

12-1-2005

# Recounting the Odds of an Even Derangement

Arthur T. Benjamin  
*Harvey Mudd College*

Curtis T. Bennett  
*Loyola Marymount University, cbennett@lmu.edu*

Florence Newberger  
*California State University, Long Beach*

---

## Repository Citation

Benjamin, Arthur T.; Bennett, Curtis T.; and Newberger, Florence, "Recounting the Odds of an Even Derangement" (2005).  
*Mathematics Faculty Works*. 108.  
[http://digitalcommons.lmu.edu/math\\_fac/108](http://digitalcommons.lmu.edu/math_fac/108)

## Recommended Citation

Benjamin, Arthur T., Curtis T. Bennett, and Florence Newberger. "Recounting the Odds of an Even Derangement." *Mathematics Magazine* 78, no. 5 (2005): 387-90. doi:10.2307/30044197.

We see the potential value of this proof as twofold. First, it appears cleaner and shorter than what is found in most texts. And our constant,  $\ln(8)$ , is modest compared to Sierpinski's 4 [7], Apostol's 6 [1], or the  $32 \ln(2)$  offered in earlier editions of Niven and Zuckerman [6]. LeVeque [5], Hardy and Wright [4], and the latest edition of Niven and Zuckerman [6] give no particular constant, merely proving that one exists. Chebychev [3] achieved a much smaller constant than ours, but with considerably more effort. We hope that our short proof will be found to have pedagogical value.

## REFERENCES

1. Tom M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1976.
2. Bela Bollobas, Erdős first paper: A proof of Bertrand's Postulate, pre-print, 1998.
3. P.L. Chebychev, *Memoire sur les nombres premiers*, *J. Math. Pures et Appl.* **17**, (1852), 366–390.
4. G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, 5th Edition, Clarendon Press, Oxford, 1979.
5. William J. LeVeque, *Fundamentals of Number Theory*, Dover Publications, New York, 1977.
6. I. Niven and H.S. Zuckerman, *An Introduction to the Theory of Numbers*, 3rd, 4th, 5th Editions, Wiley, New York, 1972, 1980, 1991.
7. Waclaw Sierpinski, *Elementary Theory of Numbers*, PWN, Warsaw, 1964.

# Recounting the Odds of an Even Derangement

ARTHUR T. BENJAMIN  
Harvey Mudd College  
Claremont, CA 91711  
benjamin@hmc.edu

CURTIS T. BENNETT  
Loyola Marymount University  
Los Angeles, CA 90045  
cbennett@lmu.edu

FLORENCE NEWBERGER  
California State University  
Long Beach, CA 90840-1001  
fnewberg@csulb.edu

Odd as it may sound, when  $n$  exams are randomly returned to  $n$  students, the probability that no student receives his or her own exam is almost exactly  $1/e$  (approximately 0.368), for all  $n \geq 4$ . We call a permutation with no fixed points, a *derangement*, and we let  $D(n)$  denote the number of derangements of  $n$  elements. For  $n \geq 1$ , it can be shown that  $D(n) = \sum_{k=0}^n (-1)^k n! / k!$ , and hence the *odds* that a random permutation of  $n$  elements has no fixed points is  $D(n)/n!$ , which is within  $1/(n+1)!$  of  $1/e$  [1].

Permutations come in two varieties: even and odd. A permutation is even if it can be achieved by making an even number of swaps; otherwise it is odd. Thus, one might *even* be interested to know that if we let  $E(n)$  and  $O(n)$  respectively denote the number of even and odd derangements of  $n$  elements, then (oddly enough),

$$E(n) = \frac{D(n) + (n-1)(-1)^{n-1}}{2}$$

and

$$O(n) = \frac{D(n) - (n-1)(-1)^{n-1}}{2}.$$

The above formulas are an immediate consequence of the equation  $E(n) + O(n) = D(n)$ , which is obvious, and the following theorem, which is the focus of this note.

