1, z_2 \in k\langle\langle t^{\mathbb{Q}}\rangle\rangle$ and $q \in \mathbb{Q}$, then $(z_1 + z_2)(q) = z_1(q) + z_2(q)$ and $(z_1 z_2)(q) = \sum_{u+v=q} z_1(u)z_2(v)$. We often write power series as formal sums: $z = \sum_{s \in \mathrm{Supp}(z)} z(s)t^s$, where $z(s)$ denotes the image of s under z.

Example 2.1. Given the series $z_1 = t^{1/2} + t^{1/4} + t^{1/8} + \cdots$ and $z_2 = 3t + 1$, their sum and product are

$$z_1 + z_2 = 3t + (t^{1/2} + t^{1/4} + t^{1/8} + \cdots) + 1$$

and

$$z_1 z_2 = (3t^{3/2} + 3t^{5/4} + 3t^{9/8} + \cdots) + (t^{1/2} + t^{1/4} + t^{1/8} + \cdots).$$

Given a series $z \in k\langle\langle t^{\mathbb{Q}}\rangle\rangle$, define the **leading exponent** of z to be the rational number given by $\mathcal{LE}(z) = \max\{s \mid s \in \mathrm{Supp}(z)\}$. If $s = \mathcal{LE}(z)$, we denote $z(s)$ by $\mathcal{LC}(z)$ and call it the **leading coefficient** of z. Note that $\mathcal{LE}(z_1 z_2) = \mathcal{LE}(z_1) + \mathcal{LE}(z_2)$ and $\mathcal{LC}(z_1 z_2) = \mathcal{LC}(z_1)\mathcal{LC}(z_2)$. Moreover, we have $\mathcal{LE}(z_1 + z_2) \leq \max(\mathcal{LE}(z_1), \mathcal{LE}(z_2))$, with equality holding in case $\mathcal{LE}(z_1) \neq \mathcal{LE}(z_2)$.

We say that a nonzero series $z \in k\langle\langle t^{\mathbb{Q}}\rangle\rangle$ is **simple** if it can be written in the form

$$z = \sum_{i=1}^{n} c_i t^{e_i},$$

where $c_i \in k^*, n \in \mathbb{N}^* \cup \{\infty\}, e_i \in \mathbb{Q}, e_i > e_{i+1}$. Whenever we write a series in this form, we implicitly assume that each c_i is nonzero and the exponents are written in descending order. We call $\mathbf{e} = (e_1, e_2, \dots)$ the **exponent sequence** of z. Now write $e_i = n_i/d_i$ where $d_i > 0$ and $\gcd(n_i, d_i) = 1$. We define $r_0 = 1$ and for $i \geq 1$, set $r_i = \mathrm{lcm}(d_1, \dots, d_i)$ and call $\mathbf{r} = (r_0, r_1, r_2, \dots)$ the **ramification sequence** of z.

Example 2.2. Consider the simple series

$$z = 2t^{1/2} + 3t^{1/3} + 4t^{1/4} + 5t^{1/5} + \cdots.$$

Here $\mathcal{LE}(z) = 1/2$ and $\mathcal{LC}(z) = 2$. The series z has exponent sequence $(1/2, 1/3, 1/4, 1/5, \cdots)$ and ramification sequence $(1, 2, 6, 12, 60, \cdots)$.

We are now in a position to define valuations on $k(x,y)$ based on Noetherian power series. Let $z \in k\langle\langle t^{\mathbb{Q}}\rangle\rangle$ be a Noetherian power series such that t and z are algebraically independent over k. Consider the embedding $\varphi_z : k(x,y) \to k\langle\langle t^{\mathbb{Q}}\rangle\rangle$, $x \mapsto t$, $y \mapsto z$. It can be shown that $\mathcal{L}E$ is a valuation on $k\langle\langle t^{\mathbb{Q}}\rangle\rangle$, and hence the composite map $\mathcal{L}E \circ \varphi_z : k(x,y) \to \mathbb{Q}$ is a valuation on $k(x,y)$. Given a valuation ν on $k(\mathbf{x})$, $V = \{f \in k(\mathbf{x})^* \mid \nu(f) \leq 0\}$ is a valuation ring with maximal ideal $\mathfrak{m} = \{f \in k(\mathbf{x})^* \mid \nu(f) < 0\}$, in which case $\dim_k(V/\mathfrak{m})$ is the **dimension** of the valuation. The **rank** of the valuation ν is defined to be the number of isolated subgroups of $\nu(k(\mathbf{x})^*)$. It follows that $\mathcal{L}E \circ \varphi_z$ is a zero-dimensional valuation of rank one.

Example 2.3. Let k be a field such that char $k \neq 2$. Given $z = t^{1/2} + t^{1/4} + t^{1/8} + \cdots$,

$$\begin{aligned}(\mathcal{L}E \circ \varphi_z)(x) &= \mathcal{L}E(t) = 1\\ (\mathcal{L}E \circ \varphi_z)(y) &= \mathcal{L}E(z) = 1/2\\ (\mathcal{L}E \circ \varphi_z)(y^2 - x) &= \mathcal{L}E(z^2 - t) = \mathcal{L}E(((t + 2t^{3/4} + 2t^{5/8} + \cdots) - t) = 3/4\end{aligned}$$

MacLane and Schilling proved the following result in [MacSch]:

Theorem 2.4. *Let $z \in k\langle\langle t^{\mathbb{Q}}\rangle\rangle$ be a simple series such that t and z are algebraically independent over k. If $\mathbf{e}$ is the exponent sequence of z, then the value group of $\mathcal{L}E \circ \varphi_z$ is*

$$(\mathcal{L}E \circ \varphi_z)(k(x,y)^*) = \mathbb{Z} + \mathbb{Z}e_1 + \mathbb{Z}e_2 + \cdots$$

One of the primary goals of this paper is to restrict the valuation to the polynomial ring $k[x,y]$ and compute

$$\Lambda = (\mathcal{L}E \circ \varphi_z)(k[x,y]^*) = \{\mathcal{L}E(f(t,z)) \mid f(x,y) \in k(x,y)^*\}, \tag{2.1}$$

which we call the **value monoid with respect to** z.

Now suppose z is a simple series with exponent sequence $\mathbf{e}$ and ramification sequence $\mathbf{r}$. The sequence obtained from the ramification sequence $\{r_i\}_{i\in\mathbb{N}}$ by removing repetitions is called the **reduced ramification sequence** and is denoted $\{r_i^{red}\}_{i\in\mathbb{N}}$. For each $i \in \mathbb{N}$, denote by $l(i)$ the smallest natural number such that $r_i^{red} = r_{l(i)}$; i.e.,

$$l(i) = \min\{j \in \mathbb{N} \mid r_j = r_i^{red}\}. \tag{2.2}$$

Example 2.5. The series

$$z = t^2 + t^{3/2} + t^{1/2} + t^{1/3} + t^{1/5} + t^{1/7} + t^{1/11} + \cdots$$

has ramification sequence

$$\mathbf{r} = (1, 2, 2, 6, 30, 210, 2310, \dots),$$

and hence has reduced ramification sequence

$$(1, 2, 6, 30, 210, 2310, \dots).$$

Thus $l(0) = 0$, $l(1) = 1$, $l(i) = i + 1$ for $i \geq 2$.

We define the **bounding sequence** $\mathbf{u} = (u_0, u_1, u_2, \dots)$ given by $u_0 = 0$, and for $i \geq 1$,

$$u_i = \sum_{j=0}^{i-1} \Big(\frac{r_i}{r_j} - \frac{r_i}{r_{j+1}}\Big) e_{j+1}. \tag{2.3}$$

For $i \geq 1$, we define the **monoid generating sequence**:

$$\rho_i = u_{l(i)-1} + e_{l(i)}. \tag{2.4}$$

We can fully describe the value monoid with respect to z in terms of the monoid generating sequence. The following result will be proved in Section 5 (in fact, it follows directly from the stronger result given in Theorem 5.8).

Theorem 2.6. *Let $z \in k\langle\langle t^{\mathbb{Q}} \rangle\rangle$ be a simple series such that t and z are algebraically independent over k. Assume further that the components of the exponent sequence are positive and no component is divisible by the characteristic of k. Then the value monoid with respect to z is*

$$\Lambda = (\mathcal{L}E \circ \varphi_z)(k[x, y]^*) = \mathbb{N} + \mathbb{N}\rho_1 + \mathbb{N}\rho_2 + \cdots$$

It is of interest to determine whether this result can be generalized. In particular, it would be nice to compute the value monoid after either removing the restriction that the exponent sequence must be positive or permitting some of the components of the exponent sequence to be divisible by the characteristic of the the ground field.

3. Associated Sequences

In this section, we prove some elementary results about the sequences described in the previous section. In particular, we will construct recurrence relations and formulas concerning the monoid generating sequence. To this end, there is one more sequence that will be needed in the sequel. Using the ramification sequence $\mathbf{r}$ of a simple series z and the formula (2.2), we define **partial ramification sequence** by

$$s_i = r_{l(i)}/r_{l(i-1)} = r_{l(i)}/r_{l(i)-1}.$$

Convention 3.1. For the remainder of this paper, we adopt the following conventions.

- The series $z \in k\langle\langle t^{\mathbb{Q}} \rangle\rangle$ is simple with positive support.
- The series z is transcendental over $k(t)$.
- The value monoid of z is denoted Λ.
- The exponent sequence of z is denoted $\mathbf{e} = (e_1, e_2, e_3, \dots)$.
- No component of the exponent sequence is divisible by char k.
- The ramification sequence of z is denoted $\mathbf{r} = (r_0, r_1, r_2, \dots)$.
- The bounding sequence of z is denoted $\mathbf{u} = (u_0, u_1, u_2, \dots)$.
- The function $l(i)$ is defined in (2.2).
- The monoid generating sequence of z is denoted $\rho = (\rho_1, \rho_2, \rho_3, \dots)$.
- The partial ramification sequence of z is given by $\mathbf{s} = (s_1, s_2, s_3, \dots)$.

