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THE AUTOMORPHISM GROUPS OF THE HYPERELLIPTIC SURFACES

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1. Introduction. In this paper we will compute the automorphism groups of the so-called hyperelliptic surfaces. These compact complex surfaces are characterized by having invariants $p_g = 0$, $q = 1$, and $12K = 0$. References for the elementary properties of these surfaces may be found in [**2**] (where they are called "bielliptic surfaces") or in [**1**]. They may all be constructed as the quotient $X = (E \times F)/G$, where E and F are elliptic curves, and G is a finite group of translations of E acting also on F *not* only as a group of translations; the action on $E \times F$ is the diagonal action.

There are seven non-isomorphic groups G which can act on $E \times F$ as above, two of which act on any $E \times F$, the other five requiring F to be a specific elliptic curve. In the following table the reader will find a list of the seven groups G, together with the elliptic curves E and F, and the action of G on $E \times F$.

Write $E = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau_1)$ and $F = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau_2)$. Throughout this article we will use the notation $i = \sqrt{-1}, \omega = e^{2\pi i/3}$, and $\zeta = e^{\pi i/3}$; note that $\omega = \zeta^2$.

In the last three cases it is technically more convenient to consider $X = (E \times F)/G$ as the quotient of $Y = (E \times F)/\langle \psi \rangle$ by a cyclic group of order $r (= 2, 3, 4, \text{ or } 6)$, generated by the automorphism $\bar{\phi}$ induced by ϕ. Since ψ is a translation of $E \times F, Y$ is also a complex torus of dimension two. For uniformity of notation we will define $Y = E \times F$ and $\psi =$ identity in the first four cases, so that in each case $X = Y/\langle \bar{\phi} \rangle$. Note that r is the order of the canonical class K_X in $\operatorname{Pic}(X)$ and Y is the etale cyclic cover of X defined by $K_X : Y = \operatorname{Spec}(\oplus_{i=0}^{r-1} \varphi_X(iK_X))$, with the multiplication in φ_Y defined by a chosen isomorphism $\theta : \varphi_X \to \varphi_X(rK_X)$. The formation of Y

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from X is functorial: if $p : T \to X$ is a scheme over X, a morphism from T to Y over X corresponds to an φ_T-map $\alpha : p^*K_X \to \varphi_T$ such that the composition $\varphi_T \xrightarrow{p^*\theta} \varphi_T(rp^*K_X) \xrightarrow{\alpha^{\otimes r}} \varphi_T^{\otimes r} \xrightarrow{\text{mult}} \varphi_T$ is the identity. This description allows us to readily conclude the lemma,

TABLE 1.1.

The seven groups G used to construct the hyperelliptic surfaces.
In all cases τ_1 is arbitrary.

τ_2	G	action of the generators of G on $E \times F$
arbitrary	$\mathbf{Z}/2 = \langle\phi\rangle$	$\phi\binom{e}{f} = \binom{e+1/2}{-f}$
ζ	$\mathbf{Z}/3 = \langle\phi\rangle$	$\phi\binom{e}{f} = \binom{e+1/3}{\omega f}$
i	$\mathbf{Z}/4 = \langle\phi\rangle$	$\phi\binom{e}{f} = \binom{e+1/4}{if}$
ζ	$\mathbf{Z}/6 = \langle\phi\rangle$	$\phi\binom{e}{f} = \binom{e+1/6}{\zeta f}$
arbitrary	$\mathbf{Z}/2 \times \mathbf{Z}/2 = \langle\phi, \psi\rangle$	$\phi\binom{e}{f} = \binom{e+1/2}{-f}; \psi\binom{e}{f} = \binom{e+\tau 1/2}{f+1/2}$
ζ	$\mathbf{Z}/3 \times \mathbf{Z}/3 = \langle\phi, \psi\rangle$	$\phi\binom{e}{f} = \binom{e+1/3}{\omega f}; \psi\binom{e}{f} = \binom{e+\tau 1/3}{f+(1+\zeta)/3}$
i	$\mathbf{Z}/4 \times \mathbf{Z}/2 = \langle\phi, \psi\rangle$	$\phi\binom{e}{f} = \binom{e+1/4}{if}; \psi\binom{e}{f} = \binom{e+\tau 1/2}{f+(1+i)/2}$

