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A Sequence of Polynomials for Approximating Arctangent

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NOTES

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1. INTRODUCTION. The Taylor series

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$$

was discovered by the Scotsman James Gregory in 1671 [**1**, chap. 12]. The series converges uniformly to $\arctan x$ on $[-1, 1]$; thus, we get

$$\{T_n(x)\} = \left\{ \sum_{k=0}^{n} (-1)^k x^{2k+1}/(2k+1) \right\},$$

the sequence of Taylor polynomials centered at 0 that converges to $\arctan x$ on $[-1, 1]$.

Like the Taylor polynomials for several other classical functions (e.g., $\cos x$, $\sin x$, and e^x), this sequence of polynomials is very easy to describe and work with; but unlike those Taylor sequences with factorials in the denominators of their coefficients, it does not converge rapidly for all "important" values of x. In particular, it converges extremely slowly to $\arctan x$ when $|x|$ is near 1. For example, if $x = 0.95$, we need to use T_{28}, a polynomial of degree 57, to get three decimal places of accuracy for $\arctan(0.95)$; if $x = 1$, we need to use T_{500}, a polynomial of degree 1001, to get three decimal places for $\arctan 1$. Indeed, for x in $[0, 1]$ it is easy to show that

$$|\arctan x - T_n(x)| \geq \frac{x^{2n+3}}{2(2n+3)}.$$

Thus, as $x \to 1$, $T_n(x)$ cannot approximate $\arctan x$ any better than

$$\frac{1}{2(2n+3)} = \frac{1}{2(\text{degree } T_n) + 4}.$$

The same is true near -1. It is only fair to note that $\{T_n\}$ converges to $\arctan x$ reasonably fast for x near 0.

In this note we present another elementary, easily-described sequence in $\mathbb{Q}[x]$ that approximates $\arctan x$ uniformly on $[0, 1]$ and converges much more rapidly than the sequence $\{T_n\}$. Such an approximating sequence provides, via the identities

$$\arctan x = -\arctan(-x) = \frac{\pi}{2} - \arctan\left(\frac{1}{x}\right),$$

a method of approximating $\arctan x$ for all x in $\mathbb{R}$. The approximating sequence arises from the family of rational functions $\{x^{4m}(1-x)^{4m}/(1+x^2)\}_{m\in\mathbb{N}}$.

2. THE SEQUENCE AND ITS RATE OF CONVERGENCE. We begin with an algebraic computation whose proof is easy via induction.

Lemma 1. *If* $p_1(x) = 4 - 4x^2 + 5x^4 - 4x^5 + x^6$ *and*

$$p_m(x) = x^4(1-x)^4 p_{m-1}(x) + (-4)^{m-1} p_1(x)$$

for $m \geq 2$, *then*

$$\frac{x^{4m}(1-x)^{4m}}{1+x^2} = p_m(x) + \frac{(-4)^m}{1+x^2} \qquad (m \in \mathbb{N}).$$

A calculus computation shows that $x(1-x) \leq 1/4$ on $[0, 1]$. Thus,

$$\frac{x^{4m}(1-x)^{4m}}{1+x^2} \leq \left(\frac{1}{4}\right)^{4m}$$

on $[0, 1]$, whence

$$\int_0^x \frac{t^{4m}(1-t)^{4m}}{1+t^2}\,dt \leq \left(\frac{1}{4}\right)^{4m} x \leq \left(\frac{1}{4}\right)^{4m} \qquad (x \in [0, 1]).$$

The result of the lemma can be rewritten as

$$\frac{x^{4m}(1-x)^{4m}}{1+x^2} = p_m(x) - \frac{(-1)^{m+1}4^m}{1+x^2}.$$

As a result,

$$\left|\int_0^x p_m(t) - \frac{(-1)^{m+1}4^m}{1+t^2}\,dt\right| \leq \left(\frac{1}{4}\right)^{4m}.$$