**THEOREM.** For  $n \geq 1$ ,

$$E(n) - O(n) = (-1)^{n-1}(n-1). \quad (1)$$

**Proof 1: Determining a Determinant** The fastest way to derive equation (1), as is done in [3], is to compute a determinant. Recall that an  $n$ -by- $n$  matrix  $A = [a_{ij}]_{i,j=1}^n$  has determinant

$$\det(A) = \sum_{\pi \in S_n} a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)} \operatorname{sgn}(\pi), \quad (2)$$

where  $S_n$  is the set of all permutations of  $\{1, \dots, n\}$ ,  $\operatorname{sgn}(\pi) = 1$  when  $\pi$  is even, and  $\operatorname{sgn}(\pi) = -1$  when  $\pi$  is odd. Let  $A_n$  denote the  $n$ -by- $n$  matrix whose nondiagonal entries are  $a_{ij} = 1$  (for  $i \neq j$ ), with zeroes on the diagonal. For example, when  $n = 4$ ,

$$A_4 = J_4 - I_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

By (2), every permutation that is not a derangement will contribute 0 to the sum (since it uses at least one of the diagonal entries), every even derangement will contribute 1 to the sum, and every odd derangement will contribute  $-1$  to the sum. Consequently,  $\det(A_n) = E(n) - O(n)$ . To see that  $\det(A_n) = (-1)^{n-1}(n-1)$ , observe that  $A_n = J_n - I_n$ , where  $J_n$  is the matrix of all ones and  $I_n$  is the identity matrix. Since  $J_n$  has rank one, zero is an eigenvalue of  $J_n$ , with multiplicity  $n-1$ , and its other eigenvalue is  $n$  (with an eigenvector of all 1s). Apply  $J_n - I_n$  to the eigenvectors of  $J_n$  to find the eigenvalues of  $A_n$ :  $-1$  with multiplicity  $n-1$  and  $n-1$  with multiplicity 1. Multiplying the eigenvalues gives us  $\det(A_n) = (-1)^{n-1}(n-1)$ , as desired. ■

A 1996 Note in the MAGAZINE [2] gave *even odder* ways to determine the determinant of  $A_n$ .

Although the proof by determinants is quick, the form of (1) suggests that there should also exist an *almost* one-to-one correspondence between the set of even derangements and the set of odd derangements.

**Proof 2: Involving an Involution** Let  $D_n$  denote the set of derangements of  $\{1, \dots, n\}$ , and let  $X_n$  be a set of  $n-1$  *exceptional* derangements (that we specify later), each with sign  $(-1)^{n-1}$ . We exhibit a *sign reversing involution* on  $D_n - X_n$ . That is, letting  $T_n = D_n - X_n$ , we find an invertible function  $f: T_n \rightarrow T_n$  such that for  $\pi$  in  $T_n$ ,  $\pi$  and  $f(\pi)$  have opposite signs, and  $f(f(\pi)) = \pi$ . In other words, except for the  $n-1$  exceptional derangements, every even derangement “holds hands” with an odd derangement, and vice versa. From this, it immediately follows that  $|E_n| - |O_n| = (-1)^{n-1}(n-1)$ .

Before describing  $f$ , we establish some notation. We express each  $\pi$  in  $D_n$  as the product of  $k$  disjoint cycles  $C_1, \dots, C_k$  with respective lengths  $m_1, \dots, m_k$  for some

$k \geq 1$ . We follow the convention that each cycle begins with its smallest element, and the cycles are listed from left to right in increasing order of the first element. In particular,  $C_1 = (1 a_2 \cdots a_{m_1})$  and, if  $k \geq 2$ ,  $C_2$  begins with the smallest element that does not appear in  $C_1$ . Since  $\pi$  is a derangement on  $n$  elements, we must have  $m_i \geq 2$  for all  $i$ , and  $\sum_{i=1}^k m_i = n$ . Finally, since a cycle of length  $m$  has sign  $(-1)^{m-1}$ , it follows that  $\pi$  has sign  $(-1)^{\sum_{i=1}^k (m_i-1)} = (-1)^{n-k}$ .

