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The Development of Notation in Mathematical Analysis

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Introduction

Modern students of mathematics take for granted the comparative ease with which difficult concepts, definitions, and theorems are related to them in university courses. Prior to the relatively recent call for the standardization of mathematical language, different symbols and words for the same idea abounded, making it nearly impossible for mathematicians from different regions to communicate effectively and slowing the growth of mathematics. The field of analysis is a newer subject in mathematics, as it only came into existence in the last 400 years. With a new field comes new notation, and in the era of universalism, analysis becomes key to understanding how centuries of mathematics were unified into a finite set of symbols, precise definitions, and rigorous proofs that would allow for the rapid development of modern mathematics. This paper will trace the introduction of subjects and the development of new notations in mathematics from the seventeenth to the nineteenth century that allowed analysis to flourish.

Beginning in the seventeenth century with the introduction of analytic geometry by René Descartes, and independently Pierre de Fermat, notations and concepts that would allow for the development of calculus, and then analysis, came into being. Descartes was the first mathematician to use numerical superscripts to denote higher powers, a key development that simplified products and later allowed for Isaac Newton and Gottfried Wilhelm Leibniz to preform calculus concisely. Fermat also helped make calculus analysis conceivable through his developments in maxima, minima, tangents, infinitesimals, and limits. From the work of these predecessors, Newton and Leibniz, were able to develop calculus in the late seventeenth century. Newton likely developed calculus first, but his notations for "fluxions and fluents" were less approachable than Leibniz's later symbols $\int$ and *d* for integrals and derivatives. Thus, despite an international argument to determine whose work was done first, Leibniz came out on top because of the legibility and adaptability of his symbols in a difficult subject.

The eighteenth century arguably witnessed the first instances of true analysis. Leonhard Euler developed the symbol f(x) to denote a function and first used Σ for summations, notations that made equations easier to work with; he also helped standardize notations through his prolific work, which was read internationally. Building on the notations that came before him, Augustin-Louis Cauchy developed the concepts of limits (denoted by “lim.”) and continuity that would allow him to introduce rigor into analysis and allow the subject to grow in the nineteenth century. Karl Weierstrass was then able to make Cauchy’s development more rigorous by introducing his ε-δ language, and Richard Dedekind later constructed the real number line through the method of “Dedekind cuts,” also giving us the notation for the set of real numbers, $\mathfrak{R}$, later to be displaced by $\mathbb{R}$. In addition, Dedekind introduced the term “irrational numbers.”

In the twentieth century, an attempt was made to standardize notations in analysis, a well-defined field, through the efforts of Nicolas Bourbaki. In their mission to “redo” mathematics and ground it in set theory, the Bourbaki group called for precision, rigor, and a unification of symbols that would make mathematical language universal. Thus, they standardized much of the earlier developments in analysis and also developed new symbols that would help advance the field, namely $\notin$ (not an element of), $\varnothing$ (the empty set), $\Rightarrow$ (implication), and $\Leftrightarrow$ (equivalence); they also introduced the term “bijection.”

By tracing the mathematical developments that led to analysis, one can see the effect notations have on modern mathematics. Understanding which symbols failed and which endured, and how mathematical language was standardized to increase communication and developments among mathematicians from different geographical regions, modern students can recognize how the achievements of the past have allowed for the rapid growth of, and the comparative ease of learning, mathematical analysis.

The Early History of Modern Analysis

The modern foundations of mathematical analysis were established in seventeenth century Europe when René Descartes and Pierre de Fermat independently developed analytic geometry. By analyzing the developments made and the notation used by each mathematician, undergraduate students can begin to recognize the trends that led to calculus and then to analysis. Furthermore, by gaining an understanding of the history and presentation of mathematics prior to the advent of analysis, modern students might realize how much easier it is to understand the difficult concepts introduced, as standardized notation permits simpler communication of the ideas. The understanding of analysis, as preceded by analytic geometry and calculus, "is easily enhanced through an ingenious interpretation of [Descartes'] statements in terms of later symbolisms and concepts, thus implying a specious modernity of viewpoint" (Boyer 458). With Descartes' and Fermat's developments, it became clear how heavily analytic geometry and calculus rely "upon a felicitous choice of notations" and how the "repeated use such notations have come to be associated implicitly with the ideas which they are now intended to represent" (Boyer 458). Thus, by uncovering how Descartes and Fermat developed and presented analytic geometry, one can see how calculus began and how notation became essential to mathematical communication and development.

Analytic geometry is essentially the application of algebra to geometry; it is concerned with defining and representing geometrical shapes in a numerical way. It appears that, before the seventeenth century, there was no conception "that in general an arbitrary *given equation* involving *two unknown quantities* can be regarded as a determining *per se*, with respect to a coordinate system, a plane curve," and the recognition of this "together with its fabrication into a formalized algorithmic procedure, constituted the decisive contribution of Fermat and Descartes" (Boyer 462). Since neither mathematician published his findings early on in their development, "both men were in independent possession of their methods well before [their publication] time––about 1619 for Descartes and 1629 for Fermat" (Boyer 462). Yet, since Descartes had a series of dreams that first led him to the Cartesian plane, "November 10, 1619, then, is the official birthday of analytic geometry and therefore also of modern mathematics" (Bell 40).