LEMMA 1.2. *Every automorphism of X lifts to Y.*

PROOF. Let $\pi : Y \to X$ be the quotient map and assume σ is an automorphism of X. Let $p : Y \to X$ be the composition $p = \sigma \circ \pi$. We require a lifting, $f : Y \to Y$ such that $\pi \circ f = p = \sigma \circ \pi$. Since σ is an automorphism of $X, \sigma^*K_X \cong K_X$; since π is unramified, $\pi^*K_X \cong K_Y$. Moreover since Y is an abelian surface, $K_Y \cong \varphi_Y$; hence $p^*K_X \cong \varphi_Y$. We may then choose an isomorphism $\alpha : p^*K_X \to \varphi_Y$ so that the composition $\text{mult} \circ \alpha^{\otimes r} \circ p^*\theta$ is the identity; in fact there are r choices for α, differing from each other by a factor which is an r^{th} root of unity. Each of these choices for α provide a lift to Y of the automorphism σ. $\square$

Since every automorphism of X lifts to Y, the standard theory of covering spaces [**3**] implies that $\mathrm{Aut}(X) \cong N/\langle\bar{\phi}\rangle$, where N is the normalizer of $\langle\bar{\phi}\rangle$ in $\mathrm{Aut}(Y)$. It is this group we will calculate in the first four cases where $Y = E \times F$; in the last three we can in fact lift automorphisms to $E \times F$ also, and make the analysis there.

There does not seem to be any standard notation for the hyperelliptic surfaces. We will use $X_r(\tau_2)$ for the first four surfaces in Table 1.1, for which Y is the product $E \times F$, and $\bar{X}_r(\tau_2)$ for the last three; if $r \neq 2$ then we will drop the τ_2, which is determined. Hence the hyperelliptic surfaces are $X_2(\tau_2), X_3, X_4, X_6, \bar{X}_2(\tau_2), \bar{X}_3$, and $\bar{X}_4$ in the order in which they appear in Table 1.1. Note that they all of course depend on τ_1 also, which we omit from the notation.

2. The lifting to $\mathbf{E} \times \mathbf{F}$. Since Y is an abelian surface, $\mathrm{Aut}(Y)$ is an extension of $\mathrm{Aut}_0(Y)$ (the subgroup of automorphisms fixing 0) by the translation subgroup. $\mathrm{Aut}_0(Y)$ has a natural representation into $\mathrm{GL}(2, \mathbf{C})$, inducing a homomorphism from $\mathrm{Aut}(Y)$ to $\mathrm{GL}(2, \mathbf{C})$; we will denote the image of an automorphism α of Y by $\alpha_* \in \mathrm{GL}(2, \mathbf{C})$. By composing with the determinant we have a homomorphism $\det :$ $\mathrm{Aut}(Y) \to \mathbf{C}^*$. These same constructions apply to $E \times F$ as well, and we will use the same notation for them.

LEMMA 2.1. *Let N be the normalizer of $\langle\bar{\phi}\rangle$ in* $\mathrm{Aut}(Y)$. *Then $\alpha \in N$ if and only if α is induced from an element of* $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$ *which normalizes G.*

PROOF. Let $\alpha \in N$. Then $\alpha\bar{\phi}\alpha^{-1} = \bar{\phi}^k$, and applying det to both sides forces $k = 1$, showing that α and $\bar{\phi}$ must in fact commute. Therefore α_* commutes with $\bar{\phi}_* = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$, where $\varepsilon = e^{2\pi i/r}$. Therefore α_* must be diagonal, since $\varepsilon \neq 1$; but this is equivalent to α lifting to an element of $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$, which must normalize G, since it descends to α, which descends to X. Conversely, if β is in $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$ and normalizes G, then $\beta\psi\beta^{-1} = \phi^i\psi^j$, and applying det to both sides forces $i = 0$, so β normalizes $\langle\psi\rangle$ and descends to some $\alpha \in \mathrm{Aut}(Y)$. Since β normalizes $G = \langle\phi, \psi\rangle, \alpha$ will normalize $\langle\bar{\phi}\rangle$. □

The elements of $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$ can be conveniently represented by 4-tuples $[p, q; a, d]$, which will denote the map sending (e, f) to $(ae + p, df + q)$; here $p \in E, q \in F, a \in \mathrm{Aut}_0(E)$, and $d \in \mathrm{Aut}_0(F)$. Note that, in this notation, $\phi = [1/r, 0; 1, e^{2\pi i/r}]$ and $\psi = [u, v; 1, 1]$ for appropriate u, v. It is easy to verify the following formulas:

$$[p_1, q_1; a_1, d_1][p_2, q_2; a_2, d_2] = [p_1 + a_1 p_2, q_1 + d_1 q_2; a_1 a_2, d_1 d_2], \tag{2.1}$$

$$[p, q; a, d]^{-1} = [-a^{-1}p, -d^{-1}q; a^{-1}, d^{-1}], \tag{2.2}$$

$$[p, q; a, d][u, v; 1, 1][p, q; a, d]^{-1} = [au, dv; 1, 1], \tag{2.3}$$

$$[p, q; a, d]\phi[p, q; a, d]^{-1}\phi^{-1} = [(a - 1)/r, (1 - e^{2\pi i/r})q; 1, 1]. \tag{2.4}$$

These allow us to prove the following refinement of Lemma (2.1):

Lemma 2.6. *Any element of* $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$ *which normalizes* G *in fact centralizes* G, *i.e., commutes with* ϕ *and* ψ.