Since $l(i)$ marks the index where the ramification index increases, we have $r_j = r_{l(i)}$ for $l(i) \leq j < l(i+1)$, and so

$$r_j/r_{j-1} = 1 \quad \text{for} \quad l(i) < j < l(i+1). \tag{3.1}$$

In particular, this yields

$$r_{l(i-1)} = r_{l(i)-1} \tag{3.2}$$

and

$$u_{l(i-1)} = u_{l(i)-1} \tag{3.3}$$

despite the fact that $e_{l(i-1)}$ and $e_{l(i)-1}$ need not be the same.

Note that the ramification sequence of a series $z \in k\langle\langle t^{\mathbb{Q}}\rangle\rangle$ increases without bound unless $z \in k((t^{1/n}))$ for some $n \in \mathbb{N}$. However, it is still possible that the ramification sequence occasionally (even infinitely many times) stabilizes for a finite number of steps. Whenever the ramification sequence stabilizes for a number of indices, the sequence $\{u_i\}_{i\in\mathbb{N}}$ also stabilizes, as seen in the next result.

Lemma 3.2. *If $r_i = r_k$ for indices i and k, then $u_i = u_k$.*

Proof. The result is trivial if $i = k$, so we assume $i < k$. Since $r_i = r_k$, it follows that $r_j = r_{j+1}$ for $i \le j \le k-1$, and so by (2.3), $u_k = \sum_{j=0}^{k-1} \left(\frac{r_k}{r_j} - \frac{r_k}{r_{j+1}}\right) e_{j+1} = \sum_{j=0}^{k-1} \left(\frac{r_i}{r_j} - \frac{r_i}{r_{j+1}}\right) e_{j+1} = u_i + \sum_{j=i}^{k-1} \left(\frac{r_i}{r_j} - \frac{r_i}{r_{j+1}}\right) e_{j+1} = u_i$. □

Since our main objective is to prove that Λ is generated by the sequence $1, \rho_1, \rho_2, \ldots$, we must first justify some elementary properties that allow us to understand better the behavior of this sequence. We begin by showing that the monoid generating sequence satisfies a simple recursive relation.

Lemma 3.3. *The monoid generating sequence in (2.4) satisfies the following recurrence relation:*

$$\begin{aligned} \rho_1 &= e_{l(1)}; \\ \rho_{i+1} &= s_i \rho_i - e_{l(i)} + e_{l(i+1)}. \end{aligned}$$

Proof. By (3.3), $u_{l(1)-1} = u_{l(1-1)} = u_0 = 0$, and so by (2.4), $\rho_1 = e_{l(1)}$. Also, we have by (2.3),

$$\begin{aligned} u_m + e_{m+1} &= \sum_{j=0}^{m-1} \left(\frac{r_m}{r_j} - \frac{r_m}{r_{j+1}}\right) e_{j+1} + e_{m+1} \\ &= \left(\frac{r_m}{r_{m-1}}\right) \sum_{j=0}^{m-2} \left(\frac{r_{m-1}}{r_j} - \frac{r_{m-1}}{r_{j+1}}\right) e_{j+1} + \left(\frac{r_m}{r_{m-1}} - \frac{r_m}{r_m}\right) e_m + e_{m+1} \\ &= \left(\frac{r_m}{r_{m-1}}\right) \sum_{j=0}^{m-2} \left(\frac{r_{m-1}}{r_j} - \frac{r_{m-1}}{r_{j+1}}\right) e_{j+1} + \left(\frac{r_m}{r_{m-1}}\right) e_m - e_m + e_{m+1} \\ &= \left(\frac{r_m}{r_{m-1}}\right) [u_{m-1} + e_m] - e_m + e_{m+1}, \end{aligned}$$

and so

$$\gamma_{m+1} = \left(\frac{r_m}{r_{m-1}}\right) \gamma_m - e_m + e_{m+1} \tag{3.4}$$

where $\gamma_m := u_{m-1} + e_m$. Replacing m by $l(i)$, we obtain

$$\gamma_{l(i)+1} = \left(\frac{r_{l(i)}}{r_{l(i)-1}}\right)\gamma_{l(i)} - e_{l(i)} + e_{l(i)+1} = s_i\gamma_{l(i)} - e_{l(i)} + e_{l(i)+1}. \tag{3.5}$$

If $l(i) < m < l(i+1)$, then $r_m/r_{m-1} = 1$ by (3.1), and so (3.4) yields

$$\gamma_{m+1} = \gamma_m - e_m + e_{m+1}.$$

Multiple applications of this formula yields a telescoping sum, and so

$$\begin{aligned}
\gamma_{l(i+1)} &= \gamma_{l(i+1)-1} - e_{l(i+1)-1} + e_{l(i+1)} \\
&= (\gamma_{l(i+1)-2} - e_{l(i+1)-2} + e_{l(i+1)-1}) - e_{l(i+1)-1} + e_{l(i+1)} \\
&= \gamma_{l(i+1)-2} - e_{l(i+1)-2} + e_{l(i+1)} \\
&\ \ \vdots \\
&= \gamma_{l(i)+1} - e_{l(i)+1} + e_{l(i+1)}.
\end{aligned}$$

This equation in conjunction with (3.5) yields

$$\begin{aligned}
\gamma_{l(i+1)} &= \gamma_{l(i)+1} - e_{l(i)+1} + e_{l(i+1)} \\
&= s_i\gamma_{l(i)} - e_{l(i)} + e_{l(i)+1} - e_{l(i)+1} + e_{l(i+1)} \\
&= s_i\gamma_{l(i)} - e_{l(i)} + e_{l(i+1)},
\end{aligned}$$

and since $\rho_i = u_{l(i)-1} + e_{l(i)} = \gamma_{l(i)}$ for all i, we have

$$\rho_{i+1} = s_i\rho_i - e_{l(i)} + e_{l(i+1)}.$$

□

We can also construct a recursive formula for the terms of the ramification sequence, as given in the next result.

Lemma 3.4. *For $i \in \mathbb{N}$,*

$$r_{l(i)} = 1 + \sum_{j=1}^{i}(s_j - 1)r_{l(j-1)}.$$

Proof. This follows from the simple computation

$$\sum_{j=1}^{i}(s_j - 1)r_{l(j-1)} = \sum_{j=1}^{i}((r_{l(j)}/r_{l(j-1)}) - 1)r_{l(j-1)} = \sum_{j=1}^{i}(r_{l(j)} - r_{l(j-1)}) = r_{l(i)} - r_{l(0)} = r_{l(i)} - 1.$$

For the case $i = 0$, we take the summation $\sum_{j=1}^{i}(s_j - 1)r_{l(j-1)}$ to be 0. □

Using Lemma 3.3, we can construct yet another recurrence relation for the terms of the monoid generating sequence.

Lemma 3.5. *For $i \geq 1$,*

$$\rho_i = \sum_{j=i}^{i-1}(s_j - 1)\rho_j + e_{l(i)}.$$

Proof. We proceed by induction. If $i = 1$, then by Lemma 3.3,

$$\rho_1 = e_{l(1)} = 0 + e_{l(1)} = \sum_{j=1}^{0}(s_j - 1)\rho_j + e_{l(1)}$$

since the summation that appears is empty. Now suppose the statement holds for the index i. Then by Lemma 3.3 and the induction hypothesis,

$$\begin{aligned} \rho_{i+1} - \sum_{j=1}^{i}(s_j - 1)\rho_j &= \rho_{i+1} - (s_i - 1)\rho_i - \sum_{j=1}^{i-1}(s_j - 1)\rho_j \\ &= \rho_{i+1} - (s_i - 1)\rho_i - (\rho_i - e_{l(i)}) \\ &= s_i\rho_i - e_{l(i)} + e_{l(i+1)} - (s_i - 1)\rho_i - (\rho_i - e_{l(i)}) \\ &= e_{l(i+1)}. \end{aligned}$$

□

Using this lemma, we can extract information about the denominators of the components of the monoid generating sequence, as shown in the next three results. Given $q \in \mathbb{Q}$, $q\mathbb{Z}$ denotes the set $\{qz \mid z \in \mathbb{Z}\}$.

Lemma 3.6. *For $i \geq 1$, $\rho_i \in (1/r_{l(i)})\mathbb{Z} - (1/r_{l(i-1)})\mathbb{Z}$.*

Proof. The result follows by a simple induction. Indeed, $\rho_1 = e_{l(1)} \in (1/r_{l(1)})\mathbb{Z} - \mathbb{Z} = (1/r_{l(1)})\mathbb{Z} - (1/r_{l(0)})\mathbb{Z}$. Now, assuming that $\rho_i \in (1/r_{l(i)})\mathbb{Z}$, we see by Lemma 3.5, $\rho_{i+1} = \sum_{j=1}^{i}(s_j - 1)\rho_j + e_{l(i+1)}$. Since $\rho_j \in (1/r_{l(j)})\mathbb{Z} \subset (1/r_{l(i)})\mathbb{Z}$ for $1 \leq j \leq i$, we have $\sum_{j=1}^{i}(s_j - 1)\rho_j \in (1/r_{l(i)})\mathbb{Z}$. Moreover, $e_{l(i+1)} \in (1/r_{l(i+1)})\mathbb{Z} - (1/r_{l(i)})\mathbb{Z}$, and so $\rho_{i+1} \in (1/r_{l(i+1)})\mathbb{Z} - (1/r_{l(i)})\mathbb{Z}$. □

Lemma 3.7. *If we write $\rho_i = c_i/r_{l(i)}$, then $gcd(c_i, s_i) = 1$.*

Proof. Rewrite the expression $\rho_i = c_i/r_{l(i)}$ in lowest terms: $\rho_i = \alpha_i/\beta_i$, $\alpha_i, \beta_i \in \mathbb{N}^*$ where $\gcd(\alpha_i, \beta_i) = 1$. Then $c_i = \alpha_i r_{l(i)}/\beta_i = \alpha_i \mathrm{lcm}(r_{l(i-1)}, \beta_i)/\beta_i = \alpha_i r_{l(i-1)}/\gcd(r_{l(i-1)}, \beta_i)$. Also $r_{l(i)}/r_{l(i-1)} = \mathrm{lcm}(r_{l(i-1)}, \beta_i)/r_{l(i-1)} = \beta_i/\gcd(r_{l(i-1)}, \beta_i)$. Therefore,

$$\gcd(c_i, r_{l(i)}/r_{l(i-1)}) = \gcd(\alpha_i r_{l(i-1)}/\gcd(r_{l(i-1)}, \beta_i), \beta_i/\gcd(r_{l(i-1)}, \beta_i)).$$

Since $\gcd(\alpha_i, \beta_i) = 1$ and $\gcd(r_{l(i-1)}/\gcd(r_{l(i-1)}, \beta_i), \beta_i/\gcd(r_{l(i-1)}, \beta_i)) = 1$, we have $\gcd(c_i, s_i) = \gcd(c_i, r_{l(i)}/r_{l(i-1)}) = 1$. □

Lemma 3.8. *If $0 \leq d_j < s_j$ for $1 \leq j \leq i$ and $d_i \neq 0$, then*

$$\sum_{j=1}^{i} d_j\rho_j \in (1/r_{l(i)})\mathbb{Z} - (1/r_{l(i-1)})\mathbb{Z}. \tag{3.6}$$

Proof. For $j \leq i$, we have by Lemma 3.6, $\rho_j \in (1/r_{l(j)})\mathbb{Z} \subset (1/r_{l(i)})\mathbb{Z}$, and so $\sum_{j=1}^{i} d_j\rho_j \in (1/r_{l(i)})\mathbb{Z}$. We now must prove $\sum_{j=1}^{i} d_j\rho_j \notin (1/r_{l(i-1)})\mathbb{Z}$ by induction.