Thus, in a sense, it was Descartes' *Geometry* that "made possible the other great mathematical breakthrough of the [seventeenth] century," namely calculus (Fauvel 337). As E. T. Bell puts it, the invention of analytic geometry allowed "*algebra and analysis … to be our pilots to the unchartered seas of 'space' and its 'geometry'*" (Bell 54). Furthermore, Descartes' algebraic contribution to mathematics aided "in the formation of generalizations which were to culminate in the formal algorithms of the calculus invented by Isaac Newton and Wilhelm Leibniz" (Fauvel 337). It is important to remember that Fermat's work also allowed for the development of calculus, as "each of them, entirely independently of the other, invented analytic geometry" (Bell 56). Descartes and Fermat did correspond "on the subject but this does not affect the preceding assertion" that each invented analytic geometry on his own (Bell 56). We will see that their correspondence actually allowed Fermat to add more to the field and come within a hair's breadth of calculus.

Aside from the overall invention of analytic geometry, some of the most important developments contributed to analysis by Descartes and Fermat were notational. Descartes had some developments that were better than others, as we will see through the rejection of his symbol for equality. However, we have Descartes to thank for the notation of higher powers that is universally recognized in modern mathematics. François Viète, Descartes' mathematical predecessor, was known for refining many notations used in algebra and giving mathematicians some symbols still used today. Like Viète, "Descartes advocated using letters and formal manipulations with symbols to analyse geometrical problems, but his algebraic analysis was both literally and conceptually easier to use" (Fauvel 336). Perhaps this improvement came about because a limitation of earlier algebra was "the lack of notation for higher powers of multiple simultaneous unknowns (rather than a single one)," a limitation Descartes' geometric and algebraic analysis required him to overcome (Manders 189). Thus, Descartes followed Viète's notation in that he "clearly displayed the powers of the unknowns as … products of the repeated multiplication of the unknown by itself," but he improved and simplified it by introducing numerical superscripts (Mahoney 43). Descartes' notation for higher powers was adopted by Newton and Leibniz, and it allowed them to integrate and differentiate more concisely, essentially making calculus legible.

Despite introducing the modern notation for higher powers, Descartes' sign for equality was rejected by both Newton and Leibniz in their calculus works. Robert Recorde invented =, the symbol for equality used today, and "the fact that both Newton and Leibniz used Recorde's symbol led to its general adoption" (Cajori I: 306). If Leibniz had "favored Descartes' ∞, then Germany and the rest of Europe would probably have joined France and the Netherlands in the use of it," and England would have adopted it when they finally converted from Newton's to Leibniz's calculus (Cajori I: 306). Although Descartes developed many new notations, only the ones favored by Leibniz lived on, and "the final victory of = over ∞ seems mainly due to the influence of Leibniz during the critical period at the close of the seventeenth century" (Cajori I: 307).

Fermat's improvements in analytic geometry were the direct consequence of a slight feud between the mathematicians, and offered more important ideas (rather than important notations) to Newton and Leibniz. Fermat criticized Descartes' work, unaware that he was contributing to a conspiracy by some of his peers to destroy Descartes' composition. Specifically, he "corrected Descartes in an essential point (that of the classification of curves by their degree)," which allowed Fermat to apply analytic geometry to three dimensions while "Descartes contented himself with two dimensions" (Bell 63-4). This lead to a debate between the two men, but "Descartes learned nothing from it, even though he was in error; Fermat, even though he was correct, gained new mathematical insights that led him to revise his methods and sharpen his tools" (Mahoney 171). In particular, the debate with Descartes was the "integral part of the history of Fermat's methods of maxima and minima and of tangents," which were central to the invention of calculus (Mahoney 171). For example, by finding the slope of the required tangent line at a point on a curve, Fermat found a limiting value for a limiting position (Bell 60). Furthermore, Fermat discovered that, at maxima and minima, this limiting value is zero, another important contribution to calculus (Bell 62). In fact, in a letter he sent, "Newton says explicitly that he got the hint of the method of the differential calculus from Fermat's method of drawing tangents" (Bell 64). Thus, "Fermat conceived and applied the leading idea of the differential calculus thirteen years before Newton was born and seventeen before Leibniz was born" (Bell 56). However, he did not invent calculus because "one cannot say with any degree of fairness or objectivity that Fermat's work in analysis of curves was … pointed toward the concept that underlies the calculus as its fundamental theorem" (Mahoney 282). In fact, his "analyses are often incomplete, stopping at a problem to which he assumes the reader knows the solution" (Mahoney 47). Overall, if Fermat was not looking for a particular answer, he would not find it; rather, he would focus on the problem he wanted a solution for and solve it. Thus, he lent much to analysis when he "began to operate with infinitesimals and limit procedures," but it would be up to mathematicians like Newton and Cauchy to synthesize that knowledge (Mahoney 47).

The invention of analytic geometry was a turning point for mathematicians, who suddenly found their aspirations and interests directed toward analysis. Without Descartes' initial idea for and development of the notation for higher powers, Newton and Leibniz would not have been able to concisely represent calculus. Furthermore, they would not have had all of the tools necessary to invent calculus at the ready without Fermat's developments in maxima, minima, tangents, infinitesimals, and limits. By beginning the trend of standardizing mathematical symbolism and by inventing the field of analytic geometry, Descartes and Fermat became the founders of analysis without realizing the full impact their works would have on modern mathematics.

