1-1-2016

The regularity of the boundary of a multidimensional aggregation patch

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1 Introduction

Active scalar problems are a wide class of research topics in fluid dynamics for which basic questions of existence and regularity pose challenging analysis problems that are often viewed as simpler 'analogues' of the famous Clay Math Prize Navier-Stokes problem. One subclass of such problems are the famous 'vortex patches' - exact L^∞ solutions of the incompressible inviscid two-dimensional fluid equations in which the scalar vorticity is the characteristic function of an evolving domain. The classical theory of general L^∞ weak solutions dates back to Yudovich [56] in the early 1960s. However, the challenging problem of the long time regularity of the patch boundary was not settled until the early 1990s by Chemin [28] using methods from para-differential calculus. A geometric harmonic analysis proof was developed later by the first author and Constantin in [6].

More recent works have studied the dynamics of active scalars with a gradient flow structure. This opens up the possibility of using variational methods, including tools from optimal transport theory. The general problem has the structure

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \quad v = -\nabla K * \rho \tag{1.1}$$

with many papers written to understand various subclasses of this problem [5, 9, 10, 8, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 25, 30, 34, 31, 35, 37, 38, 39, 40, 42, 41, 48, 49, 53, 54, 55]. This equation arises in many applications including materials science [35, 50, 51], cooperative control [34], granular flow [23, 24, 55], biological swarms[48, 47, 53, 54], vortex densities in superconductors [33, 2, 1, 32, 44] and chemotaxis [19, 37, 15, 16]. Some of the recent literature has studied finite time singularities and local vs global well-posedness for both the inviscid case (1.1) [9, 10, 11, 8, 12, 17, 13, 26, 27, 31, 36, 38] and the cases with various kinds of diffusion [4, 15, 13, 39, 40, 52] in multiple dimensions. The well-known Keller-Segel problem typically has a Newtonian potential and linear diffusion. For the non-diffusive problem (1.1), of particular interest is the transition from smooth solutions to weak and measure solutions with mass concentration.

This paper concerns (1.1) with the special case of $K = N$ the Newtonian potential. This equation is exactly orthogonal to the classical 2-D inviscid incompressible fluid equations in vorticity form, in the sense that the velocity field is perpendicular to that given by the Bio-Savart law. Because of its gradient flow structure, the model makes sense in all space dimensions and we consider this problem in general dimension greater than one. We are interested in a special class of solutions called "aggregation patches". These are particular L^∞ weak solutions in which the density ρ is a time dependent constant times the characteristic function of an evolving domain. Such solutions only exist for special kernels such as the

Newtonian potential. For this potential a previous paper [7] established a sharp C^γ and L^∞ regularity theory for solutions of (1.1) for $K = N$ in general dimension. The proof generalizes the ideas of [43] for C^γ solutions of the vorticity equation and [56] for the L^∞ theory. That work also developed a numerical method for computing aggregation patch boundaries and showed some interesting examples in both two and three dimensions. Of note is that these numerical solutions develop nontrivial geometric singularities at the blowup time - typically with mass concentrated along a "skeleton"-like structure. The simplest example with a trivial behavior is the collapsing sphere (or disk in 2-D) which due to symmetry collapses to a dirac mass at a single point at the blowup time. However elliptical initial data yield solutions that collapse onto a line segment and more complex initial conditions appear to collapse to a structure with branched arms. In an analogous fashion, the spreading solutions (backward time) were also studied both theoretically and numerically. In [7] the authors develop a rigorous theory proving L^1 convergence of the spreading patch solutions to an exact spherically symmetric similarity solution. However numerical simulations suggest that the weak L^1 convergence theory can not be made sharper due to the development of defects in the patch boundary in the approach to the similarity solution.

These nontrivial dynamics open up the natural question of the regularity of the patch boundary for these aggregation patch solutions. In this paper we establish this result, working in Hölder spaces as was done for the vortex patch boundary problem in fluids in [28, 6]. One key idea in [6] was a geometric lemma using cancellation properties of the gradient of the Biot-Savart kernel on half-disks and yielding a uniform estimate for the gradient of the velocity field with constants depending logarithmically on quantities measuring the smoothness of the boundary. We extend this logarithmic inequality to the multidimensional case involving even singular integrals and cancellation on half-spheres. It is worth mentioning that such kind of estimates have appeared before, without mention to the logarithmic dependence on constants, in connection with problems of classical analysis (see, for instance, [45] and [46]). See also [29], in which they appear in connection with the Muskat problem.

A difficulty we have to confront in this paper that did not appear in the incompressibile fluids case is finding defining functions of the patch for non-zero times. The defining function for the initial patch transported by the flow is not smooth, because the field is not divergence free. We find a genuine smooth defining function adapted to our context, which leads to a commutator formula expressing the gradient of the velocity applied to the gradient of our defining function as a commutator of matrix valued singular integrals. A special Hölder estimate in terms of the uniform norm of the gradient of the velocity field is derived. In addition, to prove our result we first develop the local existence and continuation theory for the patch boundary problem in general dimensions, which requires estimates of the transport map of the patch boundary in local coordinates. The technical apparatus needed at this point is more involved than the two dimensional case where one can parametrize by the circle, at least in the simply connected case. As a counterpart, we can work without additional effort on domains with holes and even on open sets made of finitely many pieces with disjoint closures. More details on the structure of the proof and information on the organization of the paper are given in the next subsections.

1.1 The main result

Let $d \geq 2$ and let $N(y)$ be the fundamental solution of the Laplace equation in $\mathbb{R}^d$. Thus $N(y) = \frac{1}{2\pi} \log |y|$ in dimension $d = 2$ and

$$N(y) = -\frac{1}{(d-2)\omega_{d-1}} \frac{1}{|y|^{d-2}}, \quad d \geq 3,$$

where ω_{d-1} is the $d-1$-dimensional surface measure of the unit sphere in $\mathbb{R}^d$. We consider the aggregation equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0 \tag{1.2}$$

$$v = -\nabla N * \rho \tag{1.3}$$

with initial data

$$\rho(x, 0) = \chi_{D_0} \tag{1.4}$$

where χ_{D_0} is the indicator function of a bounded domain $D_0 \subset \mathbb{R}^d$. We now fix $0 < \gamma < 1$ and take D_0 to be a bounded $C^{1+\gamma}$ domain (a domain with smooth boundary of class $C^{1+\gamma}$; a formal definition will be presented in section 2). Then we have the following Theorem.

Theorem 1.1. *If D_0 is a $C^{1+\gamma}$ domain, then the initial value problem* (1.2), (1.3) *and* (1.4) *has a solution given by*

$$\rho(x, t) = \frac{1}{1-t} \chi_{D_t}(x), \quad x \in \mathbb{R}^d, \quad 0 \leq t < 1 \tag{1.5}$$

where D_t is a $C^{1+\gamma}$ domain for all $0 \leq t < 1$.

As the proof shows, the preceding result also holds when D_0 is a union of finitely many $C^{1+\gamma}$ domains with disjoint closures. The conclusion is that D_t is of the same type for all $0 \leq t < 1$. It has been recently proved in [7] that the equation (1.2)–(1.3) has a unique solution in the weak sense for each initial condition $\varrho_0(x)$ in $L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. If the initial condition is the indicator function of a bounded domain D_0, then one has (1.5). In this case one speaks of aggregation patches, in analogy with the vortex patches for the vorticity equation associated with the planar Euler system (see [43, Chapter 8]). Thus our theorem solves the boundary regularity problem for aggregation patches. See [6], [28] or [43, Chapter 8] for the analogous result for the vorticity equation in the plane.

We describe a convenient reformulation of the problem that will be used throughout the rest of the paper.

Set $s = \log(\frac{1}{1-t})$, so that $0 \leq s < \infty$ if and only if $0 \leq t < 1$. Define

$$\tilde{\rho}(x, s) = (1-t)\rho(x, t) \quad \text{and} \quad \tilde{v}(x, s) = (1-t)v(x, t).$$

Then, if the initial condition is (1.4), (1.2) is equivalent to the transport equation

$$\frac{\partial \tilde{\rho}(x, s)}{\partial s} + \nabla \tilde{\rho}(x, s) \cdot \tilde{v}(x, s) = 0. \tag{1.6}$$

The flows (or particle trajectories) in the time variables t and s are defined respectively by the ODE

$$\frac{dX(x, t)}{dt} = v(X(x, t), t), \quad X(x, 0) = x$$

and

$$\frac{d\tilde{X}(x,s)}{ds} = \tilde{v}(\tilde{X}(x,s),s), \quad \tilde{X}(x,0) = x.$$

They are the same, in the sense that $\tilde{X}(x,s) = X(x,t)$. Hence the solution of the transport equation (1.6) with initial condition $\tilde{\rho}(x,0) = \chi_{D_0}(x)$ is

$$\tilde{\rho}(x,t) = \chi_{\tilde{D}_s}(x), \quad \tilde{D}_s = \tilde{X}(D_0,s) = X(D_0,t) = D_t = D_{1-e^{-s}}.$$

Dropping the tildes to simplify the writing and denoting again by t the new time s, we conclude that the problem (1.2)–(1.4) is equivalent to the non-linear transport equation

$$\frac{\partial\rho(x,t)}{\partial t} + \nabla\rho(x,t)\cdot v(x,t) = 0 \tag{1.7}$$

$$v(x,t) = (-\nabla N * \rho)(x) \tag{1.8}$$

with initial condition

$$\rho(x,0) = \chi_{D_0}(x). \tag{1.9}$$

Theorem 1.1 can be then reformulated as follows.

Theorem 1.2. *If D_0 is a $C^{1+\gamma}$ domain then the initial value problem* (1.7), (1.8) *and* (1.9) *has a solution given by*

$$\rho(x,t) = \chi_{D_t}(x), \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R},$$

where D_t is a $C^{1+\gamma}$ domain for all $t \in \mathbb{R}$.

The problem (1.7)–(1.9) for $d = 2$ is similar to the vorticity equation for incompressible perfect fluids. The difference is that the velocity field in the vorticity equation is given by $\nabla^{\perp}N * \rho$, which is an orthogonal gradient and, therefore, is divergence free. Instead the field (1.8) has divergence $-\rho$.

1.2 Outline of the paper

The proof of Theorem 1.2 is in two steps. First we look at the ODE giving the flow

$$\frac{dX(\alpha,t)}{dt} = v(X(\alpha,t),t), \quad X(\alpha,0) = \alpha, \quad \alpha \in \mathbb{R}^d, \quad t \in \mathbb{R}, \tag{1.10}$$

where the velocity field is

$$v(x,t) = -(\nabla N * \chi_{D_t})(x), \quad x \in \mathbb{R}^d, \quad D_t = X(D_0,t). \tag{1.11}$$

Following Yudovich [56] (see [43, Chapter 8] for a modern exposition), the authors in [7] prove that (1.10)-(1.11) has a unique solution and that for each t the mapping $\alpha \to X(\alpha,t)$ is a homeomorphism of $\mathbb{R}^d$ onto itself satisfying a Hölder condition of order $\beta(t)$, with $\beta(t)$ decreasing exponentially to 0 as t tends to ∞. This does not use the smoothness of the boundary of the initial domain D_0 and, in fact, it holds for an initial condition in $L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, with (1.11) modified appropriately. Assuming that D_t is a $C^{1+\gamma}$ domain for t in some time interval around $t = 0$, one can view equation (1.10) as an ODE in the Banach space $C^{1+\gamma}(\partial D_0, \mathbb{R}^d)$. This ODE can be solved for short times by applying the Picard theorem in $C^{1+\gamma}(\partial D_0, \mathbb{R}^d)$ and thus one gets a flow of $C^{1+\gamma}$-diffeomorphisms solving (1.10)-(1.11). By uniqueness of the Yudovich flow one concludes that the restriction of $X(\cdot,t)$ to ∂D_0 is of class $C^{1+\gamma}$ on the surface ∂D_0. In other words, the Yudovich flow is, for short times, of class $C^{1+\gamma}$ in the directions tangential to ∂D_0. This is discussed in section 2. Theorem 2.1 provides the local existence result. One should remark that the statement of Theorem 2.1 includes a precise lower bound for the size of the time interval on which the solution exists, which will be used later on when dealing with long time existence. Showing that the Picard theorem can be applied is equivalent to various estimates, which are collected in Theorem 2.2. Its proof is presented in sections 4 and 5. One needs bounds for the action of principal value singular integral operators on Hölder classes on smooth surfaces. The preliminary Section 3 gives two well-known lemmas in the precise forms that we need for the proof of Theorem 2.2.

