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The Segal–Shale–Weil representation, the indices of Kashiwara and Maslov, and quantum mechanics

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Dedicated to the late André Weil

Abstract

We produce a connection between the Weil 2-cocycles defining the local and adèlic metaplectic groups defined over a global field, i.e. the double covers of the attendant local and adèlic symplectic groups, and local and adèlic Maslov indices of the type considered by Souriau and Leray. With the latter tied to phase integrals occurring in quantum mechanics, we provide a formulation of quadratic reciprocity for the underlying field, first in terms of an adèlic phase integral, and then in terms of generalized time evolution unitary operators.

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1. Introduction

André Weil gave, in “Sur certains groupes d’opérateurs unitaires” [63], what is now regarded as the definitive explication of the analytic theory of quadratic forms presented by C.L. Siegel [51]. One of the main themes of [63] is Weil’s beautiful reformulation of the Fourier-analytic proof of quadratic reciprocity for a global field (in Weil's language, an $\mathbb{A}$-field) in terms of the unitary representations of the symplectic groups associated to the natural skew-symmetric pairings of the given field's localizations and adèlization with their Pontryagin duals. Weil accordingly reframed the classical analytic proof of Gauss–Euler reciprocity for an algebraic number field, originally given by Erich Hecke [27] almost forty years earlier, in terms of unitary group representations. In light of what follows, it is worth noting that the equivalent form of quadratic reciprocity used by Weil is Hilbert's, or that of Hilbert and Hasse.

Making a transition from Hecke's classical Fourier analysis, centered on the essentially Riemannian strategy of exploiting functional equations for ϑ-functions, to what is now often called "abstract" Fourier analysis, i.e., Weil's unitary representation theory, involves the use of a good deal of mathematical machinery originally developed in connection with quantum mechanics. To give one example, in Hecke's original treatment the central maneuver is to establish the aforementioned functional equations for what are now called Hecke ϑ-functions by means of Fourier analysis (Poisson summation can be used). It is an exercise in elementary analysis to tie each of the two ϑ-functions appearing in a functional equation to a Gauss sum that transforms nicely with respect to an obvious action of a generalized Legendre symbol. From there it is a short hop to quadratic reciprocity. The corresponding feature in Weil's treatment, working in an adèlic context, is the fact that a certain linear functional, now called the Weil Θ-functional, is invariant under the action of the rational points of the adèlic symplectic group (induced by a projective representation, the so-called Weil representation). And so it is that one can already recognize something quite familiar in this interplay between, on the one hand, Fourier series and ϑ-functions, and, on the other hand, unitary representations. This same counterpoint is at the heart of the functional analysis von Neumann was instrumental in developing [61] for the purpose of capturing Schrödinger's wave mechanics and Heisenberg's matrix mechanics under a single umbrella. Indeed, it is the case that the Heisenberg group, $Heis(V)$, carries a(n essentially) unique irreducible representation and since this group is an extension of the underlying symplectic space by the scalars, the according symplectic group, $Sp(V)$, acts on $Heis(V)$ in such a way that its elements twist the aforementioned representation into an equivalent one. In other words, we get a projective representation, one of the main foci of this article (see Section 2.3). The entry of the Heisenberg group onto the scene in this manner suggests a close connection to von Neumann's approach to quantum mechanics, as is evident from the role played by the Stone–von Neumann Theorem in both the theory of the oscillator representation and the theory of the Weil representation: but for their different settings they are quite the same. This interplay, or, more specifically, the connection it suggests between the present theme from the analytic theory of numbers and quantum physics, provides one of the major motivations for the investigations we carry out in this article.

Drawing a direct parallel between ϑ-functional equations and the aforementioned invariance regarding the Weil Θ-functional, evocative though this may be, is a bit disingenuous since it is the case that for our present purposes greater weight should be given to the Weil representation. As just mentioned, the latter is actually not a true representation at all, but a projective representation, which is to say that it is really a low-dimensional cohomological object. This can be seen by looking at the long exact sequence in cohomology attached to the short exact sequence realizing the projective unitary group attached to the irreducible representation of $Heis(V)$ just mentioned above: any homomorphism into the projective unitary group on V gives a 2-cohomology class with coefficients in the attendant 1-parameter unitary group. This having been said, as acting on the symplectic group, the Weil representation is a homomorphism into the group of unitary operators on a certain natural Hilbert space only up to multiplication by a 2-cocycle taking its values (*à priori*) in $\mathbb{C}_1^\times \approx \mathbb{S}^1$. In [63] Weil develops this theme on two fronts: locally, for every valuation $\mathfrak{p}$ on the underlying global field k, and adèlically, which is to say, over the ring $k_\mathbb{A}$ of k-adèles. In this way he obtains local 2-cocycles $c_\mathfrak{p} \in H^2(Sp(2, k_\mathfrak{p}), \mathbb{C}_1^\times)$, one for each valuation of the field, and an adèlic 2-cocycle $c_\mathbb{A} \in H^2(Sp(2, k_\mathbb{A}), \mathbb{C}_1^\times)$ *via* $c_\mathbb{A} = \prod_\mathfrak{p} c_\mathfrak{p}$, where we have written $Sp(2, k_\mathfrak{p})$ (resp. $H^2(Sp(2, k_\mathbb{A}))$) for the indicated local $\mathfrak{p}$-adic (resp. adèlic) symplectic group. Presently we will have occasion to sharpen this notation to $Sp(k_\mathfrak{p} \times k_\mathfrak{p}^*)$ (resp. $Sp(k_\mathbb{A} \times k_\mathbb{A}^*)$). The aforementioned invariance of the Θ-functional under the action of the rational points, through the agency of the Weil representation, means that $c_\mathbb{A} = 1$ when restricted to $Sp(k) \times Sp(k)$. In Weil's approach this suffices to yield the reciprocity law for the 2-Hilbert–Hasse symbol; in Kubota's supplement [31–63], featuring a particularly transparent presentation of the defining local 2-cocycle, we obtain 2-Hilbert reciprocity directly. In this regard the reader might also consult [22] and [3]; additionally [5] contains marvelous related material—see especially the articles by Roger Howe (p. 275 ff.) and Stephen Gelbart (p. 287 ff.). Additionally, there is the important article [43] by Ranga Rao where it is shown that by choosing a Haar measure in the right way there emerges a simple formula for the Weil representation relative to the Bruhat decomposition of $Sp(V)$ resulting in an explicit 2-cocycle representing the corresponding cohomology class.