Let  $\pi$  be a derangement in  $D_n$  with first cycle  $C_1 = (1 a_2 \cdots a_m)$  for some  $m \geq 2$ . We say that  $\pi$  has *extraction point*  $e \geq 2$  if  $e$  is the smallest number in the set  $\{2, \dots, n\} - \{a_2\}$  for which  $C_1$  does *not* end with the numbers of  $\{2, \dots, e\} - \{a_2\}$  written in decreasing order. Note that  $\pi$  will have extraction point  $e = 2$  unless the number 2 appears as the second term or last term of  $C_1$ . We illustrate this definition with some pairs of examples from  $D_9$ . Notice that in each pair below, the number of cycles of  $\pi$  and  $\pi'$  differ by one, and the extraction point  $e$  occurs in the first cycle of  $\pi$  and is the leading element of the second cycle of  $\pi'$ .

- $\pi = (1\ 9\ 7\ 2\ 8)(3\ 6)(4\ 5)$  and  $\pi' = (1\ 9\ 7)(2\ 8)(3\ 6)(4\ 5)$  have  $e = 2$ .
- $\pi = (1\ 2\ 9\ 7\ 3\ 8\ 5)(4\ 6)$  and  $\pi' = (1\ 2\ 9\ 7)(3\ 8\ 5)(4\ 6)$  have  $e = 3$ .
- $\pi = (1\ 9\ 7\ 3\ 8\ 5\ 2)(4\ 6)$  and  $\pi' = (1\ 9\ 7\ 2)(3\ 8\ 5)(4\ 6)$  have  $e = 3$ .
- $\pi = (1\ 9\ 4\ 8\ 5\ 3\ 2)(6\ 7)$  and  $\pi' = (1\ 9\ 3\ 2)(4\ 8\ 5)(6\ 7)$  have  $e = 4$ .
- $\pi = (1\ 4\ 9\ 5\ 8\ 3\ 2)(6\ 7)$  and  $\pi' = (1\ 4\ 9\ 3\ 2)(5\ 8)(6\ 7)$  have  $e = 5$ .
- $\pi = (1\ 3\ 8\ 6\ 9\ 7\ 5\ 4\ 2)$  and  $\pi' = (1\ 3\ 8\ 5\ 4\ 2)(6\ 9\ 7)$  have  $e = 6$ .

Observe that every derangement  $\pi$  in  $D_n$  contains an extraction point unless  $\pi$  consists of a single cycle of the form  $\pi = (1 a_2 Z)$ , where  $Z$  is the ordered set  $\{2, 3, \dots, n - 1, n\} - \{a_2\}$ , written in decreasing order. For example, the 9-element derangement  $(1\ 5\ 9\ 8\ 7\ 6\ 4\ 3\ 2)$  has no extraction point. Since  $a_2$  can be any element of  $\{2, \dots, n\}$ , there are exactly  $n - 1$  derangements of this type, all of which have sign  $(-1)^{n-1}$ . We let  $X_n$  denote the set of derangements of this form. Our problem reduces to finding a sign reversing involution  $f$  over  $T_n = D_n - X_n$ .

Suppose  $\pi$  in  $T_n$  has extraction point  $e$ . Then the first cycle  $C_1$  of  $\pi$  ends with the (possibly empty) ordered subset  $Z$  consisting of the elements of  $\{2, \dots, e - 1\} - \{a_2\}$  written in decreasing order. Our sign reversing involution  $f : T_n \rightarrow T_n$  can then be succinctly described as follows:

$$(1\ a_2\ X\ e\ Y\ Z)\sigma \xleftrightarrow{f} (1\ a_2\ X\ Z)(e\ Y)\sigma, \tag{3}$$

where  $X$  and  $Y$  are ordered subsets,  $Y$  is nonempty, and  $\sigma$  is the rest of the derangement  $\pi$ .