The second step consists in proving that the Yudovich flow is of class $C^{1+\gamma}$ in the directions tangential to ∂D_0 for all times. This requires a priori estimates for the quantities that determine the size of the local existence interval. The most relevant are those measuring the smoothness of the boundary of a $C^{1+\gamma}$ domain. The $C^{1+\gamma}$ smoothness of the boundary of a domain D_0 is encoded in a defining function, that is, a function Φ_0 on class $C^{1+\gamma}$ in $\mathbb{R}^d$, vanishing exactly on the boundary ∂D_0 and with non-zero gradient at each point of ∂D_0. By transporting Φ_0, that is, by setting $\varphi(x,t)=\Phi_0(X^{-1}(x,t))$ one obtains a function vanishing exactly on ∂D_t, but with gradient $\nabla\varphi(x,t)=\nabla\Phi_0\circ\nabla X^{-1}(x,t)$ which may have a jump at ∂D_t, just as $\nabla X^{-1}(x,t)$. Thus $\varphi(x,t)$ may not be a defining function for D_t (and, indeed, it is not), contrarily to what happens in the case of the vorticity equation in the plane, for which the velocity field is divergence free. One of the difficulties that we have to overcome is finding a correct way of changing Φ_0 by means of the flow and still getting a genuine defining function $\Phi(x,t)$ for D_t. This is done in section 8. Once this is achieved, we need to get a priori estimates for the γ-Hölder semi-norm $\|\nabla\Phi(\cdot,t)\|_\gamma$ on $\mathbb{R}^d$ and for the infimum of $|\nabla\Phi(x,t)|$ on ∂D_t. The subtlest estimate is that of $\|\nabla\Phi(\cdot,t)\|_\gamma$, which follows by bringing into the scene an appropriate commutator between a singular integral and a pointwise multiplication operator. This estimate is performed in section 7. Once the a priori estimates on the quantities determining the size of the local existence interval are available, the $C^{1+\gamma}$ smoothness of ∂D_t for all $t\in\mathbb{R}$ follows readily.

We close this section by showing that the transported defining function is already not smooth when $d=2$ and the initial patch is the unit disc $D_0=\{x\in\mathbb{R}^2 : |x|<1\}$. The solution of the aggregation equation (1.2)-(1.4) with initial condition the characteristic function of the unit disc is

$$\rho(x,t)=\frac{1}{1-t}\,\chi_{D(0,\sqrt{1-t})}(x),\quad x\in\mathbb{R}^2,\quad t<1.$$

The field is

$$v(x,t)=-\frac{1}{2}\begin{cases}\frac{x}{1-t}, & |x|<\sqrt{1-t},\\ \dfrac{x}{|x|^2}, & |x|\geq\sqrt{1-t}\end{cases}$$

and the inverse flow

$$X^{-1}(x,t)=\begin{cases}\frac{x}{\sqrt{1-t}}, & |x|<\sqrt{1-t},\\ \sqrt{|x|^2+t}\,\frac{x}{|x|}, & |x|\geq\sqrt{1-t}.\end{cases}$$

Take as defining function for D_0 the function $\varphi_0(x)=|x|^2-1$. Transporting φ_0 by the flow we obtain

$$\varphi(x,t)=\varphi_0(X^{-1}(x,t))=\begin{cases}\frac{1}{1-t}\left(|x|^2-(1-t)\right), & |x|<\sqrt{1-t},\\ |x|^2-(1-t), & |x|\geq\sqrt{1-t},\end{cases}$$

whose gradient has a jump at the circle $|x| = \sqrt{1-t}$ except for $t = 0$. To correct the jump one may take

$$\Phi(x,t) = (1-t)\chi_{D_t}(x)\varphi(x,t) + \chi_{\mathbb{R}^2\setminus D_t}(x)\varphi(x,t),$$

where $D_t = D(0, \sqrt{1-t})$.

2 A Flow of $C^{1+\gamma}$ Surfaces

Given $x \in \mathbb{R}^d$ with fixed $d \geq 2$ the cylinder with center x and radius r is

$$C(x,r) = \{x \in \mathbb{R}^d : |y' - x'| \leq r \quad \text{and} \quad |y_d - x_d| \leq r\},$$

where we use the standard notation $x' = (x_1, ..., x_{d-1}) \in \mathbb{R}^{d-1}$. We say that D is a $C^{1+\gamma}$ domain if for each $x \in \partial D$ there exists $r > 0$ such that, after possibly a rotation around x,

$$C(x,r) \cap \partial D = \{y \in C(x,r) : y_d = \varphi(y')\}$$

where φ is a function of class $C^{1+\gamma}$ in a ball $B(x', r'),\ r' > r$. In other words, the boundary of D is locally the graph of a $C^{1+\gamma}$ function and thus a surface of class $C^{1+\gamma}$. A standard argument based on a partition of unity shows that if D is a $C^{1+\gamma}$ domain then there exists a function $\Phi \in C^{1+\gamma}(\mathbb{R}^d)$ such that $D = \{x \in \mathbb{R}^d : \Phi(x) < 0\}$, $\partial D = \{x \in \mathbb{R}^d : \Phi(x) = 0\}$ and $\nabla\Phi(x) \neq 0$ for $x \in \partial D$. Such a function is called a defining function for D of class $C^{1+\gamma}$. Conversely, by the Implicit Function Theorem, if D has a defining function of class $C^{1+\gamma}$, then D is a $C^{1+\gamma}$ domain. There is a very useful quantity measuring the $C^{1+\gamma}$ character of a domain, namely,

$$q(D) = \frac{\|\nabla\Phi\|_\gamma}{|\nabla\Phi|_{\inf}} \tag{2.1}$$

where

$$|\nabla\Phi|_{\inf} = \inf\{|\nabla\Phi(x)| : x \in \partial D\}$$

and, for each set E and each function f defined on E, we denote by $\|f\|_{\gamma,E}$ (or by $\|f\|_\gamma$, if there is no ambiguity on the domain of the function) the Hölder γ-seminorm

$$\|f\|_{\gamma,E} = \sup\{\frac{|f(x) - f(y)|}{|x-y|^\gamma} : x, y \in E,\ x \neq y\}.$$

The ODE providing the particle trajectories is (1.10)-(1.11). As we mentioned before, in [7] one proves that (1.10) has a unique solution and that for each t the mapping $\alpha \to X(\alpha, t)$ is a homeomorphism of $\mathbb{R}^d$ onto itself satisfying a Hölder condition with exponent $\beta(t)$ decreasing exponentially to 0 as t tends to ∞. We call $X(\alpha,t)$ the Yudovich flow associated with the initial condition χ_{D_0}.

Assume that for t in an open interval containing 0 the restriction of $X(\cdot,t)$ to ∂D_0 is in $C^{1+\gamma}(\partial D_0, \mathbb{R}^d)$. Then ∂D_t is a $C^{1+\gamma}$ domain and we have

$$v(x,t) = -(N * \nabla\chi_{D_t})(x) = \int_{\partial D_t} N(x-y)\vec{n}(y)\, d\sigma_t(y)$$

where $\vec{n}$ is the exterior unit normal vector to ∂D_t and $d\sigma_t$ is the surface measure on ∂D_t. If $x = X(\alpha,t)$ and we make the change of variables $y = X(\beta,t)$ we get

$$v(X(\alpha,t),t) = \int_{\partial D_0} N(X(\alpha,t) - X(\beta,t))\Big(DX(\beta,t)(T_1(\beta)) \wedge \cdots \wedge DX(\beta,t)(T_{d-1}(\beta))\Big)\, d\sigma(\beta),$$

where $d\sigma$ is the surface measure on ∂D_0, DX is the differential of X as a differentiable mapping from ∂D_0 into $\mathbb{R}^d$ and $T_1(\beta), ..., T_{d-1}(\beta)$ is an orthonormal basis of the tangent space to ∂D_0 at the point $\beta \in \partial D_0$. The vector

$$\bigwedge_{j=1}^{d-1} DX(\beta, t)(T_j(\beta)) \tag{2.2}$$

is orthogonal to ∂D_t at the point $X(\beta, t)$ and a different choice of the orthonormal basis $T_j(\beta), 1 \leq j \leq d-1$, has the effect of introducing a $\pm$ sign in front of (2.2). We may choose the $T_j(\beta)$ so that $\vec{n}(\beta), T_1(\beta), \ldots, T_{d-1}(\beta)$ gives the standard orientation of $\mathbb{R}^d$. Let Ω be the set of functions $X \in C^{1+\gamma}(\partial D_0, \mathbb{R}^d)$ such that there exists a constant $\mu \geq 1$ for which

$$|X(\alpha) - X(\beta)| \geq \frac{1}{\mu}|\alpha - \beta|, \quad \alpha, \beta \in \partial D_0.$$

The smallest such μ is denoted by $\mu(X)$. Then X is bilipschitz and $\mu(X)$ is the Lipschitz constant of the inverse mapping. It is clear that Ω is an open subset of $C^{1+\gamma}(\partial D_0, \mathbb{R}^d)$. Given $X \in \Omega$ set

$$F(X)(\alpha) = \int_{\partial D_0} N(X(\alpha) - X(\beta)) \bigwedge_{j=1}^{d-1} DX(\beta)(T_j(\beta))\, d\sigma(\beta). \tag{2.3}$$

Therefore $X(\alpha, t)$ satisfies the ODE

$$\frac{dX(\alpha, t))}{dt} = F(X(\cdot, t))(\alpha), \quad X(\alpha, 0) = \alpha, \tag{2.4}$$

which is called the "contour dynamics equation". Our plan is to solve (2.4) for short times in the open subset Ω of the Banach space $C^{1+\gamma}(\partial D_0, \mathbb{R}^d)$. By uniqueness of the trajectories equation (1.10)-(1.11) (see, e.g., [3, Theorem 3.7, p.128]), we conclude that the restriction of the Yudovich flow $X(\cdot, t)$ to ∂D_0 is of class $C^{1+\gamma}$ for short times. In particular, $\partial D_t = X(\partial D_0, t)$ is a surface of class $C^{1+\gamma}$ for short times. In a second step we prove an a priori estimate which implies that ∂D_t is of class $C^{1+\gamma}$ for all times, thus proving Theorem 1.2.

Our estimates are most conveniently performed in terms of a particular norm defining the topology of $C^{1+\gamma}(\partial D_0, \mathbb{R}^d)$. Let us momentarily drop the subindex "0" and work with a bounded $C^{1+\gamma}$ domain D. Since ∂D is a compact surface of class $C^{1+\gamma}$ there exists $r = r(D) > 0$ such that for each $\alpha \in \partial D$ the set $\partial D \cap C(\alpha, r)$ is the graph $\beta_d = \varphi(\beta')$ of a $C^{1+\gamma}$ function φ (after a rotation around α if needed). The function

$$\tilde{X}(\beta') = X(\beta', \varphi(\beta')), \quad \beta' \in B(\alpha', r) \subset \mathbb{R}^{d-1},$$

is in $C^{1+\gamma}(B(\alpha', r))$. Set

$$\nu(X, \alpha) = |X(\alpha)| + \|D\tilde{X}\|_{\infty, B(\alpha', r)} + \|D\tilde{X}\|_{\gamma, B(\alpha', r)},$$

where D is the ordinary differential of $\tilde{X}$ and for a set E and a function f on E we denote by $\|f\|_{\infty, E}$ the supremum norm of f on E. Finally set

$$\|X\|_{1+\gamma} = \|X\|_{1+\gamma, \partial D} = \sup_{\alpha \in \partial D} \nu(X, \alpha).$$

Different choices of the local charts yield different but equivalent norms in $C^{1+\gamma}(\partial D, \mathbb{R}^d)$.