Going on to the particulars of the connections with quantum physics suggested above, we note, first, that the Weil representation was already introduced earlier in the context of a study [50] dealing with bosons (particles obeying Bose–Einstein statistics) and electron spin, authored by David Shale and building on work by I.E. Segal [49]. With physical space being the underlying topological space for this study, the symplectic group encountered here is $Sp(\mathbb{R}^d)$ for $d = 3, 4$. For our purposes d can be any positive integer, although eventually, as we home in on specific arithmetical goals, we will take $d = 1$, meaning that for the localization of k at a real archimedean place ($k_\infty = \mathbb{R}$) the associated symplectic group is of dimension $2 = 2d$. Furthermore, giving all places of k equal billing, the general local symplectic group will then be $Sp(k_\mathfrak{p} \times k_\mathfrak{p}^*)$, as already indicated earlier. *À propos*, physicists parochially tend to refer to the present projective representation of $Sp(\mathbb{R}^d)$ as the oscillator representation, while number theorists call it simply the Weil representation; ecumenists sometimes call it the Segal–Shale–Weil representation.

More needs to be said, however, about what physics has to offer. Well beyond this historical point, that in the hands of Gérard Lion and Michele Vergne [34] a deep connection was revealed between the Weil representation and the Kashiwara triple index of Lagrangian planes in a symplectic space (or manifold). Lion and Vergne proved that if we write $c = c_\infty \in H^2(Sp(\mathbb{R}^d), \mathbb{C}_1^\times)$ for the 2-cocycle for the Weil representation of $Sp(\mathbb{R}^d)$, and $\tau(l, l^\sigma, l^{\sigma'\sigma})$ for Kashiwara's triple index at the Lagrangian planes $l, l^\sigma, l^{\sigma'\sigma}$, where $\sigma, \sigma' \in Sp(\mathbb{R}^d)$, then c is an eighth root of unity of a special form:

$c(\sigma, \sigma') = \exp(-\frac{\pi i}{4}\tau(l, l^{\sigma}, l^{\sigma'\sigma}))$, seeing that τ takes values in $\mathbb{Z}$. Parenthetically, in [34] Lion and Vergne refer to Kashiwara's τ as Maslov's index, as the title of their book already indicates. Interestingly, Kashiwara and Schapira note in their Appendix A.3 to [29], "Inertia index", where the Kashiwara triple index is presumably introduced (cf. [29], p. 487), that this index is sometimes also called the Maslov index. Doubtless it is this convention that is followed in [34]. However, the genesis of the Maslov index proper is somewhat more ramified in the sense that Maslov's original formulation of it [35] occurred in the context of flows on certain kinds of manifolds, a theme taken up by V.I. Arnol'd [1], who apparently gave the index its name. A concomitant tradition is that this index should be written as μ, and, while τ and μ are closely related, they are not identical. Happily, the precise relationship between τ and μ is discussed at great length in the important recent paper [7] by S. Cappell, R. Lee, and E.Y. Miller, where, among other things, the authors present an explicit linear relation between these two indices. Thus we also get a relation between c_∞ and μ.

And this forms the point of departure for bringing physics' path integral formalism into the game. Specifically, at first working over $\mathbb{R}$, we can map out a trajectory from $c_\infty \in H^2(Sp(\mathbb{R}^d), \mathbb{C}_1^\times)$ to μ, which means that we obtain an intrinsic tie between the given cohomology class in H^2 and an index in the true sense of the word, ultimately *gratia* [34] and [7], as we shall see in the course of our discussion. But it is also the case that the Maslov index μ appears, e.g., in the formula for the density of states of nothing less than quantum mechanics' harmonic oscillator [2], as well as in a pair of marvelous and suggestive formulas presented respectively by Jean-Marie Souriau [53] and by Joel Robbin and Dietmar Salamon [45,44]. Souriau's formula directly addresses the time evolution of a certain quantum mechanical system, while Robbin and Salamon focus on a unitary operator formalism. Both of these formulas involve phase integrals which have physical interpretations. Therefore we can inquire after a deep structural connection between the arithmetical theme of quadratic reciprocity, couched in Weil's unitary representation theoretic language, and quantum mechanics, the conduit being the Maslov index, i.e. symplectic. We partially developed this theme for the real case in [4], which can be regarded as something of a precursor to the present work.

Recognizing that the preceding remarks are sketchy at best, we address in the pages that follow the autonomous question of relationships between, on the one hand, the indices of Kashiwara and Maslov, and, on the other hand, the aforementioned phase integrals. We subsequentially specialize to the setting of arithmetical interest and tie things with the Weil 2-cocycle and double cover of $Sp(k_{\mathbb{A}} \times k_{\mathbb{A}}^*)$ (for it is in fact the case that the Weil 2-cocycle takes its values in $\mathbb{Z}_2$ (or $\mu_2 = \{1, -1\} < \mathbb{C}_1^\times$, if we work multiplicatively, but we avoid this notation for obvious reasons); cf. [63] and [31]). This all means that we have to develop two major themes. First, we have to see how the results of Lion–Vergne [34], Cappell–Lee–Miller [7], Souriau [53], and Robbin–Salamon [45] can be adapted to arbitrary local fields (i.e. to all the places of k, both archimedean and non-archimedean), and then to adèlize everything in sight. Second, we have to look at what can be said about the splitting of the double cover of $Sp(k_{\mathbb{A}} \times k_{\mathbb{A}}^*)$ on the rational points in terms of the yoga of phase integrals. The former theme is, at least as far as Lion–Vergne's results go, covered by the thesis [40] of Patrice Perrin, presented in detail in the Appendix to Part I of [34]. Accordingly, in what follows, the bulk of truly novel results accrue to the theme of how what we might call the resulting symplectic geometry informs the behavior of the relevant phase integrals. This inevitably takes us to *avant garde* material on $\mathfrak{p}$-adic phase (and ultimately Feynman) integrals and their connection to the $\mathfrak{p}$-adic counterparts of the Maslov indices we consider with their subsequent adèlizations; cf. [47,58,59,17,16], and [14].