Dividing by $(-1)^{m+1}4^m$ and integrating the second term on the left we obtain

$$\left|\int_0^x \frac{(-1)^{m+1}}{4^m} p_m(t)\,dt - \arctan x\right| \leq \left(\frac{1}{4}\right)^{5m} \tag{1}$$

for all x in $[0, 1]$. It follows that

$$h_m(x) = \int_0^x \frac{(-1)^{m+1}}{4^m} p_m(t)\,dt \tag{2}$$

defines a sequence in $\mathbb{Q}[x]$ that converges uniformly on $[0, 1]$ to $\arctan x$. To get a better sense of the convergence rate, note that p_m has degree $8m - 2$, hence h_m has degree $8m - 1$. In (1) we write $4^{5m} = (4^{5/8})^{8m-1+1}$ and summarize our results in Theorem 1.

Theorem 1. *For* $m = 1, 2, \ldots$ *define* $p_m(t)$ *as in Lemma* 1 *and* $h_m(x)$ *as in* (2). *Then*

$$\left|h_m(x) - \arctan x\right| \leq \left(\frac{1}{4^{5/8}}\right)^{\deg(h_m)+1} \tag{3}$$

for all x *in* $[0, 1]$.

3. EXAMPLES, OBSERVATIONS, AND A CLOSED-FORM EXPRESSION FOR h_m. Evaluating

$$h_2(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{5\,x^9}{48} + \frac{x^{10}}{20} - \frac{43\,x^{11}}{176} + \frac{x^{12}}{4} - \frac{27\,x^{13}}{208} + \frac{x^{14}}{28} - \frac{x^{15}}{240}$$

at $x = 0.95$ and $x = 1$ we find that at both points the approximation to $\arctan x$ is within 2.28×10^{-7}, better than six decimal places of accuracy with a polynomial of much smaller degree than the Taylor polynomials mentioned in the introduction. If we consider h_7, a polynomial of degree 55, (3) guarantees that the approximation on [0, 1] is accurate to within 8.47×10^{-22}. Thus,

$$4\,h_7(1) = \frac{5061194335410645242554449}{1611028192858608556032000}$$

gives twenty digits of accuracy for π.

Like the Taylor polynomials, the h_m are one-sided approximations. Indeed, it is not hard to see that $h_m(x) - \arctan x$ is positive when m is odd and negative when m is even.

Taylor polynomials are constructed by matching a function and its derivatives at a point. *Hermite-interpolating* (or *osculating*) *polynomials* are constructed by matching a function and its derivatives at more than one point (see [**2**, sec. 3.3]). For example, we can construct a sequence $\{p_N\}$ such that $p_N^{(n)}(0) = \arctan^{(n)}(0)$ and $p_N^{(n)}(1) = \arctan^{(n)}(1)$ for $n = 0, 1, \ldots, N$. We note that such a Hermite-interpolating polynomial for $\arctan x$ cannot be in $\mathbb{Q}[x]$ because $\arctan 1 = \pi/4$. Thus the h_m are not Hermite-interpolating polynomials. Nevertheless, the next theorem shows that the h_m are similar to Hermite-interpolating polynomials in that they match derivatives of $\arctan x$ at both 0 and 1.

Theorem 2. *For* $m = 1, 2, \ldots h_m^{(n)}(0) = \arctan^{(n)}(0)$ *and* $h_m^{(n)}(1) = \arctan^{(n)}(1)$ *whenever* $1 \le n \le 4m$. *Moreover, if* $g(x)$ *is a polynomial of degree* $8m$ *such that* $g(0) = \arctan\ 0$, $g^{(n)}(0) = \arctan^{(n)}(0)$, *and* $g^{(n)}(1) = \arctan^{(n)}(1)$ *whenever* $1 \le n \le 4m$, *then* $g = h_m$.