We discuss now two simple facts concerning the norm of $C^{1+\gamma}(\partial D, \mathbb{R}^d)$ we just defined. The first is an estimate for the norm of the identity mapping I. In the local chart centered at $\alpha \in \partial D$ we have $\tilde{I}(\beta') = (\beta', \varphi(\beta')),\ \beta' \in B(\alpha', r)$. Hence $D\tilde{I}(\beta')$ is a matrix with d rows and $d-1$ columns. The matrix formed with the first $d-1$ rows is the identity in $\mathbb{R}^{d-1}$ and the last row is $(\partial_1\varphi(\beta'), ..., \partial_{d-1}\varphi(\beta'))$. Since we can assume that $|\nabla\varphi(\beta')| \le 1,\ \beta' \in B(\alpha', r)$, by implicit differentiation we get (see the proof of lemma 6.4 below) $\|\nabla\varphi\|_{\gamma, B(\alpha', r)} \le C_d\, q(D)$. Then

$$\nu(I, \alpha) \le |\alpha| + C_d + C_d\, q(D).$$

Let c be the center of mass of D and $\operatorname{diam}(D)$ its diameter. Then

$$\|I\|_{1+\gamma, \partial D} \le \operatorname{diam}(D) + |c| + C_d + C_d\, q(D). \tag{2.5}$$

The preceding inequality will be applied to ∂D_t in dealing with long time existence. On the one hand, the center of mass is an invariant of the motion, because the kernel ∇N is odd. Without loss of generality we assume from now on that the center of mass of D_0 (and, consequently, of D_t) is the origin. On the other hand, we will obtain a priori estimates for $\operatorname{diam}(D_t)$ and $q(D_t)$ (see (7.19) and (7.16)). We conclude that the estimate (2.5) is good for our a priori estimates.

The second fact we should discuss is how one estimates the Lipschitz constant of $X \in C^{1+\gamma}(\partial D, \mathbb{R}^d)$ in terms of $\|X\|_{1+\gamma}$. Take points $\alpha, \beta \in \partial D$. If $|\alpha - \beta| < r = r(D)$ we are in a local chart and then clearly $|X(\alpha) - X(\beta)| \le \|X\|_{1+\gamma}\, |\alpha - \beta|$. Otherwise we estimate by the uniform norm and we obtain $|X(\alpha) - X(\beta)| \le 2\|X\|_{1+\gamma} \le \frac{2\|X\|_{1+\gamma}}{r}\, |\alpha - \beta|$. Hence

$$|X(\alpha) - X(\beta)| \le (1 + \frac{2}{r(D)})\, \|X\|_{1+\gamma}\, |\alpha - \beta|, \quad \alpha, \beta \in \partial D. \tag{2.6}$$

Theorem 2.1. *The initial value problem*

$$\frac{dX(\alpha, t)}{dt} = F(X(\cdot, t))(\alpha), \quad X(\alpha, 0) = \alpha, \tag{2.7}$$

has a unique solution $X(\alpha, t) \in C^1((-t_0, t_0), C^{1+\gamma}(\partial D_0, \mathbb{R}^d))$ and t_0 depends only on d, $q(D_0), \sigma(\partial D_0)$ and $\operatorname{diam}(D_0)$.

This follows from the Picard Theorem for Banach spaces and

Theorem 2.2. *If $X \in \Omega$, then*

(a)

$$F(X) \in C^{1+\gamma}(\partial D_0, \mathbb{R}^d)$$

and

$$\|F(X)\|_{1+\gamma} \le C_0\, \mu(X)^{3d+2}(1 + \|X\|_{1+\gamma}^{2d+4}), \tag{2.9}$$

where C_0 denotes a constant that depends only on $d, q(D_0), \sigma(D_0)$ and $\operatorname{diam}(D_0)$.

(b) $X \to F(X)$ is locally Lipschitz on Ω: more precisely,

$$\|DF(X)\| \le C_0\, \mu(X)^{3d+8}\, (1 + \|X\|_{1+\gamma}^{3d+7}), \quad X \in \Omega, \tag{2.10}$$

where C_0 denotes a constant that depends only on $d, q(D_0), \sigma(D_0)$ and $\operatorname{diam}(D_0)$.

Here by $\|DF(X)\|$ we understand the norm of $DF(X)$ as a linear mapping from $C^{1+\gamma}(\partial D_0, \mathbb{R}^d)$ into itself. As the reader will realize, the precise form of the constant in the right hand side of (2.9) is crucial.

We show now that Theorem 2.2 implies Theorem 2.1. The only point that requires further discussion is the size of the interval $(-t_0, t_0)$ on which the solution exists. We need to find a ball $B(I, \rho) \subset \Omega$, of center the identity and radius ρ, so that F is Lipschitz in $B(I, \rho)$ and we have an explicit bound for F on $B(I, \rho)$. Lemma 6.4 gives for $r_0 = r(D_0)$ the inequality $r_0^{-\gamma} \leq 2q(D_0)$. Take

$$\rho = \frac{1}{2(1 + 2^{1+\frac{1}{\gamma}} q(D_0)^{\frac{1}{\gamma}})}$$

so that if $X \in B(I, \rho)$ then, by (2.6),

$$\begin{aligned} |X(\alpha) - X(\beta)| &\geq |\alpha - \beta| - |X(\alpha) - \alpha - (X(\beta) - \beta)| \\ &\geq |\alpha - \beta| \left(1 - (1 + \frac{2}{r(D_0)})\|X - I\|_{1+\gamma}\right) \\ &\geq \frac{|\alpha - \beta|}{2}, \end{aligned}$$

that is, $\mu(X) \leq 2,\ X \in B(I, \rho)$.

Clearly $\|X\|_{1+\gamma} \leq \|I\|_{1+\gamma} + \rho$ for $X \in B(I, \rho)$. By (2.10) F is Lipschitz in $B(I, \rho)$ and by (2.9) $\|X\|_{1+\gamma} \leq C_0\, 2^{3d+2} \left(1 + (\|I\|_{1+\gamma} + \rho)^{2d+4}\right)$ for $X \in B(I, \rho)$. Therefore the solution of (2.7) exists in an interval $(-t_0, t_0)$ with

$$t_0 \geq \frac{\rho}{C_0\, 2^{3d+2}\ (1 + (\|I\|_{1+\gamma} + \rho)^{2d+4})},$$

which is a quantity depending only on on $d, q(D_0), \sigma(D_0)$ and $\operatorname{diam}(D_0)$.

3 Two Lemmas

To prove Theorem 2.2 we need two elementary lemmas on integral operators acting on Hölder functions. These lemmas are well known but we prove them for the sake of the reader and because we need the precise dependence on various constants.

Lemma 3.1. *Let E be a measurable subset of $\mathbb{R}^{d-1}$ and assume that $K : E \times E \to \mathbb{R}$ is a measurable function on $E \times E$ which satisfies*

$$|K(x, y)| \leq \frac{A}{|x - y|^{d-1-\gamma}}, \tag{3.1}$$

$$|K(x_1, y) - K(x_2, y)| \leq |x_1 - x_2| \frac{A}{|x_1 - y|^{d-\gamma}}, \quad |x_1 - x_2| \leq |x_1 - y|/2. \tag{3.2}$$

Then for a constant C, which depends only on d and γ,

$$\left|\left| \int_E K(x, y) f(y)\, dy \right|\right|_{\infty, E} \leq CA \operatorname{diam}(E)^\gamma ||f||_{\infty, E} \tag{3.3}$$

and

$$\left|\left| \int_E K(x, y) f(y)\, dy \right|\right|_{\gamma, E} \leq CA ||f||_{\infty, E}. \tag{3.4}$$

Proof. Let $D = \mathrm{diam}(E)$ be the diameter of E. We assume that $\mathrm{diam}(E) < \infty$, otherwise (3.3) is trivially satisfied. If $x \in E$, then

$$\left| \int_E K(x,y) f(y)\, dy \right| \le A ||f||_{\infty,E} \int_{B(x,D)} \frac{dy}{|x-y|^{d-1-\gamma}} \le CA ||f||_{\infty,E}\, D^\gamma,$$

which is (3.3). For (3.4) take two points $x_1, x_2 \in E$ and set $\delta = 2|x_1 - x_2|$. Then

$$\begin{aligned}\left| \int_E (K(x_1,y) - K(x_2,y))\, f(y)\, dy \right| &\le \int_{E\cap B(x_1,\delta)} |K(x_1,y)|\, |f(y)|\, dy + \int_{E\cap B(x_1,\delta)} |K(x_2,y)|\, |f(y)|\, dy \\ &\quad + \int_{E\setminus B(x_1,\delta)} |K(x_1,y) - K(x_2,y)|\, |f(y)|\, dy \\ &= I_1 + I_2 + I_3.\end{aligned}$$

The first term is estimated as before :

$$I_1 \le CA ||f||_{\infty,E}\, \delta^\gamma = CA ||f||_{\infty,E}\, |x_1 - x_2|^\gamma.$$

The term I_2 can be treated as I_1 because $E \cap B(x_1, \delta) \subset B(x_2, \frac{3}{2}\delta)$. For I_3, by (3.2),

$$I_3 \le CA ||f||_{\infty,E}\, |x_1 - x_2| \int_{\mathbb{R}^{d-1}\setminus B(x_1,\delta)} \frac{dy}{|x_1 - y|^{d-\gamma}} \le CA ||f||_{\infty,E}\, |x_1 - x_2|^\gamma.$$

□

Lemma 3.2. *Let E be a measurable subset of $\mathbb{R}^{d-1}$ and let $K : E \times E \to \mathbb{R}$ be a measurable function on $E \times E$ which satisfies*

$$|K(x,y)| \le \frac{A}{|x-y|^{d-1}}, \tag{3.5}$$

$$|K(x_1,y) - K(x_2,y)| \le |x_1 - x_2| \frac{A}{|x_1 - y|^d}, \quad |x_1 - x_2| \le |x_1 - y|/2. \tag{3.6}$$

Assume that b is a compactly supported bounded measurable function on E such that for a constant B one has

$$\|b\|_{\infty,E} + \sup_{\epsilon>0} \left| \int_{\{y\in E : |x-y|>\epsilon\}} K(x,y) b(y)\, dy \right| \le B, \quad x \in E. \tag{3.7}$$

Let f be a function on E satisfying a Hölder condition of order γ. Then

$$\left\| \int_E K(x,y)\, (f(x) - f(y))\, b(y)\, dy \right\|_{\infty,E} \le CAB\, \mathrm{diam}(E)^\gamma ||f||_{\gamma,E} \tag{3.8}$$

and

$$\left\| \int_E K(x,y)\, (f(x) - f(y))\, b(y)\, dy \right\|_{\gamma,E} \le CAB \|f\|_{\gamma,E}, \tag{3.9}$$

for some constant C depending only on d and γ.

Proof. The inequality (3.8) is proved as in the previous lemma. Let us deal with (3.9). Given $x_1, x_2 \in E$ set $\delta = 2|x_1 - x_2|$. Then

$$
\begin{aligned}
&\left| \int_E K(x_1,y)\,(f(x_1) - f(y))\,b(y)\,dy - \int_E K(x_2,y)\,(f(x_2) - f(y))\,b(y)\,dy \right| \\
&\le \int_{E\cap B(x_1,\delta)} |K(x_1,y)|\,|f(x_1) - f(y)|\,|b(y)|\,dy + \int_{E\cap B(x_1,\delta)} |K(x_2,y)|\,|f(x_2) - f(y)|\,|b(y)|\,dy \\
&+ \left| \int_{E\setminus B(x_1,\delta)} K(x_1,y)\,(f(x_1) - f(y))\,b(y)\,dy - \int_E K(x_2,y)\,(f(x_2) - f(y))\,b(y)\,dy \right| \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

The first term can be estimated readily by (3.5):

$$
I_1 \le C A\, B\, \|f\|_{\gamma,E} \int_{B(x_1,\delta)} \frac{dy}{|x-y|^{d-1-\gamma}}\,dy \le C A\, B\, \|f\|_{\gamma,E}\, |x_1 - x_2|^\gamma
$$

for some constant C depending only on d and γ. For I_2 one only needs to observe that $E \cap B(x_1,\delta) \subset E \cap B(x_2, \frac{3}{2}\delta)$. For I_3 we have, by (3.5), (3.6) and (3.7),

$$
\begin{aligned}
I_3 \le &\left| (f(x_1) - f(x_2)) \int_{\{y\in E: |y-x_1|>\delta\}} K(x_1,y) b(y)\,dy \right| \\
&+ \int_{\{y\in E: |y-x_1|>\delta\}} |f(x_2) - f(y)|\,|K(x_1,y) - K(x_2,y)|\,|b(y)|\,dy \\
&\le B \|f\|_{\gamma,E}\, |x_1 - x_2|^\gamma + CAB\|f\|_{\gamma,E}\, |x_1 - x_2| \int_{\{y\in\mathbb{R}^{d-1}: |x_2-y|>\frac{\delta}{2}\}} \frac{dy}{|x_2 - y|^{d-\gamma}} \\
&\le CAB\|f\|_{\gamma,E}\, |x_1 - x_2|^\gamma.
\end{aligned}
$$

In the second inequality we applied (3.6) and then that $|y - x_1| \ge \frac{2}{3}|y - x_2|$ and $|y - x_2| \ge \frac{\delta}{2}$ for $y \in \mathbb{R}^{d-1} \setminus B(x_1,\delta)$. $\square$

4 Proof of Theorem 2.2, part (a)

For convenience of notation we assume $d \ge 3$. The case $d = 2$ has a similar proof and can also be obtained, in the simply connected case, from the argument in [43, Chapter 8] when the tangential derivative $z_\alpha(\alpha', t)$ is replaced by the normal derivative $iz_\alpha(\alpha', t)$.