First, we show that $d_j\rho_j \notin (1/r_{l(j-1)})\mathbb{Z}$ whenever $0 < d_j < s_j$. Write $\rho_j = c_j/r_{l(j)}$. Suppose, for contradiction, $d_j\rho_j = (d_jc_j)/r_{l(j)} \in (1/r_{l(j-1)})\mathbb{Z}$ where $0 < d_j < s_j$. Thus,

$r_{l(j)} \mid d_j c_j r_{l(j-1)}$. Now, $s_j = r_{l(j)}/r_{l(j-1)}$, and so $s_j \mid d_j c_j$. By Lemma 3.7, $\gcd(c_j, s_j) = 1$, and so $s_j \mid d_j$. Since $0 < d_j < s_j$, we have a contradiction.

Now we proceed to show the inductive step. Suppose $0 \le d_j < s_j$ for $1 \le j \le i+1$ and $d_{i+1} \ne 0$. We write

$$\sum_{j=1}^{i+1} d_j \rho_j = \left(\sum_{j=1}^{i} d_j \rho_j \right) + d_{i+1}\rho_{i+1}.$$

By the induction hypothesis, $\sum_{j=1}^{i} d_j \rho_j \in (1/r_{l(i)})\mathbb{Z}$. Now, $d_{i+1}\rho_{i+1} \in (1/r_{l(i+1)})\mathbb{Z}$, and by the previous paragraph, $d_{i+1}\rho_{i+1} \notin (1/r_{l(i)})\mathbb{Z}$. Thus $\sum_{j=1}^{i+1} d_j \rho_j \in (1/r_{l(i+1)})\mathbb{Z} - (1/r_{l(i)})\mathbb{Z}$. □

4. Representations of Elements of the Value Monoid

In this section, we demonstrate that certain elements of Λ have a unique representation as a sum of elements of $\{1, \rho_1, \rho_2, \dots\}$. Using these representations, we prove that Λ is generated by $\{1, \rho_1, \rho_2, \rho_3, \dots\}$. To accomplish this, we must factor each element of $k[t, y]$ completely as $f(t, y) = q(t) \prod (y - s_i)$ where s_i lies in the algebraic closure of $k(t)$.

An element of $k\langle\langle t^{\mathbb{Q}} \rangle\rangle$ is said to be **Puiseux** if it lies in $k((t^{-1/m}))$ for some positive integer m. Puiseux's Theorem states that the algebraic closure of the field of Laurent series $k((t^{-1}))$ in $k\langle\langle t^{\mathbb{Q}} \rangle\rangle$ precisely consists of all elements of $k\langle\langle t^{\mathbb{Q}} \rangle\rangle$ that are Puiseux. Using Kedlaya's characterization of the generalized power series that are algebraic over the Laurent power series field when k has positive characteristic in [Ke], we have the following characteristic-free generalization of Puiseux's Theorem.

Theorem 4.1. *Let $w \in k\langle\langle t^{\mathbb{Q}} \rangle\rangle$ such that no element of its support is divisible by char k. Then w is algebraic over $k((t^{-1}))$ iff w is Puiseux.*

The result below follows directly from techniques found in [Ab] and [D].

Proposition 4.2. *Let $w = c_1 t^{m_1/n} + \cdots + c_s t^{m_s/n}$ be a finite Puiseux expansion with ramification index n where $m_i \in \mathbb{Z}^*$, $n \in \mathbb{Z}^+$, and $c_i \in k^*$. If k has positive characteristic, then assume that n is not divisible by char k. Then the minimal polynomial of w over $k(t)$ is $p(y) = \prod_{i=0}^{n-1} (y - w_i) \in k(t)[y]$, where*

$$w_i = c_1(\zeta^i t^{1/n})^{m_1} + \cdots + c_s(\zeta^i t^{1/n})^{m_s},$$

and ζ is a primitive nth root of unity in $\overline{k}$.

The **ramification index** of a Puiseux series $w \in k\langle\langle t^{\mathbb{Q}} \rangle\rangle$ is the smallest positive integer r such that $w \in k((t^{-1/r}))$. Given $z_1, z_2 \in k\langle\langle t^{\mathbb{Q}} \rangle\rangle$, we say that z_1 and z_2 **agree to (finite) order** $m \in \mathbb{N}$ if the first m terms of z_1 and z_2 are identical, but the $(m+1)$st terms (if they exist) of z_1 and z_2 are different. If we use Theorem 4.1 in place of Puiseux's Theorem, then Proposition 4.6 of [M] can be strengthened to the following characteristic-free form, where we continue the assumption that no component of the exponent sequence is divisible by char k as stated in Convention 3.1.

Proposition 4.3. *Let w be a Puiseux series in $k\langle\langle t^{\mathbb{Q}} \rangle\rangle$. Define $p(y) \in k((t^{-1}))[y]$ to be the minimal polynomial of w over $k((t^{-1}))$ where w agrees with z to order m, and none of the* *conjugates of w agree with z to a greater order. If R is the ramification index of w, then*

$$\mathcal{L}E(p(z)) = \left(\frac{R}{r_m}\right)\Big[u_m + \mathcal{L}E(z-w)\Big] \geq \left(\frac{R}{r_m}\right)\Big[u_m + e_{m+1}\Big]. \tag{4.1}$$

The simplest polynomials to which we can apply this result are those whose roots are finite Puiseux series. We make these calculations explicit in the following lemma.

Lemma 4.4. *If $g(t,y) \in k(t)[y]$ is the minimal polynomial of*

$$c_1 t^{e_1} + \cdots + c_{l(i)-1} t^{e_{l(i)-1}}$$

over $k(t)$, then $\deg_y(g(t,y)) = r_{l(i)-1}$ and $\mathcal{L}E(g(t,z)) = \rho_i$.

Proof. Let $g(t,y) \in k((t^{-1}))[y]$ be the minimal polynomial of $\sum_{j=1}^{l(i)-1} c_j t^{e_j}$ over $k((t^{-1}))$. Since the exponent sequence $\mathbf{e}$ consists solely of positive numbers, $g(t,y) \in k[t,y]$ by Proposition 4.2. Since $\sum_{j=1}^{l(i)-1} c_j t^{e_j}$ has ramification index $r_{l(i)-1}$, it follows from Proposition 4.2 that $\deg_y g(t,y) = r_{l(i)-1}$. Moreover, by Proposition 4.3, $\mathcal{L}E(g(t,z)) = \left(\frac{r_{l(i)-1}}{r_{l(i)-1}}\right)(u_{l(i)-1} + e_{l(i)}) = \rho_i$. □

We will see that in order to generate Λ, we need only consider images of polynomials whose roots are finite Puiseux series. To demonstrate this, we first show that over the collection of polynomials of a fixed degree in y, the polynomials that have the smallest image under $\mathcal{L}E \circ \phi_z$ are those whose roots are finite Puiseux series.

Proposition 4.5. *Let k be a perfect field. For each nonzero $p(x,y) \in k[x,y]$, there exists $h(x,y) \in k[x,y]$ such that the following hold:*

(i) $\deg_y p(x,y) = \deg_y h(x,y)$,
(ii) $\mathcal{L}E(p(t,z)) \geq \mathcal{L}E(h(t,z))$,
(iii) *the roots of $h(t,y)$ in $\overline{k((t^{-1}))}[y]$ are finite Puiseux series of the form $\sum_{j=1}^{l(i)-1} c_j t^{e_j}$.*

Proof. First, factor $p(t,y)$ as a polynomial in y as $p(t,y) = q(t)\prod_{i=1}^m p_i(t,y)$, where $q(t) \in k[t]$ and $p_i(t,y)$ is a monic, irreducible element of $k((t^{-1}))[y]$. We will find $h_i(x,y) \in k[x,y]$ such that $\deg_y p_i(x,y) = \deg_y h_i(x,y)$, $\mathcal{L}E(p_i(t,z)) = \mathcal{L}E(h_i(t,z))$, and the roots of $h_i(t,y)$ are finite Puiseux series of the desired form. It then follows that $h(x,y) = q(x)\prod_{i=1}^m h_i(x,y)$ satisfies the conditions of the proposition.

Since $p_i(t,y)$ is a monic, irreducible element of $k((t^{-1}))[y]$, it is the minimal polynomial of some generalized power series $\beta \in k\langle\langle t^{\mathbb{Q}}\rangle\rangle$. If k is a field of characteristic zero, by Puiseux's Theorem (Theorem 4.1), β is Puiseux. If k has positive characteristic, β is not necessarily Puiseux and the algebraic closure of $k((t^{-1}))$ is described by Kedlaya in [Ke]. We prove the result by considering two cases:

Case 1: No element of $\mathrm{Supp}(z)$ is divisible by char k.
Case 2: Some element of $\mathrm{Supp}(z)$ is divisible by char k.