The Development of Calculus

In the field of analysis, three men from "among the seventeenth-century mathematicians active in the development of modern notations [played] a prominent rôle—… Descartes, Leibniz, and Newton" (Cajori II: 180). We have already discussed Descartes' contribution through his development of analytic geometry. His advancements unlocked the door to calculus, which Fermat pushed open, and which Newton and Leibniz stepped through together after a little pushing and shoving. Since neither Newton nor Leibniz published his work early on in its development, it is unclear who actually invented calculus first, but we could speculate that

Newton would have had the idea first. However, it was Leibniz that "invented the notation of calculus and allowed [for] its further development" (Bardi 180). For example, by the end of the seventeenth century, Leibniz's "calculus was successful in various applications used by others with Leibniz's blessing, and the fact that it continued to be developed was strong testimony to Leibniz's methods" (Bardi 180). Yet, Newton's methods were not so widespread, and he "seemed less interested in promoting his fluxions and fluents than in securing the rights of their invention for himself; moreover, his notation was inferior to Leibniz's" (Bardi 180). In order to understand why the invention of calculus was so widely contested and how Leibniz ended up "winning," the notations of each man must be studied.

Newton was undoubtedly a revolutionary genius and prolific in his mathematical advancements. However, he was less interested in distributing his work to other mathematicians, focusing instead on creating it. Thus, it came as no surprise when,

> in his 'Account,' Newton attacked and devalued one of Leibniz's greatest contribution[s] to mathematics: his invention of the symbols of calculus, which had greatly enhanced the ability of mathematicians to learn and apply the methods of calculus that are still in use today (Bardi 215).

Newton believed his work was superior because he did "not confine himself to symbols" (Bardi 215). While such an argument could be relevant in modern mathematics, where creativity may be confined by the over-standardization of symbology with the rise of machine language, Newton's argument would not have held up in the seventeenth century. This is because mathematicians desperately needed to confine their publications to a universal language equipped for distribution to the growing, as most mathematicians were inventing new symbols with every publication regardless of whether or not a symbol had been assigned to an idea.

Newton's greatest achievement for calculus was in his method, especially "his emphasis on the tangent as the instantaneous direction of motion along the curve; and his discovery of a pattern in the results which yielded him an algorithm," thoughts which allowed his use of infinite series to surpass Descartes' (Fauvel 380). Not long after this discovery, Newton realized the Fundamental Theorem of Calculus: "that quadrature problems were inverse to tangency problems, and he was then in possession of what can be called the Newtonian calculus" (Fauvel 380). More specifically, "Newton called his discovery the Method of Fluxions and described it in terms of geometry" (Schrader 509). At this point, we reach Newton's failure in calculus: notation. Florian Cajori explains how Newton denoted his fluxions and fluents and why they failed:

> he gave $\overset{||}{x}, \overset{|}{x}, x, \dot{x}, \ddot{x}, \dddot{x}$, each of these terms being the fluxion of the one preceding, and the fluent of the one that follows. The $\overset{|}{x}$ and $\overset{||}{x}$ are fluent notations. His notation for the fluxions of fractions and radicals did not meet with much favor because of the typographical difficulties (II: 197-8)

Unfortunately, upon comparison, Newton's notation was not the only thing separating his work from Leibniz's. Newton "approached the idea of the variation in a function in terms of bodies in motion and the concepts of speed and acceleration, [whereas] Leibniz used the idea of mathematical infinitesimals in his approach," making his work more comprehensive than Newton's calculus (Aczel, *Wilderness* 125). Yet, there is some irony in this, given the controversy about the difficulties and contradictions in differentials, as criticized later by George Berkeley, in both Newton and Leibniz's works (Berlinghoff 46).

Thus, we advance toward Leibniz's calculus. We have already noted that the notation Leibniz had invented was the more useful and superior notation which allowed for Johann

Bernoulli and other European mathematicians to advance calculus throughout the seventeenth century (Bardi 144). But why was this so? For Leibniz, a key ingredient in the invention of calculus "was his interest in logic and language, for it led him to think deeply about the basic processes involved and to devise a notation which, by capturing an underlying unity, made his discoveries easy to use" (Fauvel 424). He "made a prolonged study of matters of notation … [and] experimented with different symbols, corresponded with mathematicians on the subject, and endeavored to ascertain their preferences" (Cajori II: 180-1). He correctly supposed that his notation would make for the easy development of calculus, an idea proven by the fact that his symbols from 1675 are found in every modern calculus textbook (Bardi 244). Leibniz' "way of calculating with symbols—truly, a *calculus*—and its scope and power is illustrated by the range of problems Leibniz tackled with it" (Fauvel 424).

The symbols that Leibniz invented for differential and integral calculus first appeared on October 29, 1675 when he thought of the integral sign. Leibniz saw integration as "summation, which is why he gave it his symbol, '∫,' which is a fancy *S* that he invented" (Bardi 86). The symbol "provided a general way to treat infinitesimal problems of calculus" and was one of the reasons Leibniz's work spread (Bardi 86). However, despite becoming a master mathematician through his creation of a clear, compact language for calculus, Leibniz did not publish his work for nearly a decade, keeping it from wide use longer than necessary (Bardi 87).

A key development in Leibniz's calculus aside his notation was the realization that integration and differentiation "'are each other's converse'" (Bardi 123). In fact,

> once it was proved that differentiation and summation were reciprocal operations, Leibniz allowed himself to consider *d* and ∫ as fully-fledged assemblers, just as were sum and difference, product and quotient, and also powers and roots. In supplement to the six

> operations of Descartes' algebra, Leibniz thus considered that he himself added a new pair of reciprocal ones, thus constituting a calculation framework of his own, with eight operations, exceeding and encompassing the ancient scheme (Serfati 84).