Let $X \in \Omega$ and set $\mu = \mu(X)$. By (2.3)

$$
F(X)(\alpha) = \int_{\partial D_0} N(X(\alpha) - X(\beta))\, \vec{G}(\beta)\, d\sigma(\beta), \quad \alpha \in \partial D_0, \tag{4.1}
$$

where $\vec{G} : \partial D_0 \to \mathbb{R}^d$ satisfies

$$
\|\vec{G}\|_{\gamma,\partial D_0} \le C_0 \|X\|_{1+\gamma}^{d-1}
$$

Since ∂D_0 is a compact $C^{1+\gamma}$ surface there exists $r_0 > 0$ such that for each $\alpha_0 \in \partial D_0$ the part of ∂D_0 lying in the cylinder $C(\alpha_0, 6r_0)$ is the graph $\alpha_d = \varphi(\alpha')$ of a function $\varphi \in C^{1+\gamma}(B(\alpha_0', 6r_0))$, after possibly a rotation around α_0. We show in Lemma 6.4 below that we can take

$$
(6r_0)^{-\gamma} = 2q(D_0) \tag{4.2}
$$

and that one has

$$\|\nabla\varphi\|_{\gamma,\,B(\alpha_0,6r_0)} \le 2q(D_0).$$

Let $\psi \in C_0^\infty(B(\alpha_0,3r_0))$ such that $0 \le \psi \le 1$, $\psi = 1$ on $B(\alpha_0,2r_0))$ and $|\nabla\psi| \le A/r_0$, A a numerical constant. Set $F(X)(\alpha) = F_1(\alpha) + F_2(\alpha)$, $\alpha \in \partial D_0$, with

$$F_1(\alpha) = \int_{\partial D_0} N(X(\alpha) - X(\beta))\, \vec{G}(\beta)\, \psi(\beta)\, d\sigma(\beta), \quad \alpha \in \partial D_0. \tag{4.3}$$

Then F_1 is the local part of the integral in (4.1) and F_2 the far away part. For $\alpha \in \partial D_0 \cap C(\alpha_0, 6r_0)$ set, to simplify the writing, $a = (a_1, ..., a_{d-1}) = \alpha'$, so that $(a, \varphi(a)) \in \partial D_0$ for $a \in B(a_0, 6r_0)$. Define

$$\tilde{F}_j(a) = F_j(a, \varphi(a)), \quad a \in B(a_0,6r_0), \quad j = 1,2.$$

The function $\tilde{F}_2(a)$ is of class C^∞ in $B(a_0,2r_0)$ and it is easily estimated in $C^{1+\gamma}(B(a_0,r_0))$ by taking gradient twice. The result is

$$\|\tilde{F}_2\|_{1+\gamma,\,B(a_0,r_0)} \le C_0\, \mu(X)^d\, (1 + \|X\|_{1+\gamma}^d).$$

The constant C_0 in the preceding inequality contains explicitly the area of the surface $\sigma(\partial D_0)$ and negative powers of r_0, which can be estimated in terms of $q(D_0)$ by virtue of (4.2).

We turn now our attention to the more challenging term $\tilde{F}_1(a)$. To simplify notation set

$$Z(a) = X(a, \varphi(a)), \quad a \in B(a_0, 6r_0), \tag{4.4}$$

so that

$$\frac{1}{\mu(X)}\, |a - b| \le |Z(a) - Z(b)| \le C_0 \|X\|_{1+\gamma}\, |a - b|, \quad a \in B(a_0, 6r_0). \tag{4.5}$$

Define

$$M(a,b) = \frac{1}{|Z(a) - Z(b)|^{d-2}}, \quad a, b \in B(a_0, 6r_0).$$

An estimate of the norm of $\tilde{F}_1(a)$ in $C^{1+\gamma}(B(a_0,3r_0))$ is equivalent to an estimate in this space of the function

$$Tf(a) = \int M(a,b) f(b)\, db$$

where

$$f(b) = \vec{G}(b, \varphi(b))\psi(b, \varphi(b))\, (1 + |\nabla\varphi(b)|^2)^{1/2}$$

is in $C^\gamma(B(a_0,3r_0))$, has compact support in $B(a_0,3r_0)$, and satisfies

$$\|f\|_{\gamma,\,B(a_0,3r_0)} \le C_0\, \|X\|_{1+\gamma}^{d-1}. \tag{4.6}$$

Passing to components we can assume that f takes real values. Our first task is to compute the distributional derivatives of Tf. In view of the singularity of the kernel and the dimension of the space, which is $d-1$, we expect a singular integral of Calderón-Zygmund type to appear. That this is indeed the case is shown by the formula

$$\partial_j Tf(a) = \mathrm{p.\,v.} \int \frac{\partial}{\partial a_j} M(a,b) f(b)\, db, \quad 1 \le j \le d-1, \tag{4.7}$$

involving a principal value integral. The only difficulty in proving the above identity is to ascertain that the boundary term appearing in the integration by parts vanishes. If $g \in C_0^\infty(B(a, r_0))$ then

$$-\int Tf(a)\,\partial_j g(a)\,da = -\lim_{\varepsilon\downarrow 0}\int\left(\int_{|a-b|>\varepsilon} M(a,b)\,\partial_j g(a)\,da\right) f(b)\,db.$$

Fix b and integrate by parts to get

$$-\int_{|a-b|>\varepsilon} M(a,b)\,\partial_j g(a)\,da = \int_{|a-b|>\varepsilon}\frac{\partial}{\partial a_j}M(a,b)\,g(a)\,da + \int_{|a-b|=\varepsilon} g(a)\,M(a,b)\,n_j(a)\,d\sigma(a), \quad (4.8)$$

where $n_j(a) = (a_j - b_j)/|a-b|$. To handle the boundary term in (4.8), note first that since $M(a,b) = O(|a-b|^{2-d})$ and $\sigma(\partial B(b,\epsilon)) = O\big(\epsilon^{d-2}\big)$, we have

$$\lim_{\varepsilon\downarrow 0}\int_{|a-b|=\varepsilon} g(a) M(a,b) n_j(a)\,d\sigma(a) = g(b)\lim_{\varepsilon\downarrow 0}\int_{|a-b|=\epsilon} M(a,b) n_j(a)\,d\sigma(a).$$

We now exploit the fact that n_j is an odd function of $\xi = a - b$ to get

$$\begin{aligned}\int_{|b-a|=\varepsilon} M(a,b) n_j(a)\,d\sigma(a) &= \int_{|\xi|=\varepsilon} M(b+\xi,b)\frac{\xi_j}{|\xi|}\,d\sigma(\xi)\\ &= \frac{1}{2}\int_{|\xi|=\varepsilon}\Big(M(b+\xi,b) - M(b-\xi,b)\Big)\frac{\xi_j}{|\xi|}\,d\sigma(\xi)\end{aligned}$$

An elementary calculation gives

$$\Big|M(b+\xi,b) - M(b-\xi,b)\Big| \le \frac{C}{|\xi|^{d-1}}|Z(b+\xi) + Z(b-\xi) - 2Z(b)| \le \frac{C}{|\xi|^{d-2-\gamma}}$$

which yields

$$\int_{|a-b|=\varepsilon} M(a,b) n_j(a)\,d\sigma(a) = O\big(\epsilon^{\gamma}\big),$$

and, consequently, shows (4.7).

We now prove that the principal value operator in (4.7) maps boundedly $C^\gamma(B(a_0, 3r_0))$ into itself. The strategy is as follows : if there were the derivative with respect to b_j in the kernel of the operator in (4.7) we could try an integration by parts. We show that a sort of commutator changing the derivative with respect to a_j into one with respect to b_j has a kernel with extra smoothness and thus the corresponding commutator operator satisfies the C^γ estimate we are looking for. Set

$$C(a,b) = (2-d)\frac{(Z(a) - Z(b))\cdot(\frac{\partial}{\partial a_j}Z(a) - \frac{\partial}{\partial b_j}Z(b))}{|Z(a) - Z(b)|^d}.$$

We then have

$$\begin{aligned}\frac{\partial}{\partial a_j}M(a,b) &= (2-d)\frac{(Z(a) - Z(b))\cdot\frac{\partial}{\partial a_j}Z(a)}{|Z(a)-Z(b)|^d}\\ &= C(a,b) + (2-d)\frac{(Z(a) - Z(b))\cdot\frac{\partial}{\partial b_j}Z(b)}{|Z(a)-Z(b)|^d}\\ &= C(a,b) - \frac{\partial}{\partial b_j}M(a,b)\end{aligned}$$

To show that the operator

$$Cf(a) = \int C(a,b)\, f(b)\, db \tag{4.9}$$

is bounded on $C^\gamma(B(a_0, 3r_0))$ we appeal to lemma 3.2. Remark that

$$C(a,b) = K(a,b) \cdot (\frac{\partial Z}{\partial a_j}(a) - \frac{\partial Z}{\partial b_j}(b))$$

with $K(a,b)$ a kernel which satisfies the hypothesis (3.1) and (3.2) of lemma 3.2 with constant $A = C_0 \|X\|^3_{1+\gamma}\, \mu(X)^{d+2}$. To apply lemma 3.2 we need to check that

$$\left| \int_{|b-a|>\epsilon} K(a,b)\, f(b)\, db \right| \leq C, \quad \epsilon > 0, \quad a \in B(a_0, 3r_0).$$

For that write, for $a \in B(a_0, 3r_0)$,

$$\begin{aligned}\int_{\varepsilon<|b-a|} K(a,b) f(b)\, db &= \int_{\varepsilon<|b-a|<6r_0} K(a,b)(f(b) - f(a))\, db + f(a) \int_{\varepsilon<|b-a|<6r_0} K(a,b)\, db \\ &\equiv I(a) + II(a)\end{aligned}$$

The term $I(a)$ is estimated straightforwardly by

$$|I(a)| \leq C_0\, \|X\|^d_{1+\gamma}\, \mu(X)^d \int_{|a-b|<6r_0} \frac{1}{|a-b|^{d-1-\gamma}}\, db = C_0\, \|X\|^d_{1+\gamma}\, \mu(X)^d.$$

For $II(a)$ we use a "pseudo-oddness" property of the kernel. We have

$$\begin{aligned}&\int_{\varepsilon<|a-b|<6r_0} \frac{Z(b) - Z(a)}{|Z(b) - Z(a)|^d}\, db \\ &= \int_{\varepsilon<|\xi|<6r_0} \frac{Z(a+\xi) - Z(a)}{|Z(a+\xi) - Z(a)|^d}\, d\xi \\ &= \frac{1}{2} \int_{\varepsilon<|\xi|<6r_0} \left(\frac{Z(a+\xi) - Z(a)}{|Z(a+\xi) - Z(a)|^d} + \frac{Z(a-\xi) - Z(a)}{|Z(a-\xi) - Z(a)|^d} \right) d\xi \\ &= \frac{1}{2} \int_{\varepsilon<|\xi|<6r_0} (Z(a+\xi) - Z(a)) \left(\frac{1}{|Z(a+\xi) - Z(a)|^d} - \frac{1}{|Z(a-\xi) - Z(a)|^d} \right) d\xi \\ &\quad + \int_{\varepsilon<|\xi|<6r_0} \frac{Z(a+\xi) + Z(a-\xi) - 2Z(a)}{|Z(a-\xi) - Z(a)|^d}\, d\xi\end{aligned}$$

The elementary inequality

$$\left| |z|^d - |w|^d \right| \leq d \sup_{0 \leq j \leq d-1} (|z|^{d-1-j}, |w|^j)\, |z \pm w|$$

provides an estimate for the first term above and the second is estimated straightforwardly. We finally obtain

$$\begin{aligned}|II(a)| &\leq C_0\, \mu(X)^{2d}\, (1 + \|X\|^{d+1}_{1+\gamma})\, \|f\|_{\infty, B(a_0, 3r_0)} \\ &\leq C_0\, \mu(X)^{2d}\, (1 + \|X\|^{2d}_{1+\gamma}).\end{aligned}$$