Yielding to temptation a little, we suggest at this admittedly early stage that the enterprise of going through these two investigations should in due course shed light on a particularly vexing question in analytic number theory, *viz.* that of the nature of the obstacles encountered in trying to generalize the analytic proof of quadratic reciprocity to higher degrees, from, say, 2-Hilbert reciprocity to n-Hilbert reciprocity. Given the structural similarity between phase integrals and Fourier integrals, there is a possibility that we can begin to address this question directly—Hecke himself posed this challenge at the close of [27] by asking for transcendental functions to generalize his ϑ-functions. Regarding these, assuming a somewhat different standpoint for the moment, one might enquire after deep connections to algebraic geometry, given that ϑ-functions give rise to sections of line bundles over abelian varieties. Here the definitive reference is Mumford [37]. Also, still regarding ϑ-functions *per se*, the theory of the Weil representation as per [63] provides that they can be regarded as functions on a quotient of the 2-fold cover of the symplectic group, Weil's metaplectic group. But, perhaps on a more prosaic level, ϑ-functions can be viewed as sums of parameterized Gauss kernels, and the latter also figure directly into the theory of quadratic phase Feynman integrals. One may therefore ask whether the analytic proof of higher reciprocity ultimately redounds to maneuvers with higher phase Feynman integrals. However, these claims about higher reciprocity are at this point still speculative and preliminary.

1.1. The structure of this paper

Central to all of our considerations is the theory of the Weil (projective) representation of the symplectic group, and we devote the second chapter of this article to this subject. There are two themes to discuss in this connection, which, for lack of a better word, we refer to as, respectively, the arithmetical part and the quantum mechanical part.

The arithmetical part is largely concerned with the indicated material in Weil's 1964 paper [63] and its 1980 elaboration and interpretation given by Lion, Vergne, and Perrin [34]. Our objective is to have a full treatment of the Weil representation at our disposal, in the settings of all local fields $k_{\mathfrak{p}}$ arising as completions of the underlying global field k at its prime spots $\mathfrak{p}$, as well as in the setting of the k-adèles $k_{\mathbb{A}}$. As we noted in the preceding section, Lion and Vergne restructured Weil's theory in the real setting, expressly focusing on symplectic geometry, while the needed extension to local fields is covered in their Appendix to Part I, devoted to the work of Patrice Perrin [40]. These particulars set the stage for the transition to the third chapter of our paper, i.e., the discussion of the Kashiwara index (their Maslov index) in [34].

To wit, in chapter three the focus falls explicitly on the Maslov index in its different manifestations and definitions, and its connections to the 2-cocycle $c_{\mathfrak{p}} \in H^2(Sp(2, k_{\mathfrak{p}}), \mathbb{Z}_2)$ (again, with values in $\mathbb{Z}_2$ rather than $\mathbb{C}_1^{\times}$ or $\mathbb{S}^1$ thanks to Weil [63]). While [34] deals only with Kashiwara's Maslov index, i.e., $\tau_{\mathfrak{p}}$, with $\mathfrak{p} \in \mathfrak{V}_k$, where $\mathfrak{V}_k$ denotes the set of all places of k, whence we only obtain a relationship between the $c_{\mathfrak{p}}$ and the corresponding $\tau_{\mathfrak{p}}$ in this context, we need to connect the $c_{\mathfrak{p}}$ to what Maurice de Gosson calls the Arnol'd–Leray–Maslov index in [8], i.e. the data $\mu_{\mathfrak{p}}$, with $\mathfrak{p} \in \mathfrak{V}_k$. The reason for this is that, following [8], when $\mathfrak{p} = \infty_{\mathbb{R}}$ this Arnol'd–Leray–Maslov index is only off by a factor of 2 from the Maslov index m discussed by Souriau in [53], and thereafter by Leray in [33]. It falls to us, therefore, to extend this relation to a usable connection between the $c_{\mathfrak{p}}$ data and $\mu_{\mathfrak{p}}$ and $m_{\mathfrak{p}}$ data, *mutatis mutandis*.

It is in this context that we introduce the fact that, as also per the aforementioned seminal work [63] by André Weil (see also [3]), the splitting of the adèlic double cover of the symplectic group on the rational points is equivalent to the law of quadratic reciprocity for the ground field k. We give a compact discussion of this at the end of chapter three.

Next we turn to physics, at least after a fashion. First off, for the sake of clarity, when we are concerned with its arithmetical aspects, we take the liberty of referring to the Segal–Shale–Weil representation simply as the Weil representation, while we use the term oscillator representation when looking at the object from a physicist's perspective. Regardless of its name, however, the development of this projective representation is intimately tied to the unitary representations of some flavor of Heisenberg group, specifically to the according Schrödinger representations, bearing in mind, of course, that we are concerned with what transpires in the setting of the localizations of a global number field.

For our ultimate purposes the main point about the oscillator representation is its relationship to a phase integral development of quantum mechanics (and perhaps, *à la* Feynman, quantum field theory). Therefore, in the fourth chapter, we turn to the formalism of physically motivated phase integrals, also taking the liberty to discuss the position such integrals occupy relative to the Schrödinger and Heisenberg pictures of quantum mechanics; here we largely follow Prugovečki [42]. More precisely, given a quantum mechanical system (e.g., a single particle, in the simplest case), the formalism of Schrödinger's wave equation deals with the time evolution of its states, keeping the observables fixed, whereas in Heisenberg's formulation of quantum mechanics (i.e. matrix mechanics) it is the other way around. The transition between these "pictures", Schrödinger's and Heisenberg's, is given in terms of the behavior of a 1-parameter subgroup of the group of unitary operators on the Hilbert space of states of the system, and it is a relatively straightforward matter to finesse this subgroup's infinitesimal generator so as to bring in an explicit phase integral. At the same time, this infinitesimal generator comes from the system's Hamiltonian (measuring total energy), which is fundamental as far as our projected next step goes, namely, Feynman's famous idiosyncratic development of quantum mechanics, even though he starts out in [20] with Schrödinger's equation and works with a Lagrangian. In this connection we refer the reader to Faddeev's wonderful discussion of the attendant procedures in [19], and also Feynman's book [21], co-written with A.R. Higgs.