It is helpful to note that, although these discoveries and his notations were published in his 1684 and 1686 papers, "the term integral was first used in a paper by one of the Bernoulli brothers in 1690 and 'integral calculus' first appeared as a term in a paper written by Johann Bernoulli with Leibniz in 1698" (Bardi 123). Now that we know both Newton and Leibniz's methods and symbols, a side by side comparison of their work, as presented by Dorothy V. Schrader, reveals much about the two:

> Instead of the flowing quantities and velocities of Newton, Leibniz worked with infinitely small differences and sums. He used the now familiar $\frac{dy}{dx}$ instead of Newton's dotted letters for the derivate symbol; he used $\int$ for his integration symbol while Newton used either words or a rectangle enclosing the function. Newton himself asserted that his 'prickt' letters were equivalent to Leibniz's $\frac{dy}{dx}$ and that by $\boxed{\frac{aa}{64x}}$ he meant the same thing that Leibniz meant by $\int \frac{aa}{64x}$. Leibniz was more interested in developing a notation for his new method than was Newton (510).

Here, we see how closely the two were related, proving that each did have a way of doing calculus. Furthermore, we see the ease and simplicity of Leibniz's notation, which is more useable and expandable than Newton's.

Though the two mathematicians, their colleagues, and their countries fought considerably over who invented calculus first and whose method was better, "there is no evidence that Newton borrowed from Leibniz; there is little evidence that Leibniz borrowed from Newton" (Schrader 516). We may suppose that Newton invented fluxions at least ten years before Leibniz developed the calculus," but both men are honored equally as two independent inventors today (Schrader 516). As for the development of analysis, "England seems to have been the loser" (Schrader 519). As the world of mathematics progressed following the seventeenth century invention of calculus, "German and French mathematicians established reputations for themselves and their countries, while England remained insular and isolated," stuck in Newton's limited calculus if only for their reverence for their genius mathematician (Schrader 519). The delay in British mathematics prompted the publication of G. H. Hardy's *A Mathematician's Apology*, in which he lamented that the Newton-Leibniz controversy separated the British from continental European work. The international preference for Leibniz's symbols became apparent when L'Hospital published the first calculus textbook in 1696, as he used Leibniz's notation and cemented the prolonged use of $\int$ and $\frac{dy}{dx}$ in mathematics. Ultimately, "both Leibniz and Newton are equally credited with independently developing the modern theory of calculus based on the work of Eudoxus, Archimedes, Fermat, Descartes, and other mathematicians," and the work of both mathematicians were key to the beginnings of modern analysis (Aczel, Wilderness 125).

The Beginning of Analysis

While "the seventeenth century brought immense advances in mathematics, which took the pioneering work of the ancient Greeks and catapulted it into the modern age, culminating in the birth of the calculus," it was in the eighteenth century that "calculus was taken to higher levels of understanding, application, and abstraction, and mathematical analysis as we know it today was born" (Aczel, *Wilderness* 137). The first step in developing this modern analysis was applying it widely, a feat achieved through the abundant publications of Leonhard Euler. Furthermore, a new field needs new notations, another area in which Euler was able to lead other mathematicians, as seen through his development of the symbol f(x) to denote a function and Σ for summations. However, it is through Augustin-Louis Cauchy that analysis really sets itself apart from other mathematics, as he developed the concepts of limits and continuity that would allow the subject to grow. In their work, both Euler and Cauchy were able to refine calculus and introduce analysis to the mathematical world.

Developing a new field in mathematics is no easy feat, especially when publications are not widely disseminated and notations differ from one mathematician to another. Thus, the proliferation of Euler's work provided the ideal, and only possible, way to introduce analysis to the wider world; had any other mathematician attempted to reach a wide audience and make his notations conventional, he would have failed. In other words, symbols develop slowly, often depending on the success of the mathematician who presents them: "if his or her work is widely read and the symbol is appealing, other authors adopt it," if not, the symbol is lost (Stallings 232). Since Euler was "the most prolific writer ever to have written about mathematics [he] may have introduced more of the symbols used today than any other mathematician" (Stallings 235). Since seventeenth century mathematicians laid the foundation for uniform mathematical symbology, Euler was able to introduce many notions in the eighteenth century, including his most famous symbols: f(x), Σ, π, *e*, and *i*. Only f(x) and Σ will be discussed here given their relevance to modern calculus and analysis. The fact that Euler developed so many notations besides the ones discussed here is extremely impressive, as, "excepting Leibniz and Euler, no mathematician has invented more than two ideographs which are universally adopted in modern mathematics" (Cajori II: 337).

Many of the notations we use today rely on the symbology that Euler introduced in the eighteenth century. The first use of f(x) can be seen in a pre-calculus textbook that Euler published, in which he “emphasized the idea of a function,” and many of the conventions and notations we still use were introduced in his other books (Berlinghoff 45). For example, the development of the modern notion of an integral is linked to the evolution of the function, which Euler “conceived of the notion [for] in a fairly general way” by working with curves and finding that $y = f(x)$ (Bourbaki, *Elements History* 219). This notation for a function is still present in analysis, as is Euler’s use of Σ to denote a summation. This notation can be seen in his work on the infinite series, such as those for sin(x) and cos(x) (Struik 120). Without Euler’s notations of f(x) and Σ, integrals and infinite series would be much more difficult to compute, and Cauchy’s work on limits and continuity would have been much more difficult to publish, if he could have done it at all. Thus, it is fitting that Euler’s “contemporaries called him ‘analysis incarnate,’” as without his work in, and devotion to, the field, it may not have developed as quickly as it did (Aczel, *Wilderness* 142).