Case 1: Without loss of generality, we assume that no conjugate of β agrees with z to a higher order. We denote this order by m, and denote the ramification index of β by R, in which case $r_m \mid R$. As shown in [St], $p_i(t,y) \in k((t^{-1}))[y]$ must be a polynomial of degree R.

Let L be the largest index such that $r_L = r_m$, in which case $r_{L+1} > r_L$, and so $L+1$ is of the form $l(\kappa)$ for some $\kappa \in \mathbb{N}$. Let $g(t,y) \in k[t,y]$ be the minimal polynomial of $\sum_{j=1}^{l(\kappa)-1} c_j t^{e_i}$ over $k(t)$. Then by Lemma 4.4, $\deg_y(g(t,y)) = r_{l(\kappa)-1} = r_L = r_m$ and $\mathcal{LE}(g(t,z)) = \rho_\kappa$. Therefore, if we define $h(x,y) = g(x,y)^{(R/r_m)}$, then $\mathcal{LE}(h(t,z)) = (R/r_m)\rho_\kappa$ and $\deg_y(h(x,y)) = (R/r_m)\deg_y(g) = R = \deg_y(p_i(x,y))$.

Since $r_L = r_m$, we know by Lemma 3.2 that $u_L = u_m$. Moreover, $L \geq m$, and so $e_{m+1} \geq e_{L+1}$. Thus by Proposition 4.3, $\mathcal{LE}(p_i(t,z)) \geq (R/r_m)[u_m + e_{m+1}] \geq (R/r_m)[u_L + e_{L+1}] = (R/r_m)[u_{l(\kappa)-1} + e_{l(\kappa)}] = (R/r_m)\rho_\kappa = \mathcal{LE}(h(t,z))$.

Case 2 : Let char $k = p$. Let E be the normal closure of $k((t^{-1}))(\beta)/k((t^{-1}))$. As in the proof of Corollary 9 of [Ke], if M is the integral closure of k in E, then E can be expressed as a tower of Artin-Schreier extensions over $M((t^{-1/mq}))$, where q is the degree of inseparability of $E/k((t^{-1}))$. Since E is normal over $k((t^{-1}))$, and hence over $k((t^{-1/mq}))$, the normal closure of $k((t^{-1}))$ must be contained in E. The field $k(\zeta_m)((t^{-1/mq})) = k(\zeta_m, t^{-1/qm})((t^{-1}))$ is the normal closure of $k((t^{-1/mq}))$ (it is the splitting field of $X^{mq} - t^{-1}$ over $k((t^{-1}))$), and so we have the following normal extensions:

$$k((t^{-1})) \subset k(\zeta_m)((t^{-1/mq})) \subset E.$$

Define $F = k(\zeta_m)((t^{-1/mq}))$, and let $\tau_\ell \in \mathrm{Gal}(F/k((t^{-1})))$ be given by $t^{1/qm} \mapsto \zeta_m^{q\ell} t^{1/qm}$. Note that as $\zeta_m^0, \zeta_m, \ldots, \zeta_m^{m-1}$ runs through all the mth roots of unity, so does the list $\zeta_m^0, \zeta_m^q, \ldots, \zeta_m^{(m-1)q}$ since $\gcd(m,q) = 1$. Each element of $\mathrm{Gal}(F/k((t^{-1})))$ can be written as $\tau_\ell \mu$ where $\mu \in \mathrm{Gal}(k(\zeta_m)/k)$. We write the collection of all elements of $\mathrm{Gal}(F/k((t^{-1})))$ as $\{\psi_1, \ldots, \psi_b\}$.

Define a homomorphism $\lambda_\ell : \mathbb{Q} \to \overline{k}^*$ by $\lambda_\ell(ap^n/b) = \zeta_b^{a\ell s}$ where $a \in \mathbb{Z}$, $b \in \mathbb{N}^*$, $p \nmid ab$ and $s \equiv p^n \mod b$ (or, if $n < 0$, we require $sp^{-n} \equiv 1 \mod b$). It is straightforward to show that if $\lambda : \mathbb{Q} \to \overline{k}^*$ is a homomorphism whose kernel contains $\mathbb{Z}$ and $\mu \in \mathrm{Gal}(k(\zeta_m)/k)$, then

$$\sum_{i \in I} x_i t^i \mapsto \sum_{i \in I} \lambda(i)\mu(x_i) t^i \tag{4.2}$$

is a $\overline{k}((t^{-1}))$-automorphism of $\overline{k}\langle\langle t^{\mathbb{Q}} \rangle\rangle$ (where I is any Noetherian subset of $\mathbb{Q}$). Given $\psi_j \in \mathrm{Gal}(F/k((t^{-1})))$, we write $\psi_j = \tau_\ell \mu$ for some $1 \leq \ell \leq m$ and $\mu \in \mathrm{Gal}(k(\zeta_m)/k)$. In case $\lambda = \lambda_\ell$, note that the function in (4.2) is an extension of ψ_j to $\overline{k}\langle\langle t^{\mathbb{Q}} \rangle\rangle$. We denote the restriction of this function to E by ϕ_j. We will show that ϕ_j sends $\overline{k((t^{-1}))}$ to itself, and since E is a normal extension of $k((t^{-1}))$, it follows that $\phi_j \in \mathrm{Gal}(E/k((t^{-1})))$ is an extension of ψ_j.

To show that ϕ_j sends $\overline{k((t^{-1}))}$ to itself, we appeal to Kedlaya's description of the algebraic closure in Corollary 9 of [Ke]. First, we review a few key ideas from that paper. The support of any algebraic series must be a set of the form

$$S_{m,v,c} = \{(1/m)(w + b_1 p^{-1} + \cdots + b_{j-1} p^{-j+1} + p^{-n}(b_j p^{-j} + \cdots)) \mid w \leq v, \sum b_i \leq c\}$$

where $m \in \mathbb{N}, v, c \geq 0$. Note that $S_{a,b,c}$ is defined differently than the form given by Kedlaya since our support is Noetherian rather than well-ordered. We say that a sequence c_n satisfies a linearized recurrence relation (LRR) if for some $d_0, \dots, d_k$, for all $n \in \mathbb{N}$,

$$d_0 c_n + d_1 c_{n+1}^p + \cdots + d_k c_{n+k}^{p^k} = 0.$$

Let $\sum x_i t^i$ be a series with support $S_{m,v,c}$. We say $\sum x_i t^i$ is **twist-recurrent** if for each $w \le v$, $\sum b_i \le c$, the sequence $c_n = x_{(1/m)(w+b_1p^{-1}+\cdots+b_{j-1}p^{-j+1}+p^{-n}(b_jp^{-j}+\cdots))}$ satisfies an LRR. According to [Ke], the algebraic closure of $k((t^{-1}))$ consists of all twist-recurrent series $x = \sum x_i t^i$ such that the x_i lie in a finite extension of k.

Now suppose $\sum x_i t^i$ is a twist-recurrent series. We will show that $\phi_j\left(\sum x_i t^i\right)$ is also twist-recurrent, and so by the previous paragraph, ϕ_j sends $\overline{k((t^{-1}))}$ to itself. Since $\sum x_i t^i$ is twist-recurrent, it follows that $c_n = x_{(1/m)(w+b_1p^{-1}+\cdots+b_{j-1}p^{-j+1}+p^{-n}(b_jp^{-j}+\cdots))}$ satisfies an LRR of the form $d_0 c_n + d_1 c_{n+1}^p + \cdots + d_k c_{n+k}^{p^k} = 0$. To show that $\phi_j\left(\sum x_i t^i\right)$ is twist-recurrent, we must prove that $\lambda(f(n))\mu(c_n)$ satisfies an LRR where $f(n) = (1/m)(w + b_1p^{-1} + \cdots + b_{j-1}p^{-j+1} + p^{-n}(b_jp^{-j} + \cdots))$, $\lambda = \lambda_\ell$ for some ℓ and $\mu \in \mathrm{Gal}(k(\zeta_m)/k)$. If c_n satisfies the LRR $\sum_{i=0}^k d_i c_{n+i}^{p^i} = 0$, it follows that $0 = \mu\left(\sum_{i=0}^k d_i c_{n+i}^{p^i}\right) = \sum_{i=0}^k \mu(d_i)\mu(c_{n+i})^{p^i}$, and so $\mu(c_n)$ satisfies an LRR. Thus we only have to show that if c_n satisfies an LRR, then so does $c'_n = \lambda_\ell(f(n))c_n$.

Now suppose c_n satisfies the LRR $\sum_{i=0}^k d_i c_{n+i}^{p^i} = 0$. Rewrite $w + b_1p^{-1} + \cdots + b_{j-1}p^{-j+1}$ as $\frac{\alpha_1}{p^{m_1}}$ where $p \nmid \alpha_1$ and $m_1 \le j-1$. If we rewrite $b_jp^{-j} + b_{j+1}p^{-j-1}\cdots$ as $\frac{\alpha_2}{p^{m_2}}$ where $p \nmid \alpha_2$ and $m_2 \ge j$, then

$$f(n) = \frac{\alpha_1 p^{m_2+n} + \alpha_2 p^{m_1}}{mp^{m_1+m_2+n}} = \frac{\alpha_1 p^{m_2-m_1+n} + \alpha_2}{mp^{m_2+n}}.$$

If we define s_n, d_1, d_2 so that $s_n p^n \equiv 1 \mod m$, $d_1 p^{m_1} \equiv 1 \mod m$, and $d_2 p^{m_2} \equiv 1 \mod m$, then $\lambda_\ell(f(n)) = \zeta_m^{(\alpha_1 p^{m_2-m_1+n}+\alpha_2)s_n d_2} = \zeta_m^{\alpha_1 d_1} \cdot \zeta_m^{\alpha_2 d_2 s_n}$, and so if we define $d'_i = \zeta_m^{-\alpha_1 d_1 p^i} d_i$, then

$$\sum_{i=0}^k d'_i c'^{p^i}_{n+i} = \sum_{i=0}^k (\zeta_m^{-\alpha_1 d_1 p^i}) d_i (\zeta_m^{\alpha_1 d_1} \cdot \zeta_m^{\alpha_2 d_2 s_{n+i}})^{p^i} c_{n+i}^{p^i} = \sum_{i=0}^k (\zeta_m^{-\alpha_1 d_1 p^i}) d_i (\zeta_m^{\alpha_1 d_1 p^i})(\zeta_m^{\alpha_2 d_2 s_{n+i}})^{p^i} c_{n+i}^{p^i},$$

which simplifies as

$$\sum_{i=0}^k (\zeta_m^{\alpha_2 d_2 s_{n+i}})^{p^i} d_i c_{n+i}^{p^i} = \sum_{i=0}^k (\zeta_m^{\alpha_2 d_2 s_n s_i})^{p^i} d_i c_{n+i}^{p^i} = (\zeta_m^{\alpha_2 d_2 s_n}) \sum_{i=0}^k d_i c_{n+i}^{p^i} = 0,$$

and so c'_n satisfies an LRR.