Proof. Let $\beta \in \mathrm{Aut}(E) \times \mathrm{Aut}(F)$ normalize G. Then $\beta\psi\beta^{-1} = \phi^i\psi^j$, and applying det to both sides forces $i = 0$, so $\beta\psi\beta^{-1} = \psi^j$. Similarly $\beta\phi\beta^{-1} = \phi^i\psi^k$, and applying det forces $i = 1$, so that $\beta\phi\beta^{-1}\phi^{-1} = \psi^k$ for some k. We want to show that $k = 0$ and $j = 1$. In the first four cases when Y is a product, ψ is the identity and there is nothing to show; hence we must analyze only the last three cases. In these cases $\psi = [u, v; 1, 1]$, where $u = n\tau_1/r$; here $n = 1$ if $r = 2$ or 3 and $n = 2$ if $r = 4$. Write $\beta = [p, q; a, d]$ and assume $\beta\psi\beta^{-1} = \psi^j$ and $\beta\phi\beta^{-1}\phi^{-1} = \psi^k$. Then, from (2.3) and (2.4), we must have

$$(a - 1)/r = kn\tau_1/r \quad \text{and} \quad an\tau_1/r = jn\tau_1/r \tag{2.5}$$

by only considering the first coordinate in the two equalities. Recalling that $a \in \mathrm{Aut}_0(E)$ and $0 \le j, k < r/n$, one checks easily that the only solutions to (2.5) are $a = j = 1$, $k = 0$, $r = 2, 3, 4$ and $a = -1$, $j = 1$, $k = 0$, $r = 2$. In all cases $k = 0$ and $j = 1$, proving the lemma. □

3. The computation of AutX). Let M denote the centralizer of G in $\mathrm{Aut}(E) \times \mathrm{Aut}(F)$. By the above lemma, $\mathrm{Aut}(X) \cong N/\langle\bar{\phi}\rangle \cong M/G$. It is a simple matter to calculate M using formulas (2.1)–(2.4); we present the results below

PROPOSITION 3.1.

(a) $M(X_2(\tau_2)) = \{[p,q;a,d] | a = \pm 1, d \in \mathrm{Aut}(F)$, and $2q = 0$, i.e., $q = 0, 1/2, \tau_2/2$, or $(1+\tau_2)/2 \mod \Lambda_2\}$,

(b) $M(X_3) = \{[p,q;a,d] | a = 1, d \in \mathrm{Aut}(F)$, and $(\omega - 1)q = 0$, i.e., $q = 0, (1+\zeta)/3$, or $2(1+\zeta)/3 \mod \Lambda_2\}$,

(c) $M(X_4) = \{[p,q;a,d] | a = 1, d \in \mathrm{Aut}(F)$, and $(i-1)q = 0$, i.e., $q = 0$ or $(1+i)/2 \mod \Lambda_2\}$,

(d) $M(X_6) = \{[p,q;a,d] | a = 1, d \in \mathrm{Aut}(F)$, and $q = 0\}$,

(e) $M(\bar{X}_2(\tau_2)) = \{[p,q;a,d] | a = \pm 1, d = \pm 1$, and $2q = 0$, i.e., $q = 0, 1/2, \tau_2/2$, or $(1+\tau_2)/2 \mod \Lambda_2\}$,

(f) $M(\bar{X}_3) = \{[p,q;a,d] | a = 1, d = 1, \omega$, or ω^2, and $(\omega-1)q = 0$, i.e., $q = 0, (1+\zeta)/3$, or $2(1+\zeta)/3 \mod \Lambda_2\}$,

(g) $M(\bar{X}_4) = M(X_4)$.

It is evident from the above proposition that in every case M is generated by its E-translations, its F-translations, its E-automorphisms (elements of $\mathrm{Aut}_0(E)$), and its F-automorphisms. It may be convenient to the reader to present these generators for M, which we do in Table 3.1.