With this background in place we go on, in the fourth chapter, to look at the formal connection between phase integrals and 1-parameter subgroups of certain unitary Lie groups void of particular physical overtones. This is of particular interest because the work of Souriau [53] mentioned earlier, connecting his Maslov index (i.e. essentially the Arnol'd–Leray–Maslov index) to a phase integral, has found something of a modern echo in work by Joel Robbin and Dietmar Salamon [44,45] dating to the early 1990s, in which time evolution unitary operators dependent on a given Hamiltonian are featured.

Additionally, given our prevailing need for $\mathfrak{p}$-adic data for all $\mathfrak{p} \in \mathfrak{V}_k$, we look into the matter of designing suitable compatible $\mathfrak{p}$-adic and, subsequently, adèlic phase integrals, for reasons that become clear in the fifth and final chapter of our paper. Suffice it to note at this stage that since Hamiltonians are not available in these non-archimedean contexts, we employ the strategy employed by V.S. Vladimirov, V.I. Volovich and E.I. Zelenov [58] and Branko Dragovich [17,16] and shift the onus entirely to Weyl quantization.

Thus, the middle three chapters serve to lay out a coherent treatment of the connection between, on the one hand, the cohomological data afforded by the Weil 2-cocycles, locally as well as adèlically, and, on the other hand, Kashiwara and Maslov indices, and subsequently to relate the latter to quadratic phase Feynman integrals. With all this in place, the (very brief) fifth chapter is devoted to the culminating task of casting the fact that $c_{\mathbb{A}}$ is split on $Sp(k)$ in terms of generalized adèlic phase integrals and in terms of the operators discussed by Robbin and Salamon in this context.

We should add that something of a *caveat* is in order regarding what we are up to: since the prevailing thrust of this article is ultimately number theoretic, we take great pains to develop the physical themes that come into play, given their relative unfamiliarity, and in this regard we also take the liberty of providing some historical background where indicated. To mitigate the irregularity in emphasis this engenders, we provide many references. Additionally, in extending our main result from the paradigm case of the real field (as a completion of the rational numbers) to other local fields so as to include all the completions of our base field, k, and subsequently proceeding over to the case of the k-adèles as the underlying topological ring, we are faced with something of a technical imbalance. As far as our number-theoretic considerations are concerned, we are traveling well-trodden paths which were first developed by such scholars as Tate [54] and Weil [64], and which over the years took their place in the mainstream of both analytic and algebraic number theory. But when it comes to quantum physics, considerations of non-archimedean settings are both still rather novel and, as far as our purposes are concerned, only partly developed. Specifically, whereas the theory of unitary operators on a Hilbert space is certainly part of quantum mechanics' formalism as developed by von Neumann, the context is of course that of space and time as generally understood, and the presupposition is made that measurement (always a rational number, no matter how sophisticated the instrumentation) ultimately "lives" in the real numbers; indeed, this hypothesis in implicit in science as a general rule: why should one even consider other completions of the rational numbers? But in the last few decades a small number of scholars have begun to consider the possibility that a non-archimedean setting, i.e. p-adic completions of $\mathbb{Q}$, should be a proper context for quantum mechanics (cf. [57,47,58,59]), and most recently versions of quantum mechanics have been proposed in the context of none other than the $\mathbb{Q}$-adèles (cf. [14,17,16]), with, in both cases (locally as well as adèlically) Weyl quantization taking the lead and obviating the need for a Hamiltonian. Thus, over the base field $\mathbb{Q}$ there is available to us a well-developed non-archimedean formalism which we can bring into play as, so to speak, the other side of the equation (rather literally as it turns out) when it comes to our central results pitting the Weil 2-cocycle $c_{\mathbb{A}}$ against functional phase integrals. What is missing, however, is a well-developed quantum mechanics formalism over the local fields $k_{\mathfrak{p}}$, and then over the adèle ring $k_{\mathbb{A}}$ with $(k : \mathbb{Q}) > 1$: the physicists obviously have no need for such conceits. Therefore, in what we do below in this most general setting, we provide only the architecture of what our general formulas should look like (sufficient for the purposes outlined above), but leave aside for now the task of lifting p-adic and $\mathbb{Q}$-adèlic quantum mechanics to $\mathfrak{p}$-adic and k-adèlic quantum mechanics. We save this for a sequel to the present work. What we present here is autonomous.

Furthermore, specifically concerning the types of integrals that enter into our considerations, i.e., the aforementioned functional phase integrals, there is an element of ambiguity present that we should like to say a few words about now, before we get underway. Of course, on the number theoretic side the integrals that occur are entirely well-defined as (in their most general form) Haar integrals. The physics side of things, as it stands, will ultimately be seen to involve physically meaningful integrals of the type considered (over the real numbers) by Jean-Marie Souriau [53], who was following leads by V.I. Arnol'd [1] and Jean Leray [33]. The latter physically meaningful integrals are all taken over some $\mathbb{R}^n$ and, as such, avoid the notorious problem of Feynman path integrals proper, *viz.* their ill-definition due to the absence of a true measure on Feynman's spaces of paths. The integrals we reach "on the physics side of things" in what follows are, first, also integrals over $\mathbb{R}^n$ (and indeed correspond to those considered in []) and, subsequently, are generalizations of these well-defined integrals to both local and adèlic non-archimedean settings. What is not done yet, but what we hope to get to in a future study, is to address the natural next move, namely, what happens when we go from the present path integrals to Feynman path integrals; we note that this tactic fits with what Robbin and Salamon undertake in [45] and [44]. Additionally, what all this comes down to is that we take the liberty to play fast and loose with terminology, as far as the jargon we use in the present article is concerned. This is much along the lines of what seems to be conventional in the literature: generally speaking the physically meaningful integrals we deal with are functional integrals and phase integrals because there is a phase that defines them (in the usual manner), and, since the action (or Lagrangian) appearing in the integrand is multiplied by the imaginary unit, can also be called oscillatory integrals. We use all these descriptions, and what should be borne in mind is that because they set the stage for Feynman's integrals, they indeed occupy a very special role in quantum physics.

Finally, the author wishes to express his sincere gratitude to a referee, whose cogent suggestions were incorporated into this work.