So far, we have shown that ϕ_j sends $\overline{k((t^{-1}))}$ to itself, and since E is a normal extension of $k((t^{-1}))$, we know $\phi_j \in \mathrm{Gal}(E/k((t^{-1})))$ is an extension of ψ_j. Let $\{\sigma_1, \dots, \sigma_d\}$ be the complete collection of F-automorphisms of E. Since E/F and $F/k((t^{-1}))$ are normal extensions, a routine exercise shows that the collection $\{\phi_i \sigma_j \mid 1 \le i \le b, 1 \le j \le d\}$ consists of all $k((t^{-1}))$-automorphisms of E. Since q is the degree of inseparability of E over $k((t^{-1}))$, the minimal polynomial m_β of β over $k((t^{-1}))$ can be factored as

$$m_\beta(t, y) = \prod_{i=1}^d \left(\prod_{j=1}^b (y - \phi_j \sigma_i \beta) \right)^q.$$

For any series $s = \sum_{i \in I} c_i t^{e_i}$, we define an associated Puiseux series by $\mathcal{P}(s) = \sum_{i \in J} c_i t^{e_i}$ where $J = \{a/b \in I \mid a \in \mathbb{Z}, b \in \mathbb{N}^* \text{ and } p \nmid b\}$ and remainder by $\mathcal{R}(s) = s - \mathcal{P}(s)$. Since no component of the ramification sequence of z is divisible by p, we obtain

$$\mathcal{L}E(z - \phi_j \sigma_i \beta) = \mathcal{L}E(z - \mathcal{P}(\phi_j \sigma_i \beta) - \mathcal{R}(\phi_j \sigma_i \beta)) \geq \mathcal{L}E(z - \mathcal{P}(\phi_j \sigma_i \beta)). \tag{4.3}$$

Since ϕ_j is of the form (4.2), for any series $s \in k\langle\langle t^{\mathbb{Q}}\rangle\rangle$, $\mathcal{P}(\phi_j s) = \phi_j(\mathcal{P}(s))$. Applying this to (4.3), we obtain

$$\mathcal{L}E(z - \phi_j \sigma_i \beta) \geq \mathcal{L}E(z - \phi_j \mathcal{P}(\sigma_i \beta)).$$

Of all the conjugates $\phi_j(\mathcal{P}(\sigma_i\beta))$ of $\mathcal{P}(\sigma_i\beta)$ over F, choose α_i to be the one that agrees with z to the highest order. Note that $\prod_{j=1}^{b}(y - \phi_j \alpha_i)$ must be of the form $m_{\alpha_i}(t, y)^{\ell_i}$ where $m_{\alpha_i}(t, y)$ is the minimal polynomial of α_i over $k((t^{-1}))$ and $\ell_i \in \mathbb{N}$. Since α_i is a Puiseux series such that no element of its support is divisible by p, we have reduced the problem to Case 1, and the proof is complete. $\square$

Now, we define a sequence of rational numbers that give the minimal possible value of an image of a polynomial of degree d under the map $\mathcal{L}E \circ \varphi_z$.

Definition 4.6. For each natural number d,

$$\lambda_d := \min\{\mathcal{L}E(f(t, z)) \mid f \in k[x, y]^* \text{ and } \deg_y(f(x, y)) = d\}.$$

Lemma 4.7. *Let k be a perfect field. For any positive integer d,*

$$\lambda_d = \mathcal{L}E\left(\prod_{j=1}^{w} f_j(t, z)^{d_j}\right) \tag{4.4}$$

where w is a positive integer, the exponent d_j is nonnegative, and f_j is the minimal polynomial of $\sum_{i=1}^{l(j)-1} c_i t^{e_i}$ over $k(t)$. Moreover, $d = \sum d_j \deg_y(f_j(x, y))$.

Proof. By the definition of λ_d, there exists $p(x, y) \in k[x, y]$ such that $\deg_y(p(x, y)) = d$ and $\mathcal{L}E(p(t, z)) = \lambda_d$. By Proposition 4.5, there exists $h(x, y)$ such that $\lambda_d = \mathcal{L}E(p(t, z)) \geq \mathcal{L}E(h(t, z))$, $\deg_y(h(x, y)) = d$, and $h(t, y)$ has finite Puiseux series as roots. Thus, by the definition of λ_d, $\lambda_d = \mathcal{L}E(h(t, z))$. Since $h(x, y)$ is a product of minimal polynomials of finite Puiseux series, we can write h as $h(t, z) = \prod_{j=1}^{w} f_j(t, z)^{d_j}$, where w is a positive integer, and for each $1 \leq j \leq w$, the exponent d_j is nonnegative, and f_j is the minimal polynomial of $\sum_{i=1}^{l(j)-1} c_i t^{e_i}$ over $k(t)$. $\square$

Using this lemma, we can produce a unique representation for each λ_d in terms of the monoid generating sequence.

Proposition 4.8. *Let k be a perfect field. For any positive integer d, λ_d can be uniquely expressed in the form*

$$\lambda_d = \sum_{j=1}^{w} d_j \rho_j, \tag{4.5}$$

where w is a positive integer, and for each $1 \leq j \leq w$, we have

$$0 \leq d_j < s_j. \tag{4.6}$$

In this case,

$$d = \sum_{j=1}^{d} d_j r_{l(j-1)}.$$

Proof. By Lemma 4.7, there exists $h(x, y) \in k[x, y]$ such that $\lambda_d = \mathcal{L}E(h(t, z))$, $\deg_y(h(x, y)) = d$, and

$$h(t, z) = \prod_{j=1}^{w} f_j(t, z)^{d_j},$$

where w is a positive integer, and for each $1 \leq j \leq w$, the exponent d_j is nonnegative, and f_j is the minimal polynomial of $\sum_{i=1}^{l(j)-1} c_i t^{e_i}$ over $k(t)$. By Lemma 4.4, $\deg_y f_j(x, y) = r_{l(j)-1}$ and $\mathcal{L}E(f_j(t, z)) = \rho_j$, and so

$$\lambda_d = \mathcal{L}E\left(\prod_{j=1}^{w} f_j(t, z)^{d_j}\right) = \sum_{i=1}^{w} d_j \mathcal{L}E(f_j(t, z)) = \sum_{i=1}^{w} d_j \rho_j$$

and

$$d = \deg_y h(x, y) = \sum_{j=1}^{w} d_j \deg_y f_j(x, y) = \sum_{j=1}^{w} d_j r_{l(j)-1} = \sum_{j=1}^{w} d_j r_{l(j-1)}.$$

Next we show that each d_j satisfies the bounds given by (4.6). Suppose for contradiction, for some k, $d_k \geq s_k = r_{l(k)}/r_{l(k-1)}$. Define

$$D_j = \begin{cases} d_j + 1 & \text{if } j = k+1; \\ d_j - s_j & \text{if } j = k; \\ d_j & \text{otherwise.} \end{cases}$$

Using this in conjunction with the recurrence relation given in Lemma 3.3, we obtain

$$\begin{aligned} \sum_{j=1}^{w} d_j \rho_j - \sum_{j=1}^{w} D_j \rho_j &= (d_k - D_k)\rho_k + (d_{k+1} - D_{k+1})\rho_{k+1} \\ &= s_k \rho_k - \rho_{k+1} \\ &= e_{l(k)} - e_{l(k+1)}, \end{aligned}$$

and so

$$\begin{aligned} \sum_{j=1}^{w} d_j r_{l(j-1)} - \sum_{j=1}^{w} D_j r_{l(j-1)} &= (d_k - D_k) r_{l(k-1)} + (d_{k+1} - D_{k+1}) r_{l(k)} \\ &= s_k r_{l(k-1)} - r_{l(k)} \\ &= 0. \end{aligned}$$

These equations in conjunction with Lemma 4.4 yield

$$\mathcal{L}E\left(\prod_{j=1}^{w} f_j(t, z)^{D_j}\right) = \sum_{j=1}^{w} D_j \rho_j = \sum_{j=1}^{w} d_j \rho_j - e_{l(k)} + e_{l(k+1)} < \sum_{j=1}^{w} d_j \rho_j = \mathcal{L}E(h)$$

and

$$\deg\left(\prod_{j=1}^{w} f_j(t,z)^{D_j}\right) = \sum_{j=1}^{w} D_j \deg(f_j) = \sum_{j=1}^{w} D_j r_{l(j-1)} = \sum_{j=1}^{w} d_j r_{l(j-1)} = \deg(h).$$

However, $\mathcal{L}E(h) = \lambda_d$, and so we have contradicted the minimality of $\mathcal{L}E(h)$. Thus $0 \leq d_j < s_j$ for each $1 \leq j \leq w$, and so we have proved the bounds given by (4.6).