Note that, in every case, $p \in E$ is arbitrary, so that $E \subseteq M$ as the subgroup $\{[p,0;1,1]\}$; moreover $E \cap G = \{\mathrm{id}\}$. Hence E also embeds in the quotient $M/G \cong \mathrm{Aut}(X)$ as a normal subgroup and we will consider our task complete if we identify the quotient of M/G by E which is a finite group. We will also give generators for $\mathrm{Aut}(X)/E$, lifted to M. We present this information in Table 3.2.

TABLE 3.1.
Generators for M

X	translations of E	translations of F	automorphisms of E	automorphisms of F
$X_2(i)$	E	$\{0, 1/2, i/2, (1+i)/2\}$	$\{\pm 1\}$	$\{1, i, -1, -i\}$
$X_2(\zeta)$	E	$\{0, 1/2, \zeta/2, (1+\zeta)/2\}$	$\{\pm 1\}$	$\{1, \zeta, \zeta^2, -1, -\zeta, -\zeta^2\}$
$X_2(\tau_2)$	E	$\{0, 1/2, \tau_2/2, 1+\tau_2/2\}$	$\{\pm 1\}$	$\{\pm 1\}$
	(for τ_2 general, i.e., Λ_2 is neither square nor hexagonal)			
X_3	E	$\{0, (1+\zeta)/3, (2+2\zeta)/3\}$	$\{1\}$	$\{1, \zeta, \zeta^2, -1, -\zeta, -\zeta^2\}$
X_4	E	$\{0, (1+i)/2\}$	$\{1\}$	$\{1, i, -1, -i\}$
X_σ	E	$\{0\}$	$\{1\}$	$\{1, \zeta, \zeta^2, -1, -\zeta, -\zeta^2\}$
$\bar{X}_2(\tau_2)$	E	$\{0, 1/2, \tau_2/2, (1+\tau_2)/2\}$	$\{\pm 1\}$	$\{\pm 1\}$
$\bar{X}_3$	E	$\{0, (1+\zeta)/3, (2+2\zeta)/3\}$	$\{1\}$	$\{1, \omega, \omega^2\}$
$\bar{X}_4$	E	$\{0, (1+i)/2\}$	$\{1\}$	$\{1, i, -1, -i\}$

TABLE 3.2.

X	$\lvert\mathrm{Aut}\,(X)/E\rvert$	$\mathrm{Aut}\,(X)/E$	generators for $\mathrm{Aut}\,(X)/E$ in M
$X_2(i)$	16	$\mathbf{Z}/2 \times D_8$	$[0,0;-1,1]$ generates the $\mathbf{Z}/2$
		(D_8 is the dihedral	$[0,1/2;1,i]$ has order 4 in D_8
		group of order 8	$[0,0;1,i]$ has order 2 in D_8
$X_2(\zeta)$	24	$\mathbf{Z}/2 \times A_4$	$[0,0;-1,1]$ generates the $\mathbf{Z}/2$
		(A_4 is the	$[0,0;1,\zeta]$ has order 3 in A_4
		alternating group	$[0,1/2;1,1]$ and $[0,\zeta/2;1,1]$
		of order 12)	generate the 2-part of A_4
$X_2(\tau_2)$	8	$(\mathbf{Z}/2)^3$	$[0,0;-1,1]$, $[0,1/2;1,1]$, and
	(τ_2 general)		$[0,\tau_2/2;1,1]$ generate $(\mathbf{Z}/2)^3$
X_3	6	S_3	$[0,(1+\zeta)/3;1,1]$ has order 3
		(the symmetric group)	$[0,0;1,\zeta]$ has order 2
X_4	2	$\mathbf{Z}/2$	$[0,(1+i)/2;1,1]$ generates
X_σ	1	$\{1\}$	
$\bar{X}_2(\tau_2)$	4	$\mathbf{Z}/2 \times \mathbf{Z}/2$	$[0,0;-1,1]$ and $[0,\tau_2/2;1,1]$
			generate the $\mathbf{Z}/2 \times \mathbf{Z}/2$
$\bar{X}_3$	1	$\{1\}$	
$\bar{X}_4$	1	$\{1\}$	

With this table we consider our description of $\mathrm{Aut}(X)$ complete. We note the following interesting corollary:

Every automorphism of $\bar{X}_r(\tau_2)$ lifts to $X_r(\tau_2)$.

Indeed, we have proven that every automorphism of $\bar{X}_r(\tau_2)$ lifts to $E \times F$, in fact to an automorphism which commutes with ϕ. Hence that lifting descends to $X_r(\tau_2)$.

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