2. The Segal–Shale–Weil representation

2.1. Shale's paper

Let $\mathcal{H}$ be the real Hilbert space consisting of the real normalizable solutions ψ of the Klein–Gordon equation for the electron,

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \triangle\psi - \frac{m^2c^2}{\hbar^2}\psi \tag{2.1}$$

in the form given by Pauli [39, p. 146] in his elegant treatment of Dirac's relativistic wave equation for the free electron. As always, $\triangle$ is the Laplacian, c is the speed of light, $\hbar$ is Planck's constant, and m is the electron's mass. Following I.E. Segal [49], $\mathcal{H}$ admits a non-degenerate skew-symmetric bilinear form B by means of the procedure of first complexifying $\mathcal{H}$, i.e. forming the space $\mathcal{H} \otimes \mathbb{C}$, recovering $\mathcal{H}$'s original inner product by taking the real part of the inner product on $\mathcal{H} \otimes \mathbb{C}$, and then stipulating that B should be the accompanying imaginary part. Thus, we are immediately in possession of a symplectic datum $(\mathcal{H}, B)$ as part of the standard formalism of relativistic quantum mechanics for the present system.

Next, if $T : \mathcal{H} \to \mathcal{H}$ is a bounded (whence continuous) linear operator, recall that T is of Hilbert–Schmidt type (or simply Hilbert–Schmidt) if its L^2-norm relative to a(ny) orthonormal basis $\{x_\alpha\}_\alpha$ for $\mathcal{H}$ is finite: $\sum_\alpha \|T(x_\alpha)\| < \infty$. The class of Hilbert–Schmidt operators on $\mathcal{H}$ constitutes a Banach space denoted by $HS(\mathcal{H})$.

Then, as usual, write $GL(\mathcal{H})$ for the composition group of all bounded linear operators T on $\mathcal{H}$ for which T^{-1} is also bounded (i.e. continuous). If, generally, $|T|$ stands for the self-adjoint part of the polar decomposition of T, define, first, the subgroup

$$GL(\mathcal{H})_{HS} = \{T \in GL(\mathcal{H}) \mid T = I + E,\ E \text{ elementary},\ E \in HS(\mathcal{H})\} \tag{2.2}$$

of $GL(\mathcal{H})$, and, second, define what Shale calls the restricted general linear group:

$$RGL(\mathcal{H}) = \{T \in GL(\mathcal{H}) \mid |T| \in GL(\mathcal{H})_{HS}\}. \tag{2.3}$$

Shale then proves [50, p. 152] that $RGL(\mathcal{H})$ is a topological group, and if the symplectic group for $\mathcal{H}$ is the obvious isotropy group, namely,

$$Sp(\mathcal{H}) := \{T \in GL(\mathcal{H}) \mid \forall \psi, \varkappa \in \mathcal{H}, B(T\psi, T\varkappa) = B(\psi, \varkappa)\}, \tag{2.4}$$

and

$$RSp(\mathcal{H}) := Sp(\mathcal{H}) \cap RGL(\mathcal{H}), \tag{2.5}$$

then we obtain

Proposition 1. *If $\mathcal{H}$ is decomposed as*

$$\mathcal{H} = \Lambda^{-1}(\mathcal{M}) \oplus \mathcal{M} \tag{2.6}$$

where $\Lambda : \mathcal{H} \to \mathcal{H}$ and $\Lambda^2 = -I$, then

(i) *any $T \in Sp(\mathcal{H})$ can be realized as*

$$T = U(S^{-1} \oplus S)U' \tag{2.7}$$

where $S \in GL(\mathcal{H})$ is self-adjoint, $S \oplus S^{-1}$ acts on $\Lambda^{-1}(\mathcal{M}) \oplus \mathcal{M}$ in the obvious way, and U, U' are unitary operators on $\mathcal{H}$; and

(ii) *any $T \in RSp(\mathcal{H})$ can be realized similarly but with the proviso that S be an element of $GL(\mathcal{H})_{HS}$.*

It is the decomposition (2.7) that provides a rationale for the oscillator representation as a projective unitary representation, as we now shall see.

Again with Shale we write, for any natural number n, fixed and suppressed, $S(\mathcal{H}^{FC})$ for the symmetric n-fold tensor subalgebra of $(\mathcal{H}^{FC})^{\otimes n}$ associated to the Fock–Cook quantization of the Hilbert space $L^2(M^{\otimes n})$, M being the underlying space–time manifold. It was shown by I.E. Segal [48] that there is in a place a duality

$$S(\mathcal{H}^{FC}) \underset{D}{\sim} L^2(M^{\otimes n}) \tag{2.8}$$

which intertwines with the canonical unitary representation γ of the unitary group $\mathcal{U}(\mathcal{H}^{FC})$ on the algebra $S(\mathcal{H}^{FC})$. In other words, we get, for all unitary operators T on $S(\mathcal{H}^{FC})$ a mapping

$$\Gamma(T) := D \circ \gamma(T) \circ D^{-1} : L^2(M^{\otimes n}) \to L^2(M^{\otimes n}). \tag{2.9}$$

Additionally, if we denote by Ξ the mapping from $RGL(\mathcal{H})$ to $\mathcal{U}(L^2(M^{\otimes n}))$ defined by

$$\Xi(T)(f(x)^{tame}) = \delta(T)^{1/2} f(T^* x) \tag{2.10}$$

where $\delta(T)$ is a Radon–Nikodym derivative for whose specifications we refer to [50] and [49], then Ξ is weakly continuous, and, most importantly, we have all the ingredients needed to delineate the projective oscillator representation of $RSP(\mathcal{H})$. Specifically, if a generic T is decomposed as $T = U(S^{-1} \oplus S)U'$ as in (2.7), then the oscillator representation is the association

$$T \mapsto \Gamma(U) \circ \Xi(S) \circ \Gamma(U'). \tag{2.11}$$

(To be precise, (2.10) is actually to be understood as something of a germ of the oscillator representation as Shale constructs it in [50]: he works with a unitary ray realized by taking the right side of (2.10) as its representative $\mathrm{mod}^{\times}\mathbb{C}_1^{\times}$.)

2.2. Some remarks on the Heisenberg rules and symplectic forms

To be sure, the material in the preceding section introducing Shale's construction of the oscillator representation is only a framework. For one thing, a great deal of the required functional analysis, evidently due to Shale's advisor, I.E. Segal, is not mentioned; in [50, p. 149] Shale himself characterizes the content of this material as "approximately the same as [what is provided in] my doctoral dissertation". For another, the style of quantization, that of Fock–Cook, is not elaborated at all. Regarding the first shortcoming, we note that our goal in what follows is to provide an initial sketch for Shale's groundbreaking contribution, consonant with later reformulations of Shale's results in a form more amenable to number theory rather than the physics of bosons.