Finally, we demonstrate that the expression for λ_d in (4.5) is uniquely determined. Suppose we are given two representations for λ_d:

$$\lambda_d = \sum_{j=1}^{w} d_j \rho_j = \sum_{j=1}^{w} d'_j \rho_j$$

where $0 \leq d_j, d'_j < s_j$. If we define $\Delta_j = d_j - d'_j$, then $\sum_{j=1}^{w} \Delta_j \rho_j = 0$ and $|\Delta_j| < s_j$. Multiply the expression by $r_{l(w-1)}$, and we see

$$\left(\sum_{j=1}^{w-1} r_{l(w-1)} \Delta_j \rho_j\right) + r_{l(w-1)} \Delta_w \rho_w = 0.$$

However, $r_{l(w-1)} \Delta_j \rho_j \in \mathbb{Z}$ for $j \leq w-1$, and so $r_{l(w-1)} \Delta_w \rho_w \in \mathbb{Z}$. Now write ρ_w as $c_w / r_{l(w)}$ where $c_w \in \mathbb{N}$. Then $r_{l(w-1)} \Delta_w c_w / r_{l(w)} \in \mathbb{Z}$, and so $s_w = \frac{r_{l(w)}}{r_{l(w-1)}} \mid \Delta_w c_w$. Since s_w and c_w are relatively prime by Lemma 3.7, $s_w \mid \Delta_w$. However, $|\Delta_w| < s_w$, and so $\Delta_w = 0$. Thus, $\sum_{j=1}^{w-1} \Delta_j \rho_j = 0$. Repeating this argument, we find $\Delta_{w-1} = \Delta_{w-2} = \cdots = \Delta_1 = 0$, and so $d_j = d'_j$ for all $1 \leq j \leq w$. □

The idea that each λ_d has a unique representation can be extended further. In fact, there is a natural bijective correspondence between representations of natural numbers and representations of terms of the form λ_d. First, we state the following simple lemma without proof.

Lemma 4.9. *Let $b_0, b_1, b_2, b_3, \ldots$ be a sequence of positive integers such that $b_0 = 1, b_{i+1} > b_i$ and $b_i \mid b_{i+1}$ for all i. Then every positive integer $n \in \mathbb{N}$ has a unique representation of the form*

$$d = \sum_{i=0}^{w} d_i b_i,$$

where w is a positive integer, $d_w \neq 0$, and $0 \leq d_i < b_{i+1}/b_i$.

For example, if $b_i = 10^i$, then this says that every positive integer has a unique base 10 representation. Using this lemma, we produce a method for quickly computing λ_d.

Proposition 4.10. *Let k be a perfect field. Given a positive integer w and $0 \leq d_j < s_j$ for each $1 \leq j \leq w$,*

$$d = \sum_{j=1}^{w} d_j r_{l(j-1)} \;\Leftrightarrow\; \lambda_d = \sum_{j=1}^{w} d_j \rho_j.$$

Proof. The reverse implication follows directly from Proposition 4.8. For the forward implication, suppose we are given $d = \sum_{j=1}^{w} d_j r_{l(j-1)}$ where $0 \le d_j < s_j$. By Proposition 4.8, λ_d is of the form $\lambda_d = \sum_{j=1}^{w'} d'_j \rho_j$ where $d = \sum_{j=1}^{w'} d'_j r_{l(j-1)}$. By the uniqueness promised by Lemma 4.9, $w = w'$ and $d_j = d'_j$ for all $1 \le j \le w$. Thus $\lambda_d = \sum_{j=1}^{w} d_j \rho_j$. □

5. Construction of the Value Monoid

The goal of this section is to describe the value monoid Λ explicitly in terms of the sequences $\{\lambda_i\}_{i\in\mathbb{N}}$ and $\{\rho_i\}_{i\in\mathbb{N}}$. Throughout the remainder, in addition to Convention 3.1, we assume that k is a perfect field and $\{\lambda_i\}_{i\in\mathbb{N}}$ is given by Definition 4.6. We begin by showing that $\{\lambda_i\}_{i\in\mathbb{N}}$ is an increasing sequence.

Lemma 5.1. *The sequence $\lambda_0, \lambda_1, \lambda_2, \dots$ is increasing.*

Proof. We will show that $\lambda_{d+1} > \lambda_d$ for all d. By Proposition 4.8, we can write $\lambda_d = \sum_{j=1}^{w} d_j \rho_j$ where $0 \le d_j < s_j$ and

$$d = \sum_{j=1}^{w} d_j r_{l(j-1)}.$$

We now consider different cases, depending on the size of the coefficients d_j.

Case 1: First we consider the case $d_j = s_j - 1$ for all j. Then $d = \sum_{j=1}^{w}(s_j-1)r_{l(j-1)}$, and so by Lemma 3.4, $d+1 = r_{l(w)}$. Thus by Proposition 4.10, $\lambda_{d+1} = \rho_{w+1}$ and $\lambda_d = \sum_{j=1}^{w} d_j\rho_j$, and so by Lemma 3.5, $\lambda_{d+1} - \lambda_d = \rho_{w+1} - \sum_{j=1}^{w}(s_j-1)\rho_j = e_{l(w+1)} > 0$.

Case 2: Consider the case $d_1 < s_1 - 1$. Now $d+1 = (d_1+1)r_{l(0)} + \sum_{j=2}^{w} d_j r_{l(j-1)}$, and so by Proposition 4.10, $\lambda_{d+1} = (d_1+1)\rho_1 + \sum_{j=2}^{w} d_j\rho_j$. Thus $\lambda_{d+1} - \lambda_d = (d_1+1)\rho_1 - d_1\rho_1 = \rho_1 > 0$.

Case 3: Finally we consider the case where there exists an index $v > 1$ such that $d_v < s_v - 1$ and for $j < v$, $d_j = s_j - 1$. Write λ_d as $\lambda_d = \sum_{j=1}^{v-1}(s_j-1)\rho_j + \sum_{j=v}^{w} d_j\rho_j$. By Proposition 4.10, $d = \sum_{j=1}^{v-1}(s_j-1)r_{l(j-1)} + \sum_{j=v}^{w} d_j r_{l(j-1)}$, and so by Lemma 3.4,

$$d+1 = 1+\sum_{j=1}^{v-1}(s_j-1)r_{l(j-1)}+\sum_{j=v}^{w} d_j r_{l(j-1)} = r_{l(v-1)}+\sum_{j=v}^{w} d_j r_{l(j-1)} = (d_v+1)r_{l(v-1)}+\sum_{j=v+1}^{w} d_j r_{l(j-1)}.$$

Therefore, by Proposition 4.10, $\lambda_{d+1} = (d_v+1)\rho_v + \sum_{j=v+1}^{w} d_j\rho_j$, and so $\lambda_{d+1} - \lambda_d = (d_v + 1)\rho_v + \sum_{j=v+1}^{w} d_j\rho_j - (\sum_{j=1}^{v-1}(s_j-1)\rho_j + \sum_{j=v}^{w} d_j\rho_j) = \rho_v - \sum_{j=1}^{v-1}(s_j-1)\rho_j$. By Lemma 3.5, this is simply $e_l(v)$, which is positive. □

Given a submonoid M of a commutative monoid N, we define an equivalence relation on N by setting $n_1 \sim_M n_2$ if and only if there exist $m_1, m_2 \in M$ such that $m_1 + n_1 = m_2 + n_2$. Denote by N/M the collection of all equivalence classes under this relation, and define a quotient map π from N to N/M that sends n to the equivalence class containing n. The set N/M has an additive monoid structure where we define $\pi(n_1) + \pi(n_2) = \pi(n_1 + n_2)$.

Given a polynomial $f(x,y) \in k[x,y]$, we define $\deg_y(f(x,y))$ to be the smallest $d \geq 0$ such that $f(x,y) \in k[x]y^d + k[x]y^{d-1} + \cdots + k[x]y + k[x]$, and we denote

$$\Lambda_d(z) = \{\mathcal{L}E(f(t,z)) \mid f \in k[x,y]^* \text{ and } \deg_y(f(x,y)) \leq d\}. \tag{5.1}$$

Using this notation, we show that any pair of terms of the sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ are inequivalent modulo $\mathbb{Z}$.

Proposition 5.2. *For all $i \neq k$, $\lambda_i \not\sim_{\mathbb{Z}} \lambda_k$.*

Proof. Suppose $\lambda_i \sim_{\mathbb{Z}} \lambda_k$. By Proposition 4.8, for some positive integer w we can write $\lambda_i = \sum_{j=1}^{w} d_j\rho_j$ and $\lambda_k = \sum_{j=1}^{w} d'_j\rho_j$ where $0 \leq d_j, d'_j < s_j$. For each $1 \leq j \leq w$, we write $\rho_j = c_j/r_{l(j)}$, where c_j and s_j are relatively prime, as promised by Lemma 3.7.

If we define $\Delta_j = d_j - d'_j$, then $|\Delta_j| < s_j = r_{l(j)}/r_{l(j-1)}$ and $\lambda_i - \lambda_k = \sum_{j=1}^{w} \Delta_j\rho_j \sim_{\mathbb{Z}} 0$. Multiply the expression by $r_{l(w-1)}$ to obtain

$$\left(\sum_{j=1}^{w-1} r_{l(w-1)}\Delta_j\rho_j\right) + r_{l(w-1)}\Delta_w\rho_w \sim_{\mathbb{Z}} 0. \tag{5.2}$$

However, $r_{l(w-1)}\Delta_j\rho_j \in \mathbb{Z}$ for $j \leq w-1$ since $\rho_j \in (1/r_{l(j)})\mathbb{Z}$, and so by (5.2), $r_{l(w-1)}\Delta_w c_w/r_{l(w)} = r_{l(w-1)}\Delta_w\rho_w \in \mathbb{Z}$. That is, $\Delta_w c_w/s_w = r_{l(w-1)}\Delta_w c_w/r_{l(w)} \in \mathbb{Z}$, and so $s_w \mid \Delta_w c_w$. Since s_w and c_w are relatively prime, $s_w \mid \Delta_w$. However, $|\Delta_w| < s_w$, and so $\Delta_w = 0$. Thus, $\sum_{j=1}^{w-1} \Delta_i\rho_j \sim_{\mathbb{Z}} 0$. Repeating this argument, we find $\Delta_{w-1} = \Delta_{w-2} = \cdots = \Delta_1 = 0$, and so $\lambda_i = \lambda_k$. By Lemma 5.1, $i = k$. □

We quote the following result from [MoSw2].

Theorem 5.3. *For every positive integer n, the quotient Λ_d/Λ_0 has cardinality one greater than that of Λ_{d-1}/Λ_0, or equivalently, Λ_d/Λ_0 has cardinality $d+1$.*

Using this theorem in conjunction with Proposition 5.2, we compute the quotient Λ_d/Λ_0.