Proof. We deal with the case $x = 1$ first. We appeal to (2) and Lemma 1 in noting that

$$\begin{aligned} h'_m(x) &= \frac{(-1)^{m+1}}{4^m} p_m(x) = \frac{(-1)^{m+1}}{4^m}\left(\frac{x^{4m}(1-x)^{4m}}{1+x^2} - \frac{(-4)^m}{1+x^2}\right) \\ &= \left(\frac{(-1)^{m+1}}{4^m}\frac{x^{4m}}{1+x^2}\right)(1-x)^{4m} + \frac{1}{1+x^2}. \end{aligned} \tag{4}$$

Using $d/dx\,(\arctan x) = 1/(1+x^2)$ on the second term in (4) and the product rule for differentiation,

$$(f(x)g(x))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)\, g^{(k)}(x),$$

on the first, we find that

$$h_m^{(n)}(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{(-1)^{m+1}}{4^m}\frac{x^{4m}}{1+x^2}\right)^{(n-1-k)} \left((1-x)^{4m}\right)^{(k)} + \arctan^{(n)}(x).$$

If $0 \le k \le n-1$, $((1-x)^{4m})^{(k)}|_{x=1} = 0$, so $h_m^{(n)}(1) = \arctan^{(n)}(1)$ when $1 \le n \le 4m$. To prove the assertion at $x = 0$, we can rewrite the first summand in (4) as

$$\left(\frac{(-1)^{m+1}}{4^m}\frac{(1-x)^{4m}}{1+x^2}\right)x^{4m}$$

and follow the same steps.

If g is a polynomial with the properties stated, then $g - h_m$ is of degree $8m$, and because $g^{(n)}(0) - h_m^{(n)}(0) = 0$ when $0 \le n \le 4m$, its first $4m$ coefficients are 0. Hence $g - h_m = x^{4m+1}q$, where q is of degree $4m-1$. Write

$$q(x) = \sum_{k=0}^{4m-1} a_k(x-1)^k.$$

Inductive use of the product rule to compute $(g - h_m)^{(k)}(1)$ demonstrates that $a_k = C_k a_0$ when $1 \le k \le 4m-1$, with $C_k \ne 0$. Therefore its use on $(g-h_m)^{(4m)}$ shows that $(g-h_m)^{(4m)}(1) = Ca_0$, where $C \ne 0$. It follows that $a_0 = 0$ and thus that $a_k = 0$ when $1 \le k \le 4m-1$. ∎

The next lemma is the key to establishing formulas for the coefficients.

Lemma 2. *For* $m = 1, 2, \ldots$ *write*

$$\frac{(1-t)^{4m}}{1+t^2} = \sum_{j=0}^{4m-2} a_j t^j + \frac{r_m(t)}{1+t^2},$$

where r_m *is a polynomial satisfying* $\deg(r_m) < 2$. *Then the following statements hold:*

(i) $r_m(t) = (-1)^m 4^m$;

(ii) $a_{2j} = (-1)^{j+1}\sum_{k=j+1}^{2m}\binom{4m}{2k}(-1)^k$ *and* $a_{2j-1} = (-1)^{j+1}\sum_{k=j}^{2m-1}\binom{4m}{2k+1}(-1)^k$.

Proof. Write

$$\frac{(1-t)^{4m}}{1+t^2} = \frac{\sum_{k=0}^{4m}\binom{4m}{k}(-1)^k t^k}{(1+t)^2}$$
$$= \sum_{k=0}^{2m}\binom{4m}{2k}\frac{t^{2k}}{1+t^2} - \sum_{k=1}^{2m-1}\binom{4m}{2k+1}\frac{t^{2k+1}}{1+t^2},$$

where we abbreviate the terms on the left and right in the last expression by S_E and S_O, respectively. Invoking the identity

$$\frac{t^{2k}}{1+t^2} = (-1)^{k+1}\sum_{j=1}^{k}(-1)^{j+1}t^{2(j-1)} + \frac{(-1)^k}{1+t^2}$$

for $k = 1, 2, \ldots,$ we write

$$S_E = \frac{1}{1+t^2} + \sum_{k=1}^{2m} \binom{4m}{2k} \left((-1)^{k+1} \sum_{j=1}^{k} (-1)^{j+1} t^{2(j-1)} + \frac{(-1)^k}{1+t^2} \right).$$