Thus, while everything is indeed fitted into a quantum mechanics formalism, our orientation is more in the direction of the geometry provided by a skew-symmetric bilinear form, i.e. B, above, than in the direction of the mass of functional analytic details attendant to what is ultimately von Neumann's rendering of quantum mechanics (cf. [61]). This circumstance, that von Neumann's formalism can largely be replaced by symplectic geometry (surrounding B), is also our rationale for tolerating the second shortcoming, i.e. skirting any elaboration of Fock–Cook quantization. In fact, in this regard Shale himself simply notes [60] that his, and Segal's, chosen commutation relations, i.e. (*verbatim*),

$$V(z_1)V(z_2) = e^{-iB(z_1,z_2)/2} V(z_1 + z_2) \tag{2.12}$$

for all $z_1, z_2 \in \mathcal{H}$, "are essentially those given by Weyl (in [65] and) have been used by von Neumann [60] for finite systems and by Segal [49] for fields". It is standard fare, however, that Weyl quantization, which evidently engenders in (2.12) the composition law for a Lie group (cf. Proposition 2, and the remarks that follow), is not just formally equivalent to Heisenberg quantization, but is in fact directly based on Heisenberg's work of 1922, setting out his famous commutation laws which are, after all, part and parcel of canonical quantization.

Beyond this, as we shall see in Section 4.2, Weyl quantization is required for the development of non-archimedean quantum mechanics, including adèlic quantum mechanics, which ultimately constitutes the context for most of our discussion of the interplay between the yoga of 2-Hilbert reciprocity in the style of Weil and Kubota and a formalism of phase integrals.

Accordingly, capitalizing on the physical as well as mathematical equivalence between Heisenberg quantization and Weyl quantization, we can safely start with symplectic structure *per se*, taking symplectic geometry to be, for our purposes, the best context for the oscillator representation, a.k.a. the Segal–Shale representation in the present physics context. Therefore, we start with the Heisenberg picture of quantum mechanics.

First, however, a bit of history (cf. [56], specifically Van der Waerden's opening remarks). It is commonly agreed that quantum mechanics, in roughly the form in which it is now practiced, appeared on the scene in 1925 as the next evolutionary step after Heisenberg and Born's formulation of matrix mechanics. Indeed, 1925 was the year in which (in the single month of November, in fact) not only the famous *Dreimännerarbeit* [6] was submitted to Z. Physik, but Dirac presented his own formulation of quantum mechanics in terms of Poisson brackets [10] to Proc. Royal Society. We start by taking the former article as our point of departure for Heisenberg's picture of quantum mechanics and Heisenberg quantization.

So it is, then, that Born, Heisenberg, and Jordan present, in the opening pages of [6], the famous relation

$$pq - qp = \frac{h}{2\pi i}\mathbf{1}, \tag{2.13}$$

for matrices $p, q, \mathbf{1}(=id)$, the former two being the quantum mechanical counterparts to classical momentum and position, respectively. In the authors' subsequent discussions of quantum mechanical systems of, say, n degrees of freedom, these commutation relations are fleshed out to read, *verbatim*,

$$p_k q_l - q_l p_k = \frac{h}{2\pi i}\delta_{kl} \tag{2.14}$$

$$p_k p_l - p_l p_k = 0 \tag{2.15}$$

$$q_k q_l - q_l q_k = 0 \tag{2.16}$$

for all $1 \leq k, l \leq n$, with δ_{kl} Kronecker's delta: $\delta_{kl} = 1$ (resp. 0) when $k = l$ (resp. $k \neq l$). We see immediately, however, that if $[\_, \_]$ denotes the commutator (or Lie bracket), as usual, then the preceding relations can be recast as

$$[p_k, p_l] = 0 = [q_k, q_l] \tag{2.17}$$

$$[p_k, q_l] = \frac{h}{2\pi i}\delta_{kl} = -[q_l, p_k]. \tag{2.18}$$

This formalism is already amenable to interpretation as a carrier of symplectic structure, but it is traditional to normalize the Lie bracket first, so as to yield the schema

$$[p_k, p_l] = 0 = [q_k, q_l] \tag{2.19}$$

$$[p_k, q_l] = \delta_{kl} = -[q_l, p_k] \tag{2.20}$$

with $1 \leq k, l \leq n$. Evidently this normalization can also be effected at the level of the Schrödinger wave equation (see [42] for an in-context treatment) as well as that of Dirac's formulation of quantum mechanics using Poisson brackets. It is particularly apposite to single out Dirac's approach, in contrast to those of Schrödinger and Heisenberg, because of its consonance with Feynman's rendering of quantum mechanics going back to his thesis [20].

With this set-up in place, following [23] (the source for everything in this section), coordinatize $\mathbb{R}^{2n}$ *via* the symbols $\{p_k; q_l\}_{1\leq k,l\leq n}$ and realize it as the phase space for the indicated n-particle system. Hamiltonian mechanics requires that this flavor of $\mathbb{R}^{2n}$ be identified with the cotangent bundle to the particle's base space, and that the phase space transformations giving the system's time evolution and its symmetries are diffeomorphisms fixing the non-degenerate differential form

$$\omega = \sum_{j=1}^{n} dp_j \wedge dq_j. \tag{2.21}$$

But now it is standard differential geometric practice to refer to these phase space diffeomorphisms as symplectomorphisms, meaning that these mappings cut out the symplectic group $Sp(2n, \mathbb{R})$ as the subgroup of $GL(2n, \mathbb{R})$ defined by the relation

$$B(x^\sigma, y^\sigma) = B(x, y) \tag{2.22}$$

for all $x, y \in \bigoplus_{k=1}^{n} \mathbb{R}p_k \oplus (\bigoplus_{l=1}^{n} \mathbb{R}q_l) \approx \mathbb{R}^{2n}$, where B is the skew-symmetric bilinear form obtained from ω by identifying the tangent space (another isomorph of $\mathbb{R}^{2n}$, of course) with the aforementioned base space, and σ runs through $Sp(2n, \mathbb{R})$. The thrust of this identification is that if we have $x = (\vec{p}, \vec{q})$, $y = (\overrightarrow{p'}, \overrightarrow{q'})$ then the data provided by (2.21) transmogrifies to the characterization of B as

$$\begin{aligned} B = B_\omega : \left(\left(\bigoplus_{k=1}^{n} \mathbb{R}p_k\right) \oplus \left(\bigoplus_{l=1}^{n} \mathbb{R}q_l\right)\right)^2 \ni (x, y) \mapsto \vec{p} \cdot \overrightarrow{q'} - \vec{q} \cdot \overrightarrow{p'} \\ = \sum_k p_k q'_k - q_k p'_k \in \mathbb{R}. \end{aligned} \tag{2.23}$$

So, to be sure, we have arrived at the definition of one of the principals in our story, the symplectic group $Sp(2n, \mathbb{R})$, in its usual form: it is the isotropy group of B.