The constant of the kernel $K(a,b)$, as in (3.1) and (3.2), is less than $C_0\,\mu(X)^{d+2}\,\|X\|^3_{1+\gamma}$. Therefore lemma 3.2 yields

$$\|Cf\|_{\gamma,B(a_0,3r_0)} \le C_0\,\mu(X)^{3d+2}\,(1+\|X\|^{2d+4}_{1+\gamma}). \tag{4.10}$$

It remains to estimate the operator

$$Uf(a) = \mathrm{p.\,v.}\int \frac{\partial}{\partial b_j} M(a,b) f(b)\,db$$

on $C^\gamma(B(a_0,3r_0))$, for $1\le j\le d-1$. Take a function $\chi\in C_0^\infty(B(a_0,4r_0))$ such that $0\le\chi\le 1$, $\chi=1$ on $B(a_0,3r_0)$ and $|\nabla\chi|\le C_0/r_0$. Then $f=f\chi$. We have

$$\begin{aligned} Uf(a) &= \int \frac{\partial}{\partial b_j} M(a,b)(f(b)-f(a))\chi(b)\,db \\ &\quad + f(a)\ \mathrm{p.v.}\int \frac{\partial}{\partial b_j} M(a,b)\chi(b)\,db \\ &= U_1 f(a) + f(a) U_2(a). \end{aligned}$$

Integrating by parts and noticing that, as before, the boundary term vanishes we get

$$U_2(a) = -\int M(a,b)\,\frac{\partial}{\partial b_j}\chi(b)\,db, \quad a\in B(a_0,3r_0).$$

Thus, by (4.4) and $\|\nabla\chi\|_{\infty,\,B(a_0,3r_0)} \le C_0 r_0^{-1}$,

$$|U_2(a)| \le C_0\,\mu(X)^{d-2}, \quad a\in B(a_0,3r_0), \tag{4.11}$$

and by lemma 3.1

$$\|U_2\|_{\gamma,\,B(a_0,3r_0)} \le C_0\,\mu(X)^{d-1}\,\|X\|^2_{1+\gamma}.$$

Here the constant of the kernel $M(a,b)$ has been estimated by $C_0\,\mu(X)^{d-1}\,\|X\|^2_{1+\gamma}$. By (4.6)

$$\|fU_2\|_{\gamma,\,B(a_0,3r_0)} \le C_0\,\mu(X)^{d-1}\,\|X\|^{d+1}_{1+\gamma}. \tag{4.12}$$

For $U_1 f$ we apply lemma 3.2. The kernel of the operator U_1 is $\partial/\partial b_j M(a,b)$, whose constant turns out to be not greater than $C_0\,\mu(X)^{d+2}\,(1+\|X\|^4_{1+\gamma})$. Tacking into account (4.6) and (4.11) lemma 3.2 yields

$$\|U_1 f\|_{\gamma,\,B(a_0,3r_0)} \le C_0\,\mu(X)^{2d}\,(1+\|X\|^{d+3}_{1+\gamma}). \tag{4.13}$$

Combining (4.12) and (4.13)

$$\|Uf\|_{\gamma,\,B(a_0,3r_0)} \le C_0\,\mu(X)^{2d}\,(1+\|X\|^{d+3}_{1+\gamma}). \tag{4.14}$$

By (4.14) and (4.10) we finally obtain

$$\|\partial_j Tf(a)\|_{\gamma,\,B(a_0,3r_0)} \le C_0\,\mu(X)^{3d+2}\,(1+\|X\|^{2d+4}_{1+\gamma}),$$

which completes the proof of (2.9).

5 Proof of Theorem 2.2, part (b)

It is enough to show that for $X \in \Omega$ and $H \in C^{1+\gamma}(\partial D_0, \mathbb{R}^d)$

$$DF(X)(H) = \frac{d}{d\lambda} F(X + \lambda H)\Big|_{\lambda=0}$$

satisfies

$$||DF(X)(H)||_{1+\gamma} \leq C_0\, \mu(X)^{3d+8}\, (1 + \|X\|_{1+\gamma}^{3d+7})\, ||H||_{1+\gamma}. \tag{5.1}$$

To prove this we may assume that $||H||_{1+\gamma} = 1$. We first compute

$$\begin{aligned}
DF(X)(H) &= \frac{d}{d\lambda} F(X + \lambda H)\Big|_{\lambda=0} \\
&= \frac{d}{d\lambda}\Big|_{\lambda=0} \int_{\partial D_0} N\Big((X + \lambda H)(\alpha) - (X + \lambda H)(\beta)\Big) \bigwedge_{j=1}^{d-1} D(X + \lambda H)(T_j(\beta))\, d\sigma(\beta) \\
&= \int_{\partial D_0} N(X(\alpha) - X(\beta)) \sum_{j=1}^{d-1} (-1)^{j-1} DH(T_j(\beta)) \bigwedge_{k \neq j} DX(T_k(\beta))\, d\sigma(\beta) \\
&\quad + \int_{\partial D_0} \nabla N(X(\alpha) - X(\beta)) \cdot (H(\alpha) - H(\beta)) \bigwedge_{j=1}^{d-1} D(X)(T_j(\beta))\, d\sigma(\beta) \\
&\equiv A(\alpha) + B(\alpha)
\end{aligned}$$

Consider first the term $A(\alpha)$. This is a sum of $d-1$ terms, each of which looks like the function $F(X)(\alpha)$ in (2.3). The only difference is that in $A(\alpha)$ one of the factors $DX(T_j)(\beta)$ has been replaced by a vector of the type $DH(T_j)(\beta)$. Then the estimate of $A(\alpha)$ in $C^{1+\gamma}(\partial D_0, \mathbb{R}^d)$ is performed in exactly the same way as we did in the previous section for $F(X)$. There is only one difference, namely, that in the bounding terms one of the factors $\|X\|_{1+\gamma}$ should be replaced by $\|H\|_{1+\gamma} = 1$. Thus

$$\|A\|_{1+\gamma} \leq C_0\, \mu(X)^{3d+2}\, (1 + \|X\|_{1+\gamma}^{2d+3}).$$

The term $B(\alpha)$ is slightly different because of the presence of the factor $H(\alpha) - H(\beta)$ in the kernel, which compensates the higher singularity of $\nabla N(X(\alpha) - X(\beta))$. The structure of the argument is, however, the same. One performs the splitting into local and far away parts, as in (4.3). The local part, which is the most difficult, can be written in local coordinates $a = (a_1, ..., a_{d-1})$ as

$$Tf(a) = \int M(a,b)\, f(b)\, db$$

where f is a scalar function satisfying the estimate (4.6), and the kernel $M(a,b)$ is given by

$$M(a,b) = \nabla N(Z(a) - Z(b)) \cdot (h(a) - h(b)),$$

$Z(a) = X(a, \varphi(a))$ and $h(a) = H(a, \varphi(a))$. The function Z satisfies (4.5) and $\|h\|_{1+\gamma, B(a_0, 3r_0)} \leq C_0$. As before, the boundary term vanishes and we have

$$\partial_j Tf(a) = \text{p. v.} \int \frac{\partial}{\partial a_j} M(a,b) f(b)\, db, \quad 1 \leq j \leq d-1.$$

We express this operator as a commutator minus an operator with kernel $\partial_j/\partial b_j M(a,b)$. Recall that the commutator gives the worst constants. The kernel of the commutator is

$$\begin{aligned} C(a,b) &= \nabla^2 N\big(Z(a)-Z(b)\big)\big(\partial_j Z(a)-\partial_j Z(b)\big)\cdot(h(a)-h(b)) + \nabla N(Z(a)-Z(b))\cdot(\partial_j h(a)-\partial_j h(b)) \\ &= K(a,b)(\partial_j Z(a)-\partial_j Z(b)) + \nabla N(Z(a)-Z(b))\cdot(\partial_j h(a)-\partial_j h(b)), \end{aligned}$$

where the second identity defines the matrix $K(a,b)$. The operator given by the kernel in the second term

$$\nabla N(Z(a)-Z(b))\cdot(\partial_j h(a)-\partial_j h(b))$$

is estimated as we did in the previous section for (4.9). The worst constants appear in estimating the operator with the kernel

$$K(a,b)(\partial_j Z(a)-\partial_j Z(b)).$$

We follow closely the argument for the estimate of (4.9). The step that gives the largest constant happens when dealing with the quantity

$$f(a)\int_{\epsilon<|b-a|<6r_0} K(a,b)\,db, \quad a\in B(a_0,3r_0). \tag{5.2}$$

The pseudo-oddness property of K gives

$$|\int_{\epsilon<|b-a|<6r_0} K(a,b)\,db| \le C_0\,\|h\|_{1+\gamma}\,\mu(X)^{2d+4}\,(1+\|X\|_{1+\gamma}^{d+4}), \quad a\in B(a_0,3r_0),$$

which, combined with (4.6), yields the upper bound

$$C_0\,\mu(X)^{2d+4}\,(1+\|X\|_{1+\gamma}^{2d+3})$$

for the norm of the matrix in (5.2). The constant of the kernel $K(a,b)$ in applying lemma 3.2 is $C_0\,\mu(X)^{d+4}\,\|X\|_{1+\gamma}^4$. Thus lemma 3.2 finally gives

$$\|Tf\|_{1+\gamma} \le C_0\,\mu(X)^{3d+8}\,(1+\|X\|_{1+\gamma}^{2d+7})$$

and

$$||DF(X)(H)||_{1+\gamma} \le C_0\,\mu(X)^{3d+8}\,(1+\|X\|_{1+\gamma}^{3d+7}),$$

which is (5.1), because we are assuming that $||H||_{1+\gamma}=1$.

6 The logarithmic inequality for $\|\nabla v(\cdot,t)\|_\infty$

Fix a bounded $C^{1+\gamma}$ domain D and write

$$v(x) = -\nabla N * \chi_D(x). \tag{6.1}$$

In section we will establish a logarithmic estimate for $||\nabla v||_\infty$ that will be needed to get long time solutions of the problem (1.7), (1.8) and (1.9).

Lemma 6.1. *Let $x \notin \partial D$, and let $\epsilon = \epsilon(x) = \operatorname{dist}(x, \partial D)$. Then for $1 \le j, k \le d$, the vector $v = (v^1, v^2, \dots, v^d)$ satisfies*

$$\frac{\partial v^j}{\partial x_k}(x) = \left(\frac{d}{\omega_{d-1}} \frac{x_j x_k}{|x|^{d+2}} * \chi_{D \setminus B(x,\epsilon)} \right)(x), \quad j \neq k \tag{6.2}$$

and

$$\frac{\partial v^j}{\partial x_j}(x) = -\frac{1}{d}\chi_D(x) - \left(\frac{1}{\omega_{d-1}} \frac{|x|^2 - d\, x_j^2}{|x|^{d+2}} * \chi_{D \setminus B(x,\epsilon)} \right)(x) \tag{6.3}$$

where $\omega_{d-1} = \sigma(\mathbb{S}^{d-1})$ and the derivatives are distributional derivatives.

Proof. Suppose $j \neq k$. By (6.1) and Green's theorem,

$$\frac{\partial v^j}{\partial x_k}(x) = \frac{d}{\omega_{d-1}} \lim_{\eta \to 0} \int_{D \cap \{|y-x| > \eta\}} \frac{(x_j - y_j)(x_k - y_k)}{|x-y|^{d+2}}\, dy,$$

but for $0 < \eta < \epsilon(x)$

$$\int_{\eta < |y-x| < \epsilon} \frac{(x_j - y_j)(x_k - y_k)}{|x-y|^{d+2}}\, dy = 0.$$

That established (6.2), and the proof of (6.3) is similar. $\square$

Notice that the principle value kernels in (6.2) and (6.3) have the form

$$K(x) = \frac{\Omega(x)}{|x|^d}, \quad x \neq 0, \tag{6.4}$$

where

(i) Ω is homogeneous of degree 0, $\Omega(x) = \Omega(\frac{x}{|x|})$,

(ii) Ω is even, $\Omega(-x) = \Omega(x)$,

(iii) $\Omega \in C^1(\mathbb{R}^d \setminus \{0\})$,

and

(iv) $\int_{\mathbb{S}^{d-1}} \Omega(x)\, d\sigma(x) = 0$.