Notice that since the number of cycles of  $\pi$  and  $f(\pi)$  differ by one, they must be of opposite signs. The derangements on the left side of (3) are those for which the extraction point  $e$  is in the first cycle. In this case,  $Y$  must be nonempty, since otherwise “ $e\ Z$ ” would be a longer decreasing sequence and  $e$  would not be the extraction point. The derangements on the right side of (3) are those for which the extraction point  $e$  is not in the first cycle (and must therefore be the leading element of the second cycle). In this case,  $Y$  is nonempty since  $\pi$  is a derangement. Thus for any derangement  $\pi$ , the derangement  $f(\pi)$  is also written in standard form, with the same extraction point  $e$  and with the same associated ordered subset  $Z$ . Another way to see that  $\pi$  and  $f(\pi)$  have opposite signs is to notice that  $f(\pi) = (xy)\pi$  (multiplying from left to right), where  $x$  is the last element of  $X$  ( $x = a_2$  when  $X$  is empty), and  $y$  is the last element

of  $Y$ . Either way,  $f(f(\pi)) = \pi$ , and  $f$  is a well-defined, sign-reversing involution, as desired. ■

In summary, we have shown combinatorially that for all values of  $n$ , there are almost as many even derangements as odd derangements of  $n$  elements. Or to put it another way, when randomly choosing a derangement with at least five elements, the *odds* of having an even derangement are nearly *even*.

**Acknowledgment.** We are indebted to Don Rawlings for bringing this problem to our attention and we thank Magnhild Lien, Will Murray, and the referees for many helpful ideas.

## REFERENCES

1. R. A. Brualdi, *Introductory Combinatorics*, 3rd ed., Prentice-Hall, New Jersey, 1999.
2. C. A. McCarthy and A. T. Benjamin, Determinants of the tournaments, this *MAGAZINE*, **69** (1996), 133–135.
3. C. D. Olds, Odd and even derangements, Solution E907, *Amer. Math. Monthly*, **57** (1950), 687–688.

# Volumes of Generalized Unit Balls

XIANFU WANG

UBC Okanagan  
3333 University Way, Kelowna  
B.C., Canada, V1V 1V7  
shawn.wang@ubc.ca

Diamonds, cylinders, squares, stars, and balls. These geometric figures are familiar to undergraduate students, but what could they possibly have in common? One answer is: They are generalized balls. The standard Euclidean ball can be distorted into a variety of strange-shaped balls by linear and nonlinear transformations. The purpose of this note is to give a unified formula for computing the volumes of generalized unit balls in  $n$ -dimensional spaces.

A generalized unit ball in  $\mathbb{R}^n$  is described by the set

$$\mathbb{B}_{p_1 p_2 \dots p_n} = \{\mathbf{x} = (x_1, \dots, x_n) : |x_1|^{p_1} + \dots + |x_n|^{p_n} \leq 1\}, \quad (1)$$

where  $p_1 > 0, p_2 > 0, \dots, p_n > 0$ .

When the numbers  $p_1, \dots, p_n$  are all greater than or equal to 1, the unit ball  $\mathbb{B}_{p_1 \dots p_n}$  is convex. Since  $|x|^p$  is not concave on  $[-1, 1]$  for  $0 < p < 1$ ,  $\mathbb{B}_{p_1 \dots p_n}$  is not necessarily convex anymore when  $n > 1$ . When  $p_1 = p_2 = \dots = p_n = p \geq 1$ , we obtain the usual  $l_p$  ball. The  $l_2$  ball is denoted by  $\mathbb{B}$ . By choosing different numbers  $p_i$ , we can alter the appearance of the generalized balls greatly, as shown in FIGURE 1 with examples in  $\mathbb{R}^3$ .

Motivated by an article by Folland [5], I derived a unified formula for calculating the volume of these balls. Although the volume formulas for the standard Euclidean ball  $\mathbb{B}$  and simplex have been known for a long time [4, pp. 208, 220], the unified formula is (relatively) new. It is surprising that no matter how strange the balls look, the volume of any ball can be computed by a single formula, as follows:

**THEOREM.** Assume  $p_1, \dots, p_n > 0$ . The volume of the unit ball  $\mathbb{B}_{p_1 p_2 \dots p_n}$  in  $\mathbb{R}^n$  is equal to