Due to the non-degeneracy of ω, we can also identify each tangent vector X at $(\vec{p}, \vec{q})$ with the cotangent vector τ_X at the same point in such a way that for any other tangent vector Y we have that

$$\tau_X(Y) = \omega(X, Y). \tag{2.24}$$

The induced association of tangent and cotangent vectors, or, equivalently (and preferably) of vector fields and differential forms, lifts to the level of classical observables which are, by definition, functions f on the phase space $\mathbb{R}^{2n}$ in the form of the association of the Hamiltonian vector field X_f to $df \in \Omega^1(\mathbb{R}^{2n})$ with

$$\omega(Y, X_f) = df(Y). \tag{2.25}$$

In terms of the canonical basis of the cotangent space this means that

$$X_f = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial}{\partial p_j} \right). \tag{2.26}$$

It is now standard fare t to define, for smooth observables f, g, the Poisson bracket

$$\{f, g\} = \omega(X_f, X_g) = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \right), \tag{2.27}$$

which is manifestly skew-symmetric and satisfies the Jacobi identity: the space of smooth (i.e., C^∞) observables acquires a Lie algebra structure. Moreover, the association $f \longmapsto X_f$ is a Lie algebra homomorphism since $[X_f, X_g] = X_{\{f,g\}}$. Additionally we get that

$$\{p_k, p_l\} = 0 = \{q_k, q_l\} \tag{2.28}$$

$$\{p_k, q_l\} = \delta_{kl}. \tag{2.29}$$

So much for the classical Hamiltonian formalism.

Quantization within the Hamiltonian framework centers on the famous correspondences between the classical and quantum pictures stipulated by the founders of quantum mechanics. In particular, the classical (position) coordinate functions $q_1, \ldots, q_n$ are interpreted as, and, in point of fact, replaced by, self-adjoint operators $Q_1, \ldots, Q_n$ on $L^2(\mathbb{R}^{2n})$, whose projectivization gives the space of states of the quantum mechanical system. Additionally, the (momentum) coordinate functions $p_1, \ldots, p_n$ are thereupon regarded as, or replaced by, self-adjoint operators $P_1, \ldots, P_n$ on $L^2(\mathbb{R}^{2n})$. Under these correspondences, which is to say, within this new framework of operators on a Hilbert space, we now obtain, in much the same way as before (cf. (2.17) and (2.18)),

$$[P_k, P_l] = 0 = [Q_k, Q_l] \tag{2.30}$$

$$[P_k, Q_l] = \frac{h}{2\pi i} \delta_{kl} \mathbf{1}. \tag{2.31}$$

Says Folland in this connection [23, pp. 15–16]: “By the ‘quantization problem’ we …mean the problem of setting up a correspondence …between classical and quantum observables, i.e., between functions on $\mathbb{R}^{2n}$ and self-adjoint operators on $L^2(\mathbb{R}^{2n})$, such that the properties of the classical observables are reflected as much as possible in their quantum counterparts in a way consistent with the probabilistic interpretation of quantum observables ..”. So it is, then ([23], p. 17), that “[t]he Poisson bracket relations for canonical coordinates in Hamiltonian mechanics [being our (2.28) and (2.29)] and ... their quantum analogues are formally identical". However, in quantum mechanics the next move is to introduce a Lie algebra structure as follows (and we adopt de Gosson's definition 6.1 on p. 161 of [8] for our purposes):

Proposition 2. *Take P_k (resp. Q_l), with $1 \leq k, l \leq n$, to be the operators*

$$P_k\psi = -i\frac{h}{2\pi}\frac{\partial\psi}{\partial q_k} \tag{2.32}$$

$$Q_l\psi = q_l\psi. \tag{2.33}$$

Then the data $\{P_k, Q_l\}_{k,l}$ satisfies (2.30) and (2.31). Furthermore, if we add the operator

$$T\psi = i\frac{h}{2\pi}\psi \tag{2.34}$$

to the mix, with $[P_k, T] = 0 = [Q_l, T]$, for $1 \leq k, l \leq n$, then

$$\mathfrak{N} := \left(\bigoplus_{k=1}^{n}\mathbb{R}P_k\right) \oplus \left(\bigoplus_{l=1}^{n}\mathbb{R}Q_l\right) \oplus \mathbb{R}T \tag{2.35}$$

is a Lie algebra under the indicated bracket, $[\_, \_]$.

$\mathfrak{N}$ is in fact a Heisenberg Lie algebra (see (2.39)), and getting from there to a Heisenberg (Lie) group is now a routine matter involving the exponentiation mapping. This will be one of the main things to do in our next section, so we end our discussion of this form of quantization here, except for the following remark by de Gosson ([8, p. 160]): "The Heisenberg group is a simple mathematical object; its interest in quantization problems comes from the fact that its Lie algebra represents in abstract form the canonical commutation relations".