We collect the polynomial and nonpolynomial parts to obtain

$$S_E = \sum_{k=1}^{2m} \binom{4m}{2k} \left((-1)^k \sum_{j=1}^{k} (-1)^j t^{2(j-1)} \right) + \sum_{k=0}^{2m} \binom{4m}{2k} \frac{(-1)^k}{1+t^2}.$$

Because

$$\sum_{k=0}^{2m} \binom{4m}{2k} (-1)^k = (-1)^m 4^m,$$

the nonpolynomial part becomes $(-1)^m 4^m/(1+t^2)$. We change the order of summation on the polynomial part to rewrite it as

$$\sum_{j=1}^{2m} \left((-1)^j \sum_{k=j}^{2m} \binom{4m}{2k} (-1)^k \right) t^{2(j-1)}.$$

A similar procedure performed on S_O yields

$$S_O = \sum_{j=1}^{2m-1} \left((-1)^j \sum_{k=j}^{2m-1} (-1)^k \binom{4m}{2k+1} \right) t^{2j-1} + \frac{t}{1+t^2} \sum_{k=0}^{2m-1} \binom{4m}{2k+1} (-1)^k.$$

Because the second summand is zero, we have established that

$$\frac{(1-t)^{4m}}{1+t^2} = \sum_{j=1}^{2m} \left((-1)^j \sum_{k=j}^{2m} \binom{4m}{2k} (-1)^k \right) t^{2(j-1)} + \frac{(-1)^m 4^m}{1+t^2}$$
$$+ \sum_{j=1}^{2m-1} \left((-1)^{j+1} \sum_{k=j}^{2m-1} \binom{4m}{2k+1} (-1)^k \right) t^{2j-1}.$$

This identity proves both parts of the lemma. ■

Combining the results of Lemmas 1 and 2, we see that

$$p_m(t) = (-1)^m 4^m \sum_{j=1}^{2m} (-1)^j t^{2(j-1)} + \sum_{j=0}^{4m-2} a_j t^{4m+j}, \tag{5}$$

where the a_j are as in Lemma 2. The closed-form expression for h_m follows from (5) and (2).

Theorem 3. *For* $m = 1, 2, \ldots$

$$h_m(x) = \sum_{j=1}^{2m} \frac{(-1)^{j+1}}{2j-1} x^{2j-1} + \sum_{j=0}^{4m-2} \frac{a_j}{(-1)^{m+1} 4^m (4m+j+1)} x^{4m+j+1},$$

where

$$a_{2i} = (-1)^{i+1} \sum_{k=i+1}^{2m} \binom{4m}{2k} (-1)^k$$

and

$$a_{2i-1} = (-1)^{i+1} \sum_{k=i}^{2m-1} \binom{4m}{2k+1} (-1)^k.$$

The theorem makes it easy to use a computer for the computation of the h_m. The author's website (`http://myweb.lmu.edu/hmedina`) contains *Mathematica* code for this computation and more details related to this note.

4. FURTHER REMARKS AND QUESTIONS. The keys to the approximating sequence $\{h_m\}$ are that the family of polynomials $x^{4m}(1-x)^{4m}$ $(m \in \mathbb{N})$ leaves an integer remainder when divided by $1+x^2$ and that the members of the family are small for x in $[0, 1]$. There are other families of polynomials with this property, but is there another simple one that converges more quickly than $\{h_m\}$ to $\arctan x$? Is there one with the desirable factorials in the denominator of the error bound?

The results herein were stumbled upon after the author became intrigued by and curious about the fact that

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi;$$

that is, $4(h_1(1) - \arctan 1) = \frac{22}{7} - \pi$. Is there a simple closed-form expression for $4h_m(1)$? If so, it would provide a sequence in $\mathbb{Q}$ for approximating π. Another problem, probably a very difficult one, is to find an easily describable sequence $\{g_n\}$ such that $4g_n(1)$ is always a convergent in the continued fraction expansion of π.

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A Cantor-Bernstein Theorem for Paths in Graphs

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The Cantor-Bernstein theorem says that if for two infinite sets A and B there are injective functions $f : A \to B$ and $g : B \to A$, then there is a bijection $A \leftrightarrow B$. Perhaps its