As we mentioned before, a $C^{1+\gamma}$ domain D has a defining function, that is, a $C^{1+\gamma}$ function $\Phi\colon \mathbb{R}^d \to \mathbb{R}$ such that and $D = \{\Phi < 0\}$, $\partial D = \{\Phi = 0\}$ and $\nabla\Phi(x) \neq 0$, $x \in \partial D$. We set

$$|\nabla\Phi|_{\inf} = \inf_{x \in \partial D} |\nabla\Phi(x)|$$

and

$$||\nabla\Phi||_\gamma = \sup_{x_1 \neq x_2 \in \mathbb{R}^d} \frac{|\nabla\Phi(x_1) - \nabla\Phi(x_2)|}{|x_1 - x_2|^\gamma}.$$

Theorem 6.2. *Let K satisfy (6.4) and (i)–(iv) and let D be a $C^{1+\gamma}$ domain with defining function Φ. Then*

$$\left| \int_{|y-x| > \epsilon} K(x-y)\chi_D(y)\, dy \right| \le \frac{C_d\, C(\Omega)}{\gamma} \left(1 + \log^+\left(|D|^{1/d} \frac{||\nabla\Phi||_\gamma}{|\nabla\Phi|_{\inf}} \right) \right), \quad x \in \mathbb{R}^d, \quad 0 < \epsilon, \tag{6.5}$$

where C_d and $C(\Omega)$ are constants that depend only on d and Ω respectively, and $\log^+ x = \max\{\log x, 0\}$ is the positive part of the logarithm.

Corollary 6.3. *If $v(x)$ is defined by* (6.1) *and Φ is a defining function for the bounded $C^{1+\gamma}$ domain D, then*

$$||\nabla v||_\infty \le \frac{C'_d}{\gamma}\left(1+\log^+\left(|D|^{1/d}\frac{||\nabla\Phi||_\gamma}{|\nabla\Phi|_{\inf}}\right)\right). \tag{6.6}$$

The Corollary is immediate from (6.5), (6.2) and (6.3).

We need a lemma.

Lemma 6.4. *Let D be a $C^{1+\gamma}$ domain with defining function Φ. If $\delta > 0$ satisfies*

$$\delta^\gamma \frac{||\nabla\Phi||_\gamma}{|\nabla\Phi|_{\inf}} \le \frac{1}{2},$$

then for each $x \in \partial D$, $\partial D \cup B(x,\delta)$ is, after a rotation around x, the graph of a $C^{1+\gamma}$ function and $D \cap B(x,\delta)$ is the part of $B(x,\delta)$ lying below the graph.

Proof. Assume, without loss of generality, that $x = 0$ and that $\nabla\Phi(0) = (0,\dots,0,\partial_d\Phi(0))$, $\partial_d\Phi(0) > 0$. Take two points $p, q \in \partial D \cup B(0,\delta)$ and set $p = (x', x_d)$ with $x' = (x_1,\dots,x_{d-1})$, and $q = (y', y_d)$ with $y' = (y_1,\dots,y_{d-1})$. Then

$$0 = \Phi(p) = \Phi(0) + \nabla\Phi(0)\cdot p + E(p) = |\nabla\Phi(0)|x_d + E(p)$$

and similarly for q. Subtracting and taking absolute value

$$|\nabla\Phi(0)||x_d - y_d| = |E(p) - E(q)| \le \|\nabla\Phi\|_\gamma\, \delta^\gamma\,(|x'-y'| + |x_d - y_d|)$$

and thus

$$\frac{|x_d-y_d|}{|x'-y'|} \le \frac{\|\nabla\Phi\|_\gamma}{|\nabla\Phi|_{\inf}}\,\delta^\gamma\left(1+\frac{|x_d-y_d|}{|x'-y'|}\right),$$

which yields

$$\frac{|x_d-y_d|}{|x'-y'|} \le 1.$$

This says that $\partial D \cup B(x,\delta)$ is the graph of a Lipschitz function $x_d = \varphi(x')$, with domain an open subset U of $\{x' \in \mathbb{R}^{d-1} : |x'| < \delta\}$, satisfying $|\nabla\varphi(x')| \le 1$, $x' \in U$. By the implicit function theorem φ is of class $C^{1+\gamma}$ on its domain and this completes the proof of the Lemma.

Notice that U contains the ball $\{x' \in \mathbb{R}^{d-1} : |x'| < \delta/2^{1/2}\}$. We need also the estimate

$$|\nabla\varphi(x')| \le (2d)^{1/2}\frac{||\nabla\Phi||_\gamma}{|\nabla\Phi|_{\inf}}\, r^\gamma, \quad |x'| \le r < \frac{\delta}{2^{1/2}}. \tag{6.7}$$

By implicit differentiation

$$\partial_j\varphi(x') = -\frac{\partial_j\Phi(x',\varphi(x'))}{\partial_d\Phi(x',\varphi(x'))}, \quad 1 \le j \le d-1,$$

and so

$$|\partial_j\Phi(x',\varphi(x'))| \le |\partial_d\Phi(x',\varphi(x'))|, \quad 1 \le j \le d-1,$$

which gives

$$|\nabla\Phi(x',\varphi(x'))| \le d^{1/2}\,|\partial_d\Phi(x',\varphi(x'))|.$$

Since $\partial_j\Phi(0) = 0$, $1 \le j \le d-1$,

$$|\nabla\varphi(x')| \le d^{1/2}\frac{\|\nabla\Phi\|_\gamma\,(2^{1/2}r)^\gamma}{|\nabla\Phi|_{\inf}}, \quad |x'| \le r,$$

which completes the proof of (6.7). $\square$

Proof of Theorem 6.2. Assume first that $x \in \partial D$. Take $\delta > 0$ such that $\delta^\gamma \frac{||\nabla\Phi||_\gamma}{|\nabla\Phi|_{\inf}} = \frac{1}{2}$ and set $\eta = \delta/2^{1/2}$. Let ϵ satisfy $0 < \epsilon < \eta$. Then

$$\int_{|y-x|>\epsilon} K(x-y)\chi_D(y)\,dy = \int_{D\cap\{\epsilon<|y-x|<\eta\}} K(x-y)\,dy + \int_{D\cap\{\eta<|x-y|<|D|^{1/d}\}} K(x-y)\,dy$$
$$+ \int_{D\cap\{|y-x|>|D|^{1/d}\}} K(x-y)\,dy = I_1 + I_2 + I_3.$$

Thus

$$|I_3| \le \int_{D\cap\{|y-x|>|D|^{1/d}\}} \frac{||\Omega||_\infty}{|x-y|^d}\,dy \le \frac{||\Omega||_\infty}{|D|}|D| = ||\Omega||_\infty,$$

and

$$|I_2| \le ||\Omega||_\infty \omega_{d-1} \int_\eta^{|D|^{1/d}} \frac{dr}{r} = ||\Omega||_\infty \omega_{d-1} \log\left(\frac{|D|^{1/d}}{\eta}\right) \le C_d\,C(\Omega)\left(1 + \log^+\left(|D|^{1/d}\frac{||\nabla\Phi||_\gamma}{|\nabla\Phi|_{\inf}}\right)\right).$$

If $\eta \ge |D|^{1/d}$, then we let $I_2 = 0$ and this brings in again the positive part of the logarithm.

Let us turn to I_1. Assume that $x = 0$ and that we are in the situation discussed in the proof of Lemma 6.4. Taking polar coordinates we get

$$|I_1| \le \left|\int_\epsilon^\eta \int_{A(r)} \Omega(\omega)\,d\sigma(\omega)\,\frac{dr}{r}\right|,$$

where $A(r) = \{\omega \in S^{d-1} : r\omega \in D\}$. Let H stand for the half-space $\{x \in \mathbb{R}^d : x_d < 0\}$. Since Ω is even and has zero integral on S^{d-1}, we conclude that the integral of Ω on the hemisphere $S^{d-1} \cap H$ is also zero. Hence

$$\int_{A(r)} \Omega(\omega)\,d\sigma(\omega) = \int_{B(r)} \Omega(\omega)\,d\sigma(\omega) - \int_{C(r)} \Omega(\omega)\,d\sigma(\omega),$$

where $B(r) = \{\omega \in S^{d-1} : r\omega \in D \setminus H\}$ and $C(r) = \{\omega \in S^{d-1} : r\omega \in H \setminus D\}$. Let us proceed to estimate the integral on $B(r)$ (the integral on $C(r)$ is estimated similarly).

For some absolute constant C_0 (which can be taken to be $\pi/2$) one has, by (6.7),

$$\sigma(B(r)) \le C_0 \sup_{|x'|\le r} |\varphi(x')|\frac{1}{r} \le C_0 \sup_{|x'|\le r} |\nabla\varphi(x')| \le C_0(2d)^{1/2}\frac{||\nabla\Phi||_\gamma}{|\nabla\Phi|_{\inf}}\,r^\gamma.$$

Therefore

$$\left|\int_\epsilon^\eta \int_{B(r)} \Omega(\omega)\,d\sigma(\omega)\frac{dr}{r}\right| \le ||\Omega||_\infty \int_\epsilon^\eta \sigma(B(r))\frac{dr}{r} \le ||\Omega||_\infty\,C_0(2d)^{1/2}\,\frac{||\nabla\Phi||_\gamma}{|\nabla\Phi|_{\inf}}\frac{1}{\gamma}\,\eta^\gamma \le C_d\,C(\Omega)\frac{1}{\gamma},$$

which completes the proof for $0 < \epsilon < \eta$. If $\eta < \epsilon$, then $I_1 = 0$ and I_2 and I_3 are estimated as before.

Let us assume now that $x \notin \partial D$. Let ϵ_0 denote the distance from x to ∂D. If $\epsilon < \epsilon_0$, then

$$\int_{|y-x|>\epsilon} K(x-y)\chi_D(y)\,dy = \int_{|y-x|>\epsilon_0} K(x-y)\chi_D(y)\,dy$$

and so we can assume that $\epsilon_0 \le \epsilon$. Take $x_0 \in \partial D$ with $|x - x_0| = \epsilon_0$ and define

$$\Delta = \int_{|y-x|>\epsilon} K(x-y)\chi_D(y)\,dy - \int_{|y-x_0|>2\epsilon} K(x_0-y)\chi_D(y)\,dy.$$

Then $\Delta = \Delta_1 + \Delta_2$, where

$$\Delta_1 = \int_{|y-x_0|>2\epsilon} \Big(K(x-y) - K(x_0-y)\Big)\chi_D(y)\,dy$$

and

$$\Delta_2 = \int_{B(x_0,2\epsilon)\setminus B(x,\epsilon)} K(x-y)\chi_D(y)\,dy.$$

We then have

$$|\Delta_2| \le ||\Omega||_\infty \frac{|B(x_0,2\epsilon)|}{\epsilon^d} = C_d\,||\Omega||_\infty$$

and, by a gradient estimate,

$$|\Delta_1| \le C(\Omega)\,|x-x_0| \int_{2\epsilon<|y-x_0|} \frac{dy}{|x-x_0|^{d+1}} \le C_d\,C(\Omega),$$

which completes the proof of Theorem 6.2. □

7 Global Existence

We prove in this section that the Yudovich flow $X(\alpha,t)$ solving the ODE (1.10) and (1.11) is smooth in the directions tangential to ∂D_0 for all $t \in \mathbb{R}$. More precisely, the restriction of $X(\cdot,t)$ to ∂D_0 is in $C^{1+\gamma}(\partial D_0, \mathbb{R}^d)$ for all times $t \in \mathbb{R}$. In particular, ∂D_t is a domain of class $C^{1+\gamma}$ for $t \in \mathbb{R}$. The local existence Theorem 2.1 shows that $X(\cdot,t)$ is in $C^{1+\gamma}(\partial D_0, \mathbb{R}^d)$ for $t \in (-T,T)$, where T is the small time given by the Picard Theorem. Assume that T is maximal with the property that the solution $X(\cdot,t)$ is defined in $(-T,T)$. Our goal is to prove the a priori estimates on $(-T,T)$ which will let us to conclude that indeed $T=\infty$.

It is enough to prove that ∂D_t is a domain of class $C^{1+\gamma}$ for all $t \in \mathbb{R}$. In fact, if this is true, then we have

$$\|X\|_{1+\gamma,\partial D_0} < \infty, \quad t \in \mathbb{R}. \tag{7.1}$$

Otherwise, let T be a maximal time so that (7.1) holds for $t \in (-T,T)$. Taking D_T or D_{-T} as initial domain at time T or $-T$ (not at time 0 !) in Theorem 2.1 we contradict the maximality of T.