With the foundation for analysis laid and the notation for analysis offered by Euler, Cauchy was able to begin his innovative work that would mark analysis as a necessary field. In fact, one of the major interests that modern mathematics is indebted to Cauchy for is “the introduction of rigor into mathematical analysis,” something that was missing in previous work (Bell 271). He also led mathematicians toward the “‘arithmetization’ of analysis which later became the core of Weierstrass’ investigations” (Struik 152). Cauchy wanted to ground his work in absolute certainty, and thus the definitions and proofs that mark the beginning of analysis are among the most thorough of any new field. In fact, it was Cauchy’s goal to

> ‘do calculus right.’ For the first time, there were *definitions* of the derivative and the integral. For the first time, the Fundamental Theorem of Calculus was highlighted as indeed fundamental. And, much as we do today, Cauchy emphasized the algebraic side of

calculus, preferring computations to diagrams and formulas to geometric intuitions (Berlinghoff 49).

In terms of "doing calculus right," Cauchy contributed much to the standardization of the field by using notations that had been used before. For example, if we look at Cauchy's "published lessons of 1823 on the infinitesimal calculus, and his lessons of 1829 on the differential calculus, we find that he availed himself of the Leibnizian dx, dy, $\frac{dy}{dx}$ and also of the Lagrangian F′ and y′ for the first derivatives" (Cajori II: 217). By employing symbols that were already recognizable, he set a standard which has prevailed widely down to our own day (in that mathematicians build upon what has already been given).

Aside from improving calculus, Cauchy also introduced new concepts hinted at by Euler that would come to define analysis. One development of Cauchy's was the limit. Cauchy defined the limit as follows:

> When the values successively attributed to the same variable approach a fixed value indefinitely, in such a way as to end up by differing from it as little as one could wish, this last value is called the *limit* of all the others. So, for example, an irrational number is the limit of the various fractions which provide values that approximate it more and more closely (Fauvel 566).

In addition to defining the limit, Cauchy also provided mathematicians with the notation still used today. Florian Cajori traces Cauchy's use of the limit and the development of the notation, which has only been adjusted slightly for ease:

> A. L. Cauchy wrote 'lim.' and pointed out that 'lim. (sin. x)' has a unique value 0, while 'lim. (($\frac{1}{x}$))' admits of two values and 'lim. ((sin. $\frac{1}{x}$))' of an infinity of values, the double parentheses being used to designate all the values that the [enclosed] function may take,

> as x approaches zero. … The period in 'lim.' was gradually dropped, and 'lim' came to be the recognized form (II: 255).

In 1823, Cauchy used his notation and that developed by Euler to define the derivative of f(x) as the limit of the quotient of differences $\frac{f(x+h)-f(x)}{h}$ as h goes to zero, when the limit exists. He also used limits to enhance the understanding of integrals by defining the integral as the limit of sums, through which "Cauchy made a good first approximation of a real proof of the Fundamental Theorem of Calculus;" it was also Cauchy who "not only raised the question, but gave the first proof, of the existence of a solution to a differential equation" (Grabiner 616-7). In short, Cauchy was able to enhance calculus and also advance analysis through the development of the limit.

Perhaps Cauchy's greatest contributions to analysis are his convergence proofs for infinite series, many of which are named after him, as in them we find the modern concept of continuity. As before, it is helpful to see his definition of the continuity of a function y = f(x), which he gave as: "*an infinitesimal change α of the independent variable x always produces an infinitesimal change f (x + α) − f (x) of the dependent variable y*" (Borovik 246). Weierstrass would later reconstruct this infinitesimal definition in terms of epsilon and delta, yet "many historians have sought to interpret Cauchy's definition as a proto-Weierstrassian definition of continuity in terms of limits" (Borovik 246). However, it is helpful to not that "Cauchy did not at any time work with 'continuity at a point', it is always continuity 'between two limits', i.e. on an interval," a hole which would affect the discoveries of Dedekind concerning continuity (Borovik 264).

The transition from calculus to limits of sequences and functions marked a major watershed in analysis and allowed for the work of Cauchy, and later Weierstrass and Dedekind, to make significant advancements in mathematics. In a sense, "Cauchy offered at last a beginning of an answer to that series of problems and paradoxes which had haunted mathematics since the days of Zeno, and he did this not by denying or ignoring them, but by creating a mathematical technique with which it was possible to do them justice" (Struik 152). Thus, it is because of Euler and Cauchy that modern analysis is so far advanced and easily communicated. Without the notational developments of f(x) and lim, for example, further studies in analysis would have been an uphill battle; luckily, Cauchy paved the way for other mathematicians to study continuity.

Approaching the Real Numbers

Following Cauchy, analysis became much wider, allowing for improvements in earlier work as well as for new developments during the nineteenth century. Karl Weierstrass was famous for his rigorous approach to mathematics, and through his work, Cauchy's definition of continuity was greatly improved by the epsilon-delta techniques. Furthermore, Richard Dedekind offered new work in continuity by constructing the real number line without any holes, a problem that had been unsolved since the Greeks rejected the idea of real numbers. He did this through the method of Dedekind cuts, in which each real number is proven to be on the number line through a widely applicable method. Through their developments in analysis, Weierstrass and Dedekind were able to introduce new notations that would become widely recognized and used in modern mathematics.