Corollary 5.4. *The quotient Λ_d/Λ_0 consists precisely of the images of $\lambda_0, \ldots, \lambda_d$.*

Proof. Since $\lambda_0, \ldots, \lambda_d \in \Lambda_d$, we know by Proposition 5.2 that the images of $\lambda_0, \ldots, \lambda_d$ are distinct in Λ_d/Λ_0. By Theorem 5.3, these images constitute the entire quotient Λ_d/Λ_0. □

For each $m \in \Lambda$, we make the following definition:

$$\lambda(m) = \min\{r \in \Lambda \mid r \sim_{\mathbb{Z}} m\}. \tag{5.3}$$

The next two results allow us to relate terms of the sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ with elements in the image of the map $\lambda : \Lambda \to \Lambda$.

Proposition 5.5. *For all $i \in \mathbb{N}$, there exists $m \in \Lambda$ such that $\lambda_i = \lambda(m)$.*

Proof. We prove the following equivalent statement: for all $i \in \mathbb{N}, m \in \Lambda$, if $m \sim_{\mathbb{Z}} \lambda_i$, then $\lambda_i \leq m$. Let $i \in \mathbb{N}$, $m \in \Lambda$ such that $m \sim_{\mathbb{Z}} \lambda_i$. Let j be the smallest index such that $m \in \Lambda_j$. Suppose, for contradiction, $j < i$. Since the image of m must lie in the quotient Λ_j/Λ_0, by Corollary 5.4 it follows that $m \sim_{\mathbb{Z}} \lambda_t$ for some $t \leq j < i$. Thus, $\lambda_i \sim_{\mathbb{Z}} \lambda_t$, which contradicts Proposition 5.2. Therefore, $j \geq i$, and so by Lemma 5.1, $m \geq \lambda_j \geq \lambda_i$. □

Proposition 5.6. *For all $m \in \Lambda$, there exists $i \in \mathbb{N}$ such that $\lambda_i = \lambda(m)$.*

Proof. Let $m \in \Lambda$. Now $m \in \Lambda_j$ for some $j \in \mathbb{N}$, and so by Corollary 5.4, $m \sim_{\mathbb{Z}} \lambda_i$ for some $i \in \mathbb{N}$. By Proposition 5.5, $\lambda_i = \lambda(m')$ for some $m' \in \Lambda$. Thus $\lambda_i \sim_{\mathbb{Z}} m \sim_{\mathbb{Z}} m'$, and so $\lambda_i = \lambda(m') = \lambda(m)$. □

We are now in a position to decompose the value monoid as a disjoint union of cosets of $\mathbb{N}$.

Theorem 5.7. *If the exponent sequence of $z \in k\langle\langle t^{\mathbb{Q}}\rangle\rangle$ is strictly positive, then the value monoid is the disjoint union*

$$\Lambda = \bigcup_{d=0}^{\infty}(\mathbb{N} + \lambda_d).$$

Proof. Given $m \in \Lambda$, there exists an index d such that $\lambda_d = \lambda(m)$ by Proposition 5.6. Therefore, $m - \lambda_d \in \mathbb{N}$, and so $m \in \mathbb{N} + \lambda_d$. The reverse containment follows directly from the fact that $\lambda_d \in \Lambda$. The sets are disjoint due to Proposition 5.2. □

Combining Theorem 5.7 and Proposition 4.8, we obtain the following.

Theorem 5.8. *Each element $m \in \Lambda$ has a unique representation of the form*

$$m = n + \sum_{j=1}^{w} d_j \rho_j, \tag{5.4}$$

where $n \in \mathbb{N}$ and for each $1 \leq j \leq w$, $0 \leq d_j < s_j$.

A weaker form of this theorem was stated earlier as Theorem 2.6.

6. Algorithms

In this section, we develop algorithms to make computations involving the value monoid Λ. It was shown in [M] that Λ is well-ordered, and so $\mathcal{L}E \circ \varphi_z$ is suitable relative to $k[\mathbf{x}]$ as described in Definition 1.1, and we can use $\mathcal{L}E \circ \varphi_z$ in the algorithms described in Section 1. Throughout this section we refer to the composite maps $\mathcal{L}E \circ \varphi_z$ and $\mathcal{L}C \circ \varphi_z$ as $\mathcal{L}E_z$ and $\mathcal{L}C_z$, respectively.

To begin, given a rational number $m \in \mathbb{Q}$, we would like to decide whether $m \in \Lambda$, and in case it is, express it in terms of the generators $1, \rho_1, \rho_2, \dots$. To accomplish this, we first prove a lemma.

Definition 6.1. For each $i \in \mathbb{N}$, define

$$\Omega_i = \{n + \sum_{j=1}^{i} d_j \rho_j \;|, n \in \mathbb{N}, 0 \leq d_j < s_j\}.$$

Lemma 6.2.

$$\Lambda \cap \mathbb{Z} \cdot \{1, \rho_1, \rho_2, \rho_3, \dots, \rho_i\} = \Omega_i.$$

Proof. The containment '$\supset$' being obvious, we only consider the case '$\subset$'. Let $m \in \Lambda \cap \mathbb{Z} \cdot \{1, \rho_1, \rho_2, \rho_3, \ldots, \rho_i\}$. By Theorem 5.7, there is a unique pair $n, d \in \mathbb{N}$ such that $m = n + \lambda_d$. Thus $\lambda_d \in \mathbb{Z} \cdot \{1, \rho_1, \rho_2, \ldots, \rho_i\}$, and so by Lemma 3.6, $\lambda_d \in (1/r_{l(i)})\mathbb{Z}$.

By Theorem 5.8, there exists a smallest $k \in \mathbb{N}$ such that $\lambda_d \in \Omega_k$. Suppose, for contradiction, that $k > i$. Then by Lemma 3.8, $\lambda_d \in (1/r_{l(k)})\mathbb{Z} - (1/r_{l(k-1)})\mathbb{Z} \subset (1/r_{l(k)})\mathbb{Z} - (1/r_{l(i)})\mathbb{Z}$, which contradicts our assertion that $\lambda_d \in (1/r_{l(i)})\mathbb{Z}$. Therefore, $i = k$, and so $\lambda_d \in \mathbb{N} \cdot \{1, \rho_1, \ldots, \rho_i\}$. $\square$

We have the following corollary.

Corollary 6.3. *The set Ω_i is closed under addition.*

Given a positive rational number m, write m as a/b where a, b are relatively prime positive integers. If $m \in \mathbb{N}$, then it is automatically in Λ, and so we can assume that $b > 1$. Our goal is to decide using modular arithmetic whether it is possible that $m \in \Lambda$. First, find the smallest i such that $b \mid r_{l(i)}$. The set of all $\mathbb{Z}$-linear combinations of $1, \rho_1, \ldots, \rho_{i-1}$ is precisely the set $\frac{1}{r_{l(i-1)}}\mathbb{Z}$. Since b does not divide $r_{l(i-1)}$, it cannot possibly be an $\mathbb{N}$-linear combination of $1, \rho_1, \ldots, \rho_{i-1}$. Now suppose m is a $\mathbb{Z}$-linear combination of $1, \rho_1, \ldots, \rho_j$ where $j > i$. However, since $b \mid r_{l(i)}$, it follows that $m \in (1/r_{l(i)})\mathbb{Z} = \mathbb{Z} \cdot \{1, \rho_1, \ldots, \rho_i\}$. If $m \in \Lambda$, then by Lemma 6.2, there exist $n, d_1, \ldots, d_i \in \mathbb{N}$ such that

$$m = n + \sum_{j=1}^{i} d_j \rho_j$$

where $0 \le d_j < s_j$ for $1 \le j \le i$ and $d_i \ne 0$. From this discussion, we have the following algorithm.

Algorithm 6.4. Let m be a positive rational number. The following algorithm determines whether $m \in \Lambda$. If $m \in \Lambda$, then the algorithm produces a decomposition of m as a linear combination of $1, \rho_1, \ldots, \rho_i$. Set $\rho_i = c_i / r_{l(i)}$.

(1) Write m as a/b where a, b are relatively prime, positive integers.
(2) Define i to be the smallest index such that $b \mid r_{l(i)}$.
(3) Define $m^{(i)} = m$.
(4) Try to solve the congruence $c_i d_i \equiv r_{l(i)} \mod s_i$ for d_i where $0 \le d_i < s_i$. If there are no solutions, then $m \notin \Lambda$.
(5) For $j = i-1, i-2, \ldots, 1$, define $m^{(j)} = m^{(j+1)} - d_{j+1}\rho_{j+1}$ and try to solve the congruence $c_j d_j \equiv r_{l(j)} \mod s_j$ for d_j where $0 \le d_j < s_j$. If any of the congruences fail to yield a solution, then $m \notin \Lambda$.
(6) Define $n = m^{(1)} - d_1\rho_1$. Then $m = n + \sum_{j=1}^{i} d_j \rho_j$. If $n \notin \mathbb{N}$, then $m \notin \Lambda$. If $n \in \Lambda$, then we have a decomposition of the desired form.

Once we have a test for whether a rational number is in the value monoid, we need to be able to determine one of its preimages under the valuation. The following algorithm accomplishes this task.

Algorithm 6.5. Let $m \in \Lambda$. This algorithm constructs $p(x, y) \in k[x, y]$ such that $\mathcal{L}E_z(p(x, y)) = m$.

(1) Using Algorithm 6.4, write $m = n + \sum_{j=1}^{i} d_j \rho_j$.
(2) For each $1 \le j \le i$, use Proposition 4.2 to compute $p_j(x,y)$, the minimal polynomial of $\sum_{j=1}^{l(i)-1} c_j x^{e_j}$ over $k(x,y)$.
(3) Define $p(x,y) = x^n \prod_{j=1}^{i} p_j(x,y)^{d_j}$. By Lemma 4.4, $\mathcal{L}E_z(p(x,y)) = m$.

The following algorithm describes how to perform division in $k[x,y]$ relative to $\mathcal{L}E_z$.

Algorithm 6.6. Let $f, g \in k[\mathbf{x}]$. This algorithm constructs $h \in k[x,y]$ such that $\mathcal{L}E_z(f - gh) < \mathcal{L}E_z(f)$ provided that such an h exists.