To show that ∂D_t is a domain of class $C^{1+\gamma}$ for all $t \in \mathbb{R}$ take a defining function Φ_0 for D_0. Then Φ_0 is in $C^{1+\gamma}(\mathbb{R}^d)$, $D_0 = \{\Phi_0 < 0\}$, $\partial D_0 = \{\Phi_0 = 0\}$ and $\nabla\Phi_0(x) \neq 0$, $x \in \partial D_0$. Consider the equation

$$\frac{\partial}{\partial t}\Phi(x,t) + \nabla\Phi(x,t)\cdot v(x,t) = -\Phi(x,t)\chi_{D_t}(x), \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}, \tag{7.2}$$

with initial condition $\Phi(x,0) = \Phi_0(x)$. Then

$$\Phi(x,t) = \Phi_0(X^{-1}(x,t)), \quad x \in \mathbb{R}^d \setminus D_t,$$

and

$$\Phi(x,t) = e^{-t}\,\Phi_0(X^{-1}(x,t)), \quad x \in D_t,$$

where, for a fixed time t, $X^{-1}(x,t)$ is the inverse of the mapping $x \to X(x,t)$. Notice that $\Phi(x,t)$ is continuously differentiable in the open sets D_t and $\mathbb{R}^d \setminus \overline{D_t}$, but that $\nabla\Phi(x,t)$ may, a priori, have a jump at the boundary of D_t, just as $\nabla X^{-1}(x,t)$ or $\nabla X(x,t)$. We will show in the next section that $\Phi(x,t)$ is of class $C^{1+\gamma}(\mathbb{R}^d)$ and thus a defining function for D_t. We use this fact freely in this section.

The a priori estimates we need are collected in the following statement.

Theorem 7.1. *Let $\Phi(\cdot,t)$ the defining function for D_t determined by* (7.2)*. Then*

$$\|\nabla\Phi(\cdot,t)\|_\infty \le \|\nabla\Phi(\cdot,0)\|_\infty \exp\int_0^t (1+\|\nabla v(\cdot,s)\|_\infty)\,ds, \tag{7.3}$$

$$|\nabla\Phi(\cdot,t)|_{\inf} \ge |\nabla\Phi(\cdot,0)|_{\inf} \exp(-\int_0^t (1+\|\nabla v(\cdot,s)\|_\infty)\,ds) \tag{7.4}$$

and

$$\|\nabla\Phi(\cdot,t)\|_\gamma \le \|\nabla\Phi(\cdot,0)\|_\gamma \exp(C\int_0^t (1+\|\nabla v(\cdot,s)\|_\infty)\,ds). \tag{7.5}$$

Proof. Taking derivatives in (7.2) we obtain that the material derivative of $\nabla\Phi$ is

$$\frac{D}{Dt}(\nabla\Phi) = -\nabla v(\nabla\Phi) - \chi_{D_t}\,\nabla\Phi. \tag{7.6}$$

We have used here that, since $\Phi(x,t)$ vanishes on ∂D_t,

$$\Phi(x,t)\,\nabla\chi_{D_t}(x) = \Phi(x,t)\,\vec{n}(x)\,d\sigma(x) = 0.$$

By (7.6)

$$|\nabla\Phi(x,t)| \le |\nabla\Phi(x,0)| + \int_o^t (1+\|\nabla v(\cdot,s)\|_\infty)\,\|\nabla\Phi(\cdot,s)\|_\infty\,ds$$

and (7.3) follows from Grönwall.

For (7.4) take $x\in\partial D_t$. Then

$$\begin{aligned}\frac{D}{Dt}\log|\nabla\Phi(x,t)| &= \frac{1}{|\nabla\Phi(x,t)|^2}\,\nabla\Phi(x,t)\cdot\frac{D}{Dt}(\nabla\Phi(x,t))\\ &\ge -\frac{1}{|\nabla\Phi(x,t)|}\left|\frac{D}{Dt}(\nabla\Phi(x,t))\right|\\ &\ge -(1+\|v(\cdot,t)\|_\infty)\end{aligned}$$

and so

$$|\nabla\Phi(x,t)| \ge |\nabla\Phi(x,0)|\,\exp\left(-\int_0^t (1+\|\nabla v(\cdot,s)\|_\infty)\,ds\right),$$

which yields (7.4) at once.

For (7.5) we need two lemmas. The first one is an elementary remark.

Lemma 7.2. *If D is a bounded $C^{1+\gamma}$ domain and Φ is a defining function for D, then*

$$\chi_D(x)\Phi(x) = \nabla N * (\chi_D\nabla\Phi)(x), \quad x\in\mathbb{R}^d. \tag{7.7}$$

Proof. On one hand, the functions in either side of (7.7) are continuous functions. On the other hand, we have the distributional identities

$$\nabla N * (\chi_D\nabla\Phi) = \nabla N * \nabla(\chi_D\Phi) = \Delta N * \chi_D\Phi = \chi_D\Phi.$$

□

Denote by HN the distributional Hessian matrix of N. If $i \neq j$ the entry in HN corresponding to the $i-$th row and $j-$th column is the distribution

$$\text{p. v. } \partial^2_{ij} N(x) = \text{p. v.} -\frac{d}{\omega_{d-1}} \frac{x_i x_j}{|x|^{d+2}}, \tag{7.8}$$

while the diagonal term corresponding to the indexes $i = j$ is the distribution

$$\text{p. v.}\, \partial^2_{ii} N(x) + \frac{1}{d}\delta_0 = \text{p. v.} \frac{1}{\omega_{d-1}} \frac{|x|^2 - d x_i^2}{|x|^{d+2}} + \frac{1}{d}\delta_0, \tag{7.9}$$

where δ_0 stands for the Dirac delta at the origin. In (7.8) and (7.9) the second order partial derivatives of N are taken pointwise for $x \neq 0$. Hence

$$HN = \text{p. v.}\, \nabla^2 N + \frac{1}{d} I_0, \tag{7.10}$$

where I_0 stands for the diagonal matrix with δ_0 in the diagonal. In the next lemma we establish a commutator formula which is crucial in what follows. In the statement below $\nabla^2 N(x),\ x \neq 0$, stands for the $d \times d$ square matrix with entries the pointwise partial derivatives $\partial_{ij} N(x)$. The integrand in the right hand side is absolutely integrable because the vector $\nabla\Phi$ satisfies a Hölder condition of order γ.

Lemma 7.3.
$$\frac{D}{Dt}(\nabla\Phi(x,t)) = \int_{D_t} \nabla^2 N(x-y)\,(\nabla\Phi(x) - \nabla\Phi(y))\; dy, \quad x \in \mathbb{R}^d. \tag{7.11}$$

Proof. Since $v = -\nabla N * \chi_{D_t}$, taking gradient we get

$$\nabla v = -HN * \chi_{D_t} = -\,\text{p. v.}\, \nabla^2 N * \chi_{D_t} - \frac{1}{d}\chi_{D_t} I, \tag{7.12}$$

with I the identity matrix. Identities (7.6) and (7.12) yield

$$\frac{D}{Dt}(\nabla\Phi) = \left(\text{p. v.}\, \nabla^2 N * \chi_{D_t}\right)(\nabla\Phi) + \frac{1}{d}\chi_{D_t}\nabla\Phi - \chi_{D_t}\nabla\Phi. \tag{7.13}$$

Taking gradient in (7.7)

$$\chi_{D_t}\nabla\Phi = HN * (\chi_{D_t}\nabla\Phi) = \text{p. v.}\, \nabla^2 N * (\chi_{D_t}\nabla\Phi) + \frac{1}{d}\chi_{D_t}\nabla\Phi. \tag{7.14}$$

Combining (7.13) and (7.14) completes the proof of the lemma. □

Lemma 7.3 yields the a priori estimate (7.5) exactly as in [6] or [43]. Theorem 7.1 is then proved. □

Inserting in (6.6) (the logarithmic estimate of $\|\nabla v(\cdot,t)\|_\infty$ in terms of $q(D_t)$) the a priori estimates (7.4) and (7.5) we get

$$||\nabla v(x,t)||_\infty \leq C + C \int_0^t ||\nabla v(\cdot,\tau)||_\infty \, d\tau,$$

which yields by Grönwall's inequality

$$||\nabla v(x,t)||_\infty \leq C\, e^{Ct}, \quad -T < t < T. \tag{7.15}$$

Upon introducing this in (7.3) and (7.4) we conclude that we have the double exponential estimate

$$q(D_t) = \frac{\|\nabla\Phi(\cdot,t)\|_\gamma}{|\nabla\Phi(\cdot,t)|_{inf}} \le C\exp(e^{Ct}), \quad -T < t < T. \tag{7.16}$$

For fixed t, the flux $\alpha \to X(\alpha,t)$ is a bilipschitz homeomorphism. In particular, we have [43, (4.47),p.149]

$$||\nabla X(\alpha,t)||_\infty \le \exp\left(\int_0^t ||\nabla v(\cdot,s)||_\infty \, ds\right), \tag{7.17}$$

and so

$$\begin{aligned}\sigma(\partial D_t) &= \int_{\partial D_0} \Big| \bigwedge_{j=1}^{d-1} DX(\beta,t)(T_j(\beta)) \Big| \, d\sigma(\beta) \\ &\le \int_{\partial D_0} (d-1)^{\frac{1}{2}} \, \|DX(\cdot,t)\|_\infty^{d-1} \, d\sigma(\beta) \\ &\le (d-1)^{\frac{1}{2}} \, \exp\left((d-1)\int_0^t ||\nabla v(\cdot,s)||_\infty \, ds\right) \sigma(\partial D_0).\end{aligned}$$

Hence, by (7.15),

$$\sigma(\partial D_t) \le (d-1)^{\frac{1}{2}} \, \sigma(\partial D_0) \, \exp(C\, e^{Ct}), \quad -T < t < T. \tag{7.18}$$

A similar estimate of the diameter of D_t follows from (7.15) and (7.17):

$$\operatorname{diam}(D_t) \le \operatorname{diam}(D_0) \, \exp(C\, e^{Ct}), \quad -T < t < T. \tag{7.19}$$

We can combine the estimates (7.16), (7.18) and (7.19) with Theorem 2.1 to show that D_t is a $C^{1+\gamma}$ domain for all $t \in \mathbb{R}$ and thus complete the proof of our main result. For that, assume that $T < \infty$ is the maximal time for which D_t is a $C^{1+\gamma}$ domain for all $t \in (-T,T)$. Since the size of the interval in which the local solution of Theorem 2.1 exists depends only on the quantities $q(D_0), \sigma(\partial D_0)$ and $\operatorname{diam}(D_0)$ and these quantities are uniformly bounded for D_t with $t \in (-T,T)$, we can apply Theorem 2.1 with initial condition D_{τ_0} at time τ_0 (not at time 0!) with τ_0 sufficiently close to T so that the new interval of existence goes beyond T. This contradicts the maximality of T.

Notice that we have concluded the proof without proving any a priori estimate for $\|DX(\cdot,t)\|_{\gamma,\partial D_0}$. This is the reason why we took particular care in estimating the local time of existence in terms of quantities related to the smoothness of the initial domain D_0. A final remark on inequality (7.19) is in order. One can easily prove the much better estimate $\operatorname{diam}(D_t) \le C_d \operatorname{diam}(D_0)$, with C_d a dimensional constant, using the straightforward potential theoretic estimate $|v(x,t)| \le C_d \, |D_t|^{1/d}, \ x \in \mathbb{R}^d, \ t \in \mathbb{R}$, and the fact that $|D_t| = e^{-t}|D_0|$.

8 The gradient of Φ has no jump

In this section we prove that the function $\Phi(x,t)$ defined by (7.2) is of class $C^{1+\gamma}(\mathbb{R}^d)$.