Anyone familiar with Weierstrassian rigor knows "the expressions 'ε method of proof' and 'ε-definition,' Weierstrass having begun in his early papers to use ε in his arithmetized treatment of limits" (Cajori II: 256). While Weierstrass was the first to popularize ε-δ proofs, "the epsilon was similarly used by Cauchy in 1821 and later, but sometimes he wrote δ instead, [so] Cauchy's δ is sometimes associated with Weierstrass' ε in phrases like 'ε and δ methods' of demonstration" (Cajori II: 256). The use of epsilon and delta was innovative and allowed Weierstrass to use algebraic inequalities rather than words in his theorems for analysis (Grabiner 617). In fact, he used his "own clear distinction between pointwise and uniform convergence along with Cauchy's delta-epsilon techniques to present a systematic and thoroughly rigorous treatment of the calculus" in his lectures (Grabiner 617). Thus, a great achievement of Weierstrass was his complete transformation of the basis of calculus. His "clear, precise definitions removed any trace of mystery or geometric intuition from calculus, putting it all on a logical foundation that depended only on algebra and arithmetic" (Berlinghoff 49). However, Weierstrass' approach wasn't easy "as students who have had to learn his 'epsilon-delta' approach to limits will still testify" (Berlinghoff 49).

One place where it is easy to see the achievement of Weierstrass' ε-δ language is in the revised definition of continuity. As we saw previously, Cauchy defined continuity as "*an infinitesimal change α of the independent variable x always produces an infinitesimal change f (x + α) − f (x) of the dependent variable y*" for a function y = f(x) (Borovik 246). About 50 years later, "Weierstrass reconstructed Cauchy's infinitesimal definition in the following terms: for every ε > 0 there exists a δ > 0 such that for every real α, if |α| < δ then |f(x+α) − f(x)| < ε" (Borovik 246). Instead of relying on words to define continuity, Weierstrass assigned epsilon and delta to algebraic inequalities, allowing for a more rigorous definition that reflected the ideals of analysis.

In the nineteenth century, mathematicians like Weierstrass "were in the final stages of tightening up the logical underpinnings of calculus, a process that had been going on for almost

200 years" (Berlinghoff 185). As such, they came "to a much deeper understanding of the *real numbers*, the numbers that can be used to label all the points on a coordinate axis" (Berlinghoff 185). Central to this work on real numbers and the number line was Richard Dedekind. Due to his modesty, "he did not publish his construction of the real numbers for fourteen years after he discovered it, but nonetheless it was an important event" (Fauvel 572). His modesty likely came from his position as a professor at the Technische Hochschule in Brunswick (which was equivalent to teaching at a modern high school, humbler than a research position at university), but he nevertheless constructed the rigorous theory of the irrational (Struik 162). In his papers on continuity, Dedekind was able to repair "the largest single omission in Cauchy's work, and sketched how it was now possible for the first time to give a rigorous account of the basic theorems about continuous functions" (Fauvel 572). His unique approach to solving this problem "was to use the fact that the rational numbers are ordered to create sets of rationals which define new numbers, the *real* numbers … it is rather like using families of parallel lines to define points at infinity in the projective plane" (Fauvel 572). This method was called the "Dedekind cut." Furthermore, the Dedekind cut method of defining irrational numbers is very similar to Eudoxus theory as presented in the fifth book of Euclid's *Elements*, showing how long it took to prove that the real number line is continuous even though the solution was hinted at in previous works (Struik 162).

Dedekind's theory relied greatly on rational numbers, which he used to define the irrational numbers, essentially filling the holes in the real number line. Specifically, he extended the rationals to define "new numbers by use of sets of the 'old' rational numbers," but instead of using convergent sequences like Cauchy, "he talked of partitions of the set of all rationals into two sets" (Jones 670). The central aspect of this theory of irrational numbers was the concept of the “cut” or “schnitt,” where “a cut separates *all* rational numbers into *two* classes, so that each number in the *first* class is *less than* each number in the *second* class; every such cut which does not ‘correspond’ to a rational number ‘defines’ an irrational number” (Bell 520). Thus, every such cut, “by definition was identified with a real number” (Jones 670). Through the method of Dedekind cuts, Dedekind was able to construct the real number line without any holes, creating the branch from analysis to real analysis and setting the stage for Cantor’s work with infinite sets.

Most notable here about Dedekind cuts is the notation developed with them. Rational numbers were already recognized in mathematics, but since Dedekind was dealing with numbers that were not rational, he “went on to call such cuts *irrational numbers* [emphasis added],” the first use of the term which has become common in modern mathematics (Fauvel 576). Furthermore, “the set of all cuts he called the real numbers,” which was denoted by $\mathfrak{R}$, which later became the $\mathbb{R}$ we recognize today as the set of all real numbers (Fauvel 576). Thus, Dedekind gave us the terminology for irrational numbers and the symbol for the set of real numbers.