(1) Compute $m = \mathcal{L}E_z(f) - \mathcal{L}E_z(g)$.
(2) Use Algorithm 6.4 to determine whether $m \in \Lambda$. If $m \notin \Lambda$, then h does not exist.
(2) Using Algorithm 6.5, find $p(x,y) \in k[x,y]$ such that $\mathcal{L}E_z(p) = m$.
(3) Define $h(x,y) = (\mathcal{L}C_z(f)/\mathcal{L}C_z(gp))p(x,y)$. Then $\mathcal{L}C_z(f) = \mathcal{L}C_z(gh)$, and since $\mathcal{L}E_z(f) = \mathcal{L}E_z(gh)$, it follows that $\mathcal{L}E_z(f - gh) < \mathcal{L}E_z(f)$.

To compute syzygy families, we first need the following lemma.

Lemma 6.7. *Let M be a monoid such that $\mathbb{Z} \subset M \subset \mathbb{Q}$, and let q be an element of the quotient group of M (i.e., the set of differences of elements of M). Then for $n \gg 0$, $q + n \in M$.*

We now prove that the intersection of principal ideals in Λ, both generated by elements of Ω_i, must be finitely generated by elements of Ω_i.

Lemma 6.8. *Given $f, g \in k[\mathbf{x}]^*$ such that $\mathcal{L}E_z(f), \mathcal{L}E_z(g) \in \Omega_i$, there exists a finite subset of Ω_i that generates $\langle \mathcal{L}E_z(f)\rangle \cap \langle \mathcal{L}E_z(g)\rangle$.*

Proof. By Lemma 6.7, for each element σ of Ω_i, there exists a minimal $\eta_\sigma \in \mathbb{Z}$ such that $\sigma - \mathcal{L}E_z(f) + \eta_\sigma, \sigma - \mathcal{L}E_z(g) + \eta_\sigma \in \Lambda$; that is, $\sigma + n_\sigma \in \langle \mathcal{L}E_z(f)\rangle \cap \langle \mathcal{L}E_z(g)\rangle$. Define Υ_i to be the finite collection $\{\sigma + \eta_\sigma \mid \sigma \in \Omega_i\}$. We will show that Υ_i generates $\langle \mathcal{L}E_z(f)\rangle \cap \langle \mathcal{L}E_z(g)\rangle$.

Let $m \in \langle \mathcal{L}E_z(f)\rangle \cap \langle \mathcal{L}E_z(g)\rangle$. By Theorem 5.8, $\Lambda = \bigcup_{j=0}^{\infty} \Omega_j$, and so for some index I, there exist $\alpha_f, \alpha_g \in \Omega_I$ such that $m = \mathcal{L}E_z(f) + \alpha_f = \mathcal{L}E_z(g) + \alpha_g$. Write α_f as $\alpha'_f + \sum_{j=i+1}^{I} d_j \rho_j$ and α_g as $\alpha'_g + \sum_{j=i+1}^{I} d'_j \rho_j$ where $\alpha'_f, \alpha'_g \in \Omega_i$ and $0 \le d_j, d'_j < s_j$. By Corollary 6.3, $\mathcal{L}E_z(f) + \alpha'_f, \mathcal{L}E_z(g) + \alpha'_g \in \Omega_i$. By the uniqueness of representation promised by Theorem 5.8, since $m = (\mathcal{L}E_z(f) + \alpha'_f) + \sum_{j=i+1}^{I} d_j \rho_j = (\mathcal{L}E_z(g) + \alpha'_g) + \sum_{j=i+1}^{I} d'_j \rho_j$, we have $d_j = d'_j$ for $i + 1 \le j \le I$. Thus $\mathcal{L}E_z(f) + \alpha'_f = \mathcal{L}E_z(g) + \alpha'_g$. So by Theorem 5.8, $m' := \mathcal{L}E_z(f) + \alpha'_f = \mathcal{L}E_z(g) + \alpha'_g = n + \sum_{j=1}^{i} \delta_j \rho_j$, where $n \in \mathbb{N}$ and $0 \le \delta_j < s_j$. Define $\sigma = \sum_{j=1}^{i} \delta_j \rho_j$, and let n_σ be the smallest $n_\sigma \in \mathbb{Z}$ such that $\sigma + n_\sigma \in \langle \mathcal{L}E_z(f)\rangle \cap \langle \mathcal{L}E_z(g)\rangle$. Since $m' = \sigma + n \in \langle \mathcal{L}E_z(f)\rangle \cap \langle \mathcal{L}E_z(g)\rangle$, it follows that $n \ge n_\sigma$. Thus $m' = (n - n_\sigma) + (\sigma + n_\sigma) \in \mathbb{N} + \Upsilon_i$, and so $m = m' + \sum_{j=i+1}^{I} d_j \rho_j = (n - n_\sigma) + (\sigma + n_\sigma) + \sum_{j=i+1}^{I} d_j \rho_j \in \mathbb{N} + \Upsilon_i + \Lambda = \Upsilon_i + \Lambda$. $\square$

The following algorithm uses the lemma above to produce a syzygy family for a pair of polynomials.

Algorithm 6.9. Let $f, g \in k[x,y]$. This algorithm will produce $m_1, \ldots, m_\ell \in \Lambda$ such that $\langle \mathcal{L}E_z(f)\rangle \cap \langle \mathcal{L}E_z(g)\rangle = \langle m_1, \ldots, m_\ell\rangle$. In addition $a_j, b_j \in k[\mathbf{x}]$ will be produced such that $\mathcal{L}E_z(a_j f - b_j g) < m_j$ for each $1 \le j \le \ell$.

(1) Using Algorithm 6.4, write $\mathcal{L}E_z(f) = n + \sum_{j=1}^{i} d_j\rho_j$ and $\mathcal{L}E_z(g) = n' + \sum_{j=1}^{i} d'_j\rho_j$ where $n, n' \in \mathbb{N}$ and $0 \le d_j, d'_j < s_j$.
(2) Let $\sigma_1, \ldots, \sigma_\ell$ be the elements of $\{\sum_{j=1}^{i} d_j\rho_j \mid 0 \le d_j < s_j\}$. For each $1 \le t \le \ell$, find a minimal η_t such that $\sigma_t - \mathcal{L}E_z(f) + \eta_t, \sigma_t - \mathcal{L}E_z(g) + \eta_t \in \Lambda$. To accomplish this, begin with $\eta = 0$ and keep incrementing η_t until $\sigma_t - \mathcal{L}E_z(f) + \eta_t, \sigma_t - \mathcal{L}E_z(g) + \eta_t \in \Lambda$ by Algorithm 6.4.
(3) For each t, define $m_t = \eta_t + n_t$. By Lemma 6.8, $\{m_1, \ldots, m_\ell\}$ generates $\langle \mathcal{L}E_z(f)\rangle \cap \langle \mathcal{L}E_z(g)\rangle$.

Below is an example of a generalized Gröbner basis with respect to a valuation that is not a Gröbner basis with respect to any monomial order.

Example 6.10. Let k be a field that is not of characteristic two. Define $f_1 = y^2 - x$ and $f_2 = xy$. Then one can check that the set $B = \{f_1, f_2\}$ is a Gröbner basis for the ideal $I = \langle f_1, f_2\rangle$ with respect to the valuation induced by $z = t^{1/2} + t^{1/4} + t^{1/8} + t^{1/16} + \cdots$ using Algorithm 1.7.

We now demonstrate that B is not a Gröbner basis with respect to any monomial order. Suppose, for contradiction, that B is a Gröbner basis with respect to some monomial order '$<$'. Note that $x^2, y^3 \in I$ since $x^2 = yf_2 - xf_1$ and $y^3 = yf_1 + f_2$. We consider two cases, depending on whether $x > y^2$ or $x < y^2$. If $x < y^2$, then $\mathrm{lt}(f_1) = y^2$ and $\mathrm{lt}(f_2) = xy$. However, $x^2 \in I$, and so if B were a Gröbner basis with respect to '$<$', then either $y^2 \mid x^2$ or $xy \mid x^2$, a contradiction. Now suppose $x > y^2$, in which case $\mathrm{lt}(f_1) = x$ and $\mathrm{lt}(f_2) = xy$. However, $y^3 \in I$, and so if B were a Gröbner basis, then either $x \mid y^3$ or $xy \mid y^3$, a contradiction.

Lastly, we note by example that some ideals do not have finite Gröbner bases with respect to a given valuation. We first prove a short lemma.

Lemma 6.11. *The sequence $\rho_0, \rho_1, \rho_2, \ldots$ is increasing.*

Proof. Since $s_j > 1$ for each index j, by Lemma 3.5, $\rho_i = \sum_{j=1}^{i-1}(s_j - 1)\rho_j + e_{l(i)} > \sum_{j=1}^{i-1}\rho_j + e_{l(i)} > \rho_{i-1}$. ☐

Example 6.12. Consider the ideal $\langle x, y\rangle$ of $k[x, y]$, and let G be a Gröbner basis with respect to the series $z \in k\langle\langle t^{\mathbb{Q}}\rangle\rangle$. For each ρ_i, let $p_i(x, y) \in k[x, y]$ such that $\mathcal{L}E_z(p_i) = \rho_i$. Since G is a Gröbner basis, there exists $g_i \in G$ such that $\mathcal{L}E_z(g_i) \mid \mathcal{L}E_z(p_i)$. That is, for some $h_i \in k[x, y]$, $\mathcal{L}E_z(g_ih_i) = \rho_i$. Since $G \cap k = \emptyset$, $\mathcal{L}E_z(g_i) > 0$, and so $\mathcal{L}E_z(h_i) < \rho_i$. Suppose, for contradiction, $\mathcal{L}E_z(g_i) \ne \rho_i$. Then $\mathcal{L}E_z(g_i) < \rho_i$, and so by Theorem 5.8 and Lemma 6.11, $\mathcal{L}E_z(g_i) = n + \sum_{j=1}^{i-1} d_j\rho_j$ and $\mathcal{L}E_z(h_i) = n' + \sum_{j=1}^{i-1} d'_j\rho_j$. Thus, $\rho_i = \mathcal{L}E_z(g_ih_i) \in (1/r_{l(i-1)})\mathbb{Z}$, which contradicts Lemma 3.6. Therefore, $\mathcal{L}E_z(g_i) = \rho_i$, and thus G is infinite.

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