Let $X(\alpha,t)$ be the Yudovich flow (1.10) and (1.11). The initial domain D_0 is bounded and has boundary of class $C^{1+\gamma}$, $0 < \gamma < 1$. By Theorem 2.1 we know that for some $T > 0$

$$X(\alpha,t) \in C^1\big((-T,T), C^{1+\gamma}(\partial D_0, \mathbb{R}^d)\big). \tag{8.1}$$

Fix a time t, $0 < t < T$. By (6.6) $\|\nabla v(\cdot,t)\|_\infty$ is finite and controlled by the constant $q(D_t)$ describing the $C^{1+\gamma}$ character of ∂D_t. In view of (8.1) this constants are uniformly bounded for $|\tau| \le t$. Hence, by (6.6),

$$\|\nabla v(\cdot,\tau)\|_\infty \le q, \quad |\tau| \le t, \tag{8.2}$$

for a positive constant q depending only on t. We know from Lemma 6.1 that the entries of the matrix $\nabla v(\cdot,t)$ are given by singular integrals with even kernels plus a scalar multiple of χ_{D_t}. By the main lemma of [45] $\nabla v(\cdot,t)$ satisfies a Hölder condition of order γ on each of the open sets D_t and $\mathbb{R}^d \setminus \overline{D_t}$, in spite of having a jump at ∂D_t. Again the constants of this Hölder conditions are controlled by the $C^{1+\gamma}$ character of ∂D_t. Therefore, for some other constant q depending only on t,

$$\|\nabla v(x_1,\tau) - \nabla v(x_2,\tau)\| \le q\, |x_1 - x_2|^\gamma, \quad |\tau| \le t, \tag{8.3}$$

provided $x_1, x_2 \in D_\tau$ or $x_1, x_2 \in \mathbb{R}^d \setminus \overline{D_\tau}$. The estimates (8.2) and (8.3) imply that $\nabla X(\cdot,t)$ extends continuously to ∂D_0 from either side. In the same vein, $\nabla X^{-1}(\cdot,t)$ extends continuously to ∂D_t from either side. This follows from standard estimates and Grönwall's inequality, as we recall below for the sake of the reader. One starts by

$$X(\alpha,t) = \alpha + \int_0^t v(X(\alpha,\tau),\tau)\, d\tau.$$

Using (8.2) we obtain

$$|X(\alpha,t) - X(\beta,t)| \le |\alpha-\beta| + q \int_0^t |X(\alpha,\tau) - X(\beta,\tau)|\, d\tau,$$

which yields by Grönwall

$$|X(\alpha,t) - X(\beta,t)| \le |\alpha-\beta|\, e^{qt}.$$

Since

$$\nabla X(\alpha,t) = I + \int_0^t \nabla v(X(\alpha,\tau),\tau) \circ \nabla X(\alpha,\tau)\, d\tau,$$

we get

$$\|\nabla X(\cdot,t)\|_\infty \le e^{qt}$$

and

$$\|\nabla X(\alpha,t) - \nabla X(\beta,t)\| \le q\, |\alpha-\beta|^\gamma \int_0^t e^{q(\gamma+1)\tau}\, d\tau + \int_0^t q \|\nabla X(\alpha,\tau) - \nabla X(\beta,\tau)\|\, d\tau,$$

provided $\alpha,\beta \in D_0$ or $\alpha,\beta \in \mathbb{R}^d \setminus \overline{D_0}$. Thus

$$\|\nabla X(\alpha,t) - \nabla X(\beta,t)\| \le \frac{1}{\gamma} e^{q(\gamma+1)t} |\alpha-\beta|^\gamma,$$

for $\alpha,\beta \in D_0$ or $\alpha,\beta \in \mathbb{R}^d \setminus \overline{D_0}$.

Since

$$\frac{d}{d\tau} X^{-1}(x,\tau) = -v(X^{-1}(x,\tau), t-\tau)$$

the same argument applied to X^{-1} yields

$$\|\nabla X^{-1}(x,t) - \nabla X^{-1}(y,t)\| \le \frac{1}{\gamma} e^{q(\gamma+1)t} |x-y|^\gamma,$$

for $x, y \in D_t$ or $x, y \in \mathbb{R}^d \setminus \overline{D_t}$.

For $x \in \partial D_t$ set

$$M = M(x,t) = \lim_{D_t \ni y \to x} \nabla X^{-1}(y,t)$$

and

$$N = N(x,t) = \lim_{\mathbb{R}^d \setminus \overline{D_t} \ni y \to x} \nabla X^{-1}(y,t).$$

Then $M(x,t)$ and $N(x,t)$ are linear mappings from $\mathbb{R}^d$ into itself that depend continuously on $x \in \partial D_t$. Let $\mathrm{Tan}(\partial D_t, x)$ stand for the tangent space to ∂D_t at the point $x \in \partial D_t$.

Lemma 8.1. *For $x \in \partial D_t$ the linear mappings M and N coincide on $\mathrm{Tan}(\partial D_t, x)$ with the differential at x of $X^{-1}(x,t)$, viewed as a differentiable mapping from ∂D_t onto ∂D_0. In particular, M and N map $\mathrm{Tan}(\partial D_t, x)$ into $\mathrm{Tan}(\partial D_0, X^{-1}(x,t))$.*

Proof. Let T be a tangent vector to ∂D_t at x, $a > 0$ and $z\colon (-a,a) \to \partial D_t$ a curve of class C^1 such that $z(0) = x$ and $z'(0) = T$. Let $\vec{n}$ be the exterior unit normal vector to ∂D_t at x. Let $0 < \varepsilon < a$ so small that $z(\theta) - \eta\vec{n} \in D_t$ provided $|\theta| < \varepsilon$ and $0 < \eta < \varepsilon$. Then

$$X^{-1}(z(\theta) - \eta\vec{n}, t) \xrightarrow{\eta \to 0} X^{-1}(z(\theta), t) \text{ uniformly in } \theta \in (-\varepsilon, \varepsilon).$$

Hence

$$\frac{d}{d\theta} X^{-1}(z(\theta) - \eta\vec{n}, t) \xrightarrow{\eta \to 0} \frac{d}{d\theta} X^{-1}(z(\theta), t)$$

as distributions on $(-\varepsilon, \varepsilon)$.

On the other hand, for $\theta \in (-\varepsilon, \varepsilon)$ we have

$$\frac{d}{d\theta} X^{-1}(z(\theta) - \eta\vec{n}, t) = \nabla X^{-1}(z(\theta) - \eta\vec{n}, t)\left(\frac{dz(\theta)}{d\theta}\right) \xrightarrow{\eta \to 0} M(z(\theta), t)\left(\frac{dz(\theta)}{d\theta}\right)$$

pointwise and boundedly. Thus

$$\frac{d}{d\theta} X^{-1}(z(\theta), t) = M(z(\theta), t)\left(\frac{dz(\theta)}{d\theta}\right), \quad \theta \in (-\epsilon, \epsilon),$$

which, for $\theta = 0$, gives

$$DX^{-1}(T) = M(T),$$

where DX^{-1} is the differential at x of X^{-1} as a smooth map from ∂D_t into ∂D_0. The argument for N is similar. The proof is complete. $\square$

Let Φ_0 be a defining function for D_0. Then $\Phi_0 \in C^{1+\gamma}(\mathbb{R}^d)$, $D_0 = \{x \in \mathbb{R}^d : \Phi_0(x) < 0\}$, $\partial D_0 = \{x \in \mathbb{R}^d : \Phi_0(x) = 0\}$ and $\nabla\Phi_0(x) \neq 0$, $x \in \partial D_0$. Set

$$\varphi(x,t) = \Phi_0(X^{-1}(x,t)), \quad x \in \mathbb{R}^d,$$

so that $D_t = \{x \in \mathbb{R}^d : \varphi(x,t) < 0\}$, $\partial D_t = \{x \in \mathbb{R}^d : \varphi(x,t) = 0\}$ and $\nabla\varphi(y,t) = \nabla\Phi_0 \circ \nabla X^{-1}(y,t)$, for $y \notin \partial D_t$. Our next task is computing the jump of $\nabla\varphi$ at ∂D_t.

Let $\vec{n}$ and $\overrightarrow{n_0}$ be the exterior unit normal vectors to ∂D_t and ∂D_0 respectively, at the points $x \in \partial D_t$ and $X^{-1}(x,t) = \alpha \in \partial D_0$. Given any vector $\vec{u} \in \mathbb{R}^d$, we have

$$\lim_{D_t \ni y \to x} \langle \nabla\varphi(y,t), \vec{u} \rangle = \langle \nabla\Phi_0(\alpha), \lim_{D_t \ni y \to x} \nabla X^{-1}(y,t)(\vec{u}) \rangle = \langle \nabla\Phi_0(\alpha), M(\vec{u}) \rangle.$$

Set $\vec{u} = \lambda \vec{n} + T$, $\lambda \in \mathbb{R}$ and $T \in \operatorname{Tan}(\partial D_t, x)$, and

$$M(\vec{n}) = A\overrightarrow{n_0} + T_0, \quad A \in \mathbb{R}, \quad T_0 \in \operatorname{Tan}(\partial D_0, \alpha).$$

Then

$$M(\vec{u}) = \lambda M(\vec{n}) + M(T) = \lambda A\overrightarrow{n_0} + \lambda T_0 + M(T)$$

and, since $\lambda T_0 + M(T) \in \operatorname{Tan}(\partial D_0, \alpha)$,

$$\langle \nabla \Phi_0(\alpha), M(\vec{u}) \rangle = \lambda A \langle \nabla \Phi_0(\alpha), \overrightarrow{n_0} \rangle = \lambda A |\nabla \Phi_0(\alpha)| = A |\nabla \Phi_0(\alpha)| \langle \vec{n}, \vec{u} \rangle.$$

Therefore

$$\lim_{D_t \ni y \to x} \nabla \varphi(y,t) = A |\nabla \Phi_0(\alpha)| \vec{n}.$$

Take an orthonormal basis $\{\tau_1, \dots, \tau_{d-1}\}$ of $\operatorname{Tan}(\partial D_t, x)$ and an orthonormal basis $\{\tau_1^0, \dots, \tau_{d-1}^0\}$ of $\operatorname{Tan}(\partial D_0, \alpha)$ so that

$$\det(\vec{n}, \tau_1, \dots, \tau_{d-1}) = \det(\overrightarrow{n_0}, \tau_1^0, \dots, \tau_{d-1}^0) = 1.$$

Call D the differential of $X^{-1}(x,t)$ at x as a smooth mapping from ∂D_t into ∂D_0. Then the matrix of M in the above basis is

$$M = \begin{pmatrix} A & 0 & \dots & 0 \\ A_1 & & & \\ \vdots & & \text{Matrix of } D & \\ A_{d-1} & & & \end{pmatrix}.$$

Taking determinants

$$\det M = A \det D.$$

Now $\det M$ is the limit of $\det \nabla X^{-1}(y,t)$ as $y \in D_t$ tends to x, which turns out to be e^t. This is so because $\det \nabla X(\alpha, t) = e^{-t}$ for $\alpha \in D_0$, which in turn is due to the the well-known fact that ([43, p. 5])

$$\frac{d}{dt} \det \nabla X(\alpha,t) = \operatorname{div} v(X(\alpha,t),t) \, \det \nabla X(\alpha,t) = -\det \nabla X(\alpha,t).$$

Therefore

$$\lim_{D_t \ni y \to x} \nabla \varphi(y,t) = e^t \frac{|\nabla \Phi_0(\alpha)|}{\det D} \vec{n}.$$

Arguing similarly for the exterior side, where $\det \nabla X^{-1}(y,t)$ is 1, we conclude that

$$\lim_{\mathbb{R}^d \setminus \overline{D_t} \ni y \to x} \nabla \varphi(y,t) = \frac{|\nabla \Phi_0(\alpha)|}{\det D} \vec{n}.$$

Then, clearly, the function

$$\Phi(y,t) = e^{-t} \varphi(y,t) \chi_{D_t}(y) + \varphi(y,t) \chi_{\mathbb{R}^d \setminus \overline{D_t}}(y)$$

has no gradient jump at ∂D_t and so $\Phi(y,t)$ is a defining function for D_t of class $C^{1+\gamma}$. The material derivative of $\Phi(y,t)$ is

$$\frac{D\Phi(y,t)}{Dt} = -\Phi(y,t) \chi_{D_t}(y)$$

and $\Phi(y,0) = \Phi_0(y)$. Hence the function determined by (7.2) is of class $C^{1+\gamma}$, as desired.

Acknowledgements. *A.Bertozzi was partially supported by NSF grants CMMI-1435709 and DMS-0907931 ; J. Garnett by NSF grant DMS 1217239; T. Laurent by NSF DMS 1414396; J. Verdera by MTM2013-44699-P (MINECO) and 2014SGR75 (Generalitat de Catalunya). J.Verdera is grateful to the Department of Mathematics of UCLA for their hospitality in October 2012.*

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