Weierstrass and Dedekind were able to take the initial work done by Cauchy in analysis and make it more rigorous, allowing ε-δ proofs to become typical in the field and for real numbers to be employed in new ways. From the initial work of Descartes and Fermat, the calculus of Newton and Leibniz, and the beginning of analysis introduced by Euler and Cauchy, Weierstrass and Dedekind were able to expand analysis and make it one of the dominant fields in modern mathematics. But most importantly, they developed notations (namely ε, δ, $\mathfrak{R}$, and the term “irrational”) that would continue to thrive as analysis was taken deeper by later mathematicians.

Standardization of Notation

With all of the developments in analysis in the nineteenth century, mathematics was beginning to be overcrowded with differences once more. Thus, the twentieth century offered an opportunity to standardize mathematics and help students learn the same foundational material, no matter which universities they attended and professors they had. Such unification would also help standardize notation, which had developed many new symbols since the seventeenth century when Descartes first pointed mathematicians toward analysis. Thus, Nicolas Bourbaki was born, and "his" sole purpose was to revolutionize mathematics by creating a textbook, called the *Elements of Mathematics*, containing all of the fundamental details of the subject.

In the late 1930s, a group of young French mathematicians came together to revolutionize the subject, as "they felt that new ideas had not been sufficiently internalized by the mathematical community, especially in France" (Berlinghoff 55). Six brilliant mathematical men promoted their goals by collectively writing a multivolume textbook, which they called the *Elements of Mathematics* "with a nod toward Euclid" (Berlinghoff 55). Since the textbook was written collectively, they adopted a pseudonym: Nicolas Bourbaki. Bourbaki was meant to be viewed as a single entity. With their work, the Bourbaki group "had a serious common purpose: to overthrow the stagnant educational regime of the time" (Aczel, *Wilderness* 244). Their ambitious goal was to use the *Elements of Mathematics* to "*redo*" all of mathematics and prompt the abandonment of the old textbooks, which continued to create differences among mathematicians (Aczel, *Wilderness* 244).

Aside from their purpose to standardize mathematics and encourage university professors to teach students the same material despite their geographical differences, Bourbaki had many rules for their collective publications. For example,

> The founders of Bourbaki believed that creative mathematics is primarily a young person's sport, so they agreed to retire from the society before the age of fifty and elect younger colleagues to replace them. Thus, Nicolas Bourbaki [became] a renowned but mysterious international scholar, always at the peak of his professional productivity, supplying the scientific world with a continuing series of modern, clear, accurate expositions in all fields of contemporary mathematics (Berlinghoff 56).

Furthermore, in order to remain a secret society and produce high-quality publications, the group "held regular meetings in French resort towns" (Aczel, *Wilderness* 244). After a chapter was written by one member, the entire group would convene and argue every detail to ensure the work would be universally adopted, allowing for precise definitions, concise theorems, and preeminent notations. Once the group perfected their first chapter, "Nicolas Bourbaki began publishing mathematical papers and textbooks" that would replace the old and ineffectual ones (Aczel, *Wilderness* 244).

Bourbaki's influence on mathematics was "mostly felt through the *Elements* [*of Mathematics*, but, since] these took many years to write, their impact came around mid-century or later" (Berlinghoff 56). Overall, the books take an abstract point of view which is typically rejected in modern mathematics, but nevertheless "give a precise and reliable account of each of the fields they cover," leading to their twentieth-century adoption (Berlinghoff 56). Bourbaki's "volumes were meant to bring together, formalize, and make precise large chunks of mathematics, and this they did (mostly) successfully," as mathematicians saw the need to standardize the subject, and Bourbaki's work provided an ideal starting point (Berlinghoff 56).

In order to produce the effects of "'a profound faith in the unity of mathematics, and wishing to be 'universal mathematicians', Bourbaki undertook to derive the whole of the mathematical universe from a single starting point.' That staring point was the theory of sets," which had been introduced by Georg Cantor a few decades earlier (Aczel, *Artist* 99). The set theory book became the first full text written by the group, and "it set the stage for the development of all of mathematics based on the notions of a set, set membership, inclusion, and the elementary set operations of union, intersection, and symmetric difference" (Aczel, *Artist* 106). Bourbaki decided to ignore the inconsistencies in the structure of set theory, and based "all of mathematics as practiced by its members, and inherent in its writings, on the foundation of the theory of sets" (Aczel, *Artist* 106).

Bourbaki also included many other topics in their books, but they specifically sought to discuss the "classical topics of analysis such as analytic functions, Fourier series, differential equations, and integration, almost all of which were included in the existing textbooks" in order to modernize the discussion of these topics (Mashaal 50). Along with their modern discussion came modern notations, new as well as old (which became the norm after their publication in Bourbaki's wide-read works). In fact, Bourbaki can "be considered the mathematician who had the greatest influence on the training, working methods, and writing style of the majority of today's mathematicians" (Bolondi 125). Their goal to use

> the most rigorous and simple language possible led them to develop numerous new terms. For example, they invented the word 'bijection,' which refers to a correspondance between two sets that associates each element of the first set to an element of the second set and vice versa. In addition to terminology, Bourbaki invented new notation. The most famous example is the symbol ∅, which represents the empty set (that is, the set with no elements). This symbol was invented by [André] Weil, the only Bourbaki to be familiar with the Norwegian alphabet (Mashaal 56).

Some of their terms were more successful than others, but many were adopted by mathematicians around the world, including $\notin$, $\varnothing$, $\Rightarrow$, and $\Leftrightarrow$.

All of the above notations became widespread in analysis, especially in the discussion of sets. Bourbaki defined $\notin$ by "the relation 'not (T $\in$ U)' is written T $\notin$ U" (Bourbaki, *Elements Mathematics* II.1, translation Dr. Berg). More importantly, the empty set notation was introduced through the *Elements of Mathematics*, which stated that "the assertion $\tau_x((\forall x)(x \notin X))$ [the set with no elements] corresponds to the … functional symbol $\varnothing$, which we refer to as *the empty set*; the relation $(\forall x)(x \notin X)$, which is equivalent to $X = \varnothing$, reads: '*the set X is empty*'" (Bourbaki, *Elements Mathematics* II. 6, translation Dr. Berg). Bourbaki also gave modern mathematicians the implication symbol, claiming that "the combination $\vee \neg$ is represented by $\Rightarrow$," meaning that $P \Rightarrow Q$ is equivalent to $P \vee (\neg Q)$ (Bourbaki, *Elements Mathematics* I.15, translation Dr. Berg). Finally, the group introduced the equivalence symbol through set theory, asserting: "Let A and B be assertions. The assertion $(A \Rightarrow B)$ and $(B \Rightarrow A)$ is represented by $A \Leftrightarrow B$" (Boubaki I.30, translation Dr. Berg).

Bourbaki was able to combine these new notations to make older notions easier to read, as seen by relations, which are still represented by the Bourbaki method. In the discussion of set theory in the *Elements of Mathematics*, they denote relations as follows:

Let **R** be a relation, and let x and y be letters. The relations

$$(\forall x)(\forall y)\mathbf{R} \Leftrightarrow (\forall y)(\forall x)\mathbf{R}$$

$$(\exists x)(\exists y)\mathbf{R} \Leftrightarrow (\exists y)(\exists x)\mathbf{R}$$

$$(\exists x)(\forall y)\mathbf{R} \Rightarrow (\forall y)(\exists x)\mathbf{R}$$

are theorems of $\mathscr{T}$,

where $\mathscr{T}$ is a theory (Bourbaki, *Elements Mathematics* I.35, translation Dr. Berg). Thus, one can see the lasting impact of Bourbaki's work in mathematics, especially in modern notation.

Unfortunately, not all of Bourbaki's work was as enduring as these notations. In fact, in their effort to standardize mathematics, they may have been "overly formal, too abstract, and much more rigorous than necessary, thus making it unnecessarily difficult to read and understand mathematics, and to use it in a meaningful way" (Aczel, *Artist* 121). Such a criticism is somewhat valid, since "Bourbaki's main aim had been to improve and deepen the the understanding of mathematical concepts, not to make them more obscure" (Aczel, *Artist* 121). Furthermore, as modern mathematics continued to develop, it became clear that "'the unity of mathematics is not based on a single root, on set theory, as Bourbaki advocated, but rather on the fact that the various branches of mathematics communicate among each other,'" reducing the reputation of *Elements of Mathematics* (Mashaal 151).

Despite their absence from modern mathematical teaching, Bourbaki led the charge for standardizing what students learned in school, which has led to the general acceptance that students of mathematics gain the same knowledge of the subject. Furthermore, their structure and call for strict proofs helped "to make mathematics more rigorous, more precise, and more proof-based," another sign that Bourbaki did affect the teaching of mathematics (Aczel, *Wilderness* 245). Also, as previously mentioned, the notations developed by Bourbaki (namely $\notin$, $\varnothing$, $\Rightarrow$, and $\Leftrightarrow$) have become standardized, marking another success in their mission to improve mathematics. By recalling the mathematical community to the "goals of axiomatizing mathematics, stressing structure, and promoting rigor in a discipline," the Bourbaki group made themselves nearly obsolete and began their "decline after the end of the 1960s precisely because

[they] had achieved [their] goal so marvelously" (Aczel, *Artist* 204). With the rigor, precision, and notations of Bourbaki, mathematics became the standardized subject we recognize today.

Conclusion

As mathematics progresses, so does the language used to communicate the advancements. The notations discussed above have allowed mathematicians to work globally rather than locally, to achieve the precision necessary in their subject, and to produce rigorous proofs. The standardization of these symbols has brought clarity to a difficult field, especially for students. The twenty-first century will likely become central to determining the success of this standardization. Perhaps mathematicians will discover that, excepting new developments, the notations we possess cover all necessary items. Perhaps the increase in machine language will show that mathematical language can be contained in even more symbols. The texts produced today will determine which makes math more legible and will likely impact the publication of mathematical papers in the future.

By following the development of analysis from the seventeenth to the twentieth century, students of mathematics gain a better understanding of how modern mathematics came to be and the struggles that earlier mathematicians faced in order to improve the subject. Modern students likely take for granted the ease of Descartes' higher powers, Fermat's maxima and minima, Newton's fluxions and fluents, Leibniz's $\int$ and *d*, Euler's Σ and f(x), Cauchy's *lim.*, Weierstrass' ε-δ proofs, Dedekind's "cuts" and $\Re$, and Bourbaki's $\notin$, $\varnothing$, $\Rightarrow$, and $\Leftrightarrow$. Any mathematics student could explain what these symbols mean, but most do not know where the notations come from. By missing these key historical facts, young mathematicians are unaware of the work that came before them and the developments that allowed them to master difficult subjects in a matter of years rather than decades. Notations allow mathematics to advance by conveying clear, concise meanings that are universally understood, knocking down language barriers that plague other fields. Through the standardization of mathematical writing, mathematicians have created a truly universal subject that will continue to unify the world one notation at a time.

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