2.3. From the Heisenberg Lie algebra and group to the oscillator representation of the real symplectic group

Given any number field that is not totally imaginary, the field of real numbers appears as a localization of the number field k at a real archimedean prime: $\mathbb{R}$ is itself a local field. Seeing that in connection with what follows, *viz.* adèlization, we must treat all places of the number field (or $\mathbb{A}$-field, in Weil's vernacular [64]) on an equal footing, be they archimedean or not, we emphasize at the start of this section that working over $\mathbb{R}$ should be seen as something of a model for what follows: in due course we will be working over an arbitrary local field $k_\mathfrak{p}$. Additionally, in accord with our primary text [34], we work first in dimension $2n$, even though our ultimate goal is to apply our results to the case considered in [63], i.e., $Sp(2, k_\mathfrak{p}) = Sp(k_\mathfrak{p} \times k_\mathfrak{p}^*)$, meaning that $n = 1$. In light of future maneuvers it is useful, however, to have the indicated preliminaries taken care of for arbitrary dimensions. But when we reach the point where Weil's result to the effect that his 2-cocycle, $c_\mathfrak{p}$, *à priori* an element of $H^2(Sp(2n, k_\mathfrak{p}), \mathbb{C}_1^\times)$, in fact takes values in $\{1, -1\}$ for all places $\mathfrak{p}$, so that we are always dealing with a double cover of the (local) symplectic group $Sp(2n, k_\mathfrak{p})$, we restrict to the case of arithmetical interest and set $n = 1$. This is done so as to be able to get to the adèlization of the collective data $\{c_{\mathfrak{p}}\}_{\mathfrak{p}}$ as quickly as possible, directly using the formula provided by Kubota in [31] instead of, say, the Matsumoto construction (cf. [36] and [30]), or even Weil's own original discussion. But for the moment, the discussion that follows, dealing with the general n-dimensional case, is adapted from [34] and [23].

The ambient vector space is

$$V := \left(\bigoplus_{i=1}^{n} \mathbb{R}P_i\right) \oplus \left(\bigoplus_{j=1}^{n} \mathbb{R}Q_j\right), \tag{2.36}$$

where the set $\{P_i; Q_j\}_{1\leq i,j\leq n}$ is a symplectic basis with respect to the non-degenerate skew-symmetric bilinear form B, meaning that

$$B(P_i, P_j) = 0 = B(Q_i, Q_j) \tag{2.37}$$

$$B(P_i, Q_j) = \delta_{ij} = -B(Q_j, P_i), \tag{2.38}$$

with B mapping bilinearly into $\mathbb{R}$. Under this regime, introduce a formal symbol E and stipulate that

$$\mathfrak{N} := V \oplus \mathbb{R}E, \tag{2.39}$$

where, in the presence of an obvious Lie bracket $[\_, \_]$, we have that

$$[\mathfrak{N}, E] = 0 \tag{2.40}$$

$$\forall x, y \in V, \quad [x, y] = B(x, y)E. \tag{2.41}$$

Thus, $(\mathfrak{N}, [\_, \_])$ acquires the structure of a Lie algebra: the Heisenberg Lie algebra. It is then standard that $\mathfrak{N}$ is associated to

$$\exp(\mathfrak{N}) =: N, \tag{2.42}$$

the corresponding Heisenberg group, also written as $Heis(V; B)$, where, with the exponential map in the game, the group law on N is given by

$$\exp(x + tE)\exp(x' + t'E) = \exp\left(x + x' + \frac{1}{2}B(x, x')E\right). \tag{2.43}$$

Accordingly we also have (using the usual notational convention that if G is a Lie group and $\mathfrak{g}$ is its Lie algebra, then $Lie(G) = \mathfrak{g}$ and $\exp(\mathfrak{g}) = G$) that

$$\mathfrak{N} = Lie(N) = Lie(Heis(V; B)). \tag{2.44}$$

Next, we define a subspace l of V to be (a) Lagrangian (plane) if it is self-dual with respect to B, expressed compactly as $l^{\perp} = l$, where

$$l^{\perp} = \{x \in V \mid B(x, y) = 0, \forall y \in l\}. \tag{2.45}$$

One proves quickly that any Lagrangian plane has dimension $n = \frac{1}{2}\dim(V)$. We say that two Lagrangian planes, $l, l^{\perp}$ are transverse if $l \cap l^{\perp} = 0$. It is standard that V admits a decomposition into pairwise transverse Lagrangian planes:

$$V = l \oplus l', \quad \text{where } l \cap l^{\perp} = 0 \text{ and } l' \cap (l')^{\perp} = 0. \tag{2.46}$$

Now, fixing such a decomposition, let

$$L = \exp(l \oplus \mathbb{R}E) < N, \tag{2.47}$$

and consider the group character

$$\chi_L : L \to \mathbb{C}_1^{\times} \quad via \quad \exp(x + tE) \mapsto e^{2\pi i t}. \tag{2.48}$$

It is easy to see that if we define the following Hilbert space naturally associated to the chosen Lagrangian plane l

$$\mathcal{H}(l) := \{\varphi \in L^2(\mathfrak{N}) \mid \forall y \in l, \forall x \in \mathfrak{N}, \varphi(yx) = \chi_L^{-1}(y)\varphi(x)\}, \tag{2.49}$$

where we abuse language a bit by just writing χ_L for $\chi_L \circ \exp$, then χ_L can be realized as acting in the algebra $\mathcal{U}(\mathcal{H}(l))$ of unitary operators on $\mathcal{H}(l)$ as follows:

$$\chi_L : L \ni \exp(x + tE) \mapsto [f \mapsto e^{2\pi i t} f] \in \mathcal{U}(\mathcal{H}(l)). \tag{2.50}$$

In other words, we have, simply, that

$$\chi_L(\exp(x + tE)) = e^{2\pi i t} \cdot id_{\mathcal{H}(l)}. \tag{2.51}$$

The point to be taken is that we can regard χ_L as a central character, seeing that the center $Z(\mathfrak{N})$ of $\mathfrak{N}$ is just the group $\exp(\mathbb{R}E)$ (cf. [23]). This is of huge significance as we proceed to invoke the theorem of Stone and von Neumann.

To wit, we define the Schrödinger representation of the Heisenberg group $Heis(V; B) = N$ as the induced representation

$$Ind_L^N(\chi_L) : N \to \mathcal{U}(\mathcal{H}(l)), \tag{2.52}$$

and one generally uses the Stone–von Neumann Theorem to infer that we have an irreducible representation [23, p. 35 ff]. For our present purposes it is apposite, however, to cite the following equivalent phrasing of the Stone–von Neumann Theorem at this stage:

Proposition 3. *If $\varrho : Heis(V; B) \to \mathcal{U}(\mathfrak{H})$ is any unitary representation of the Heisenberg group (just N) in some Hilbert space $\mathfrak{H}$, and if*

$$\varrho|_{Z(N)} = \chi_L, \tag{2.53}$$

then ϱ and the Schrödinger representation $Ind_L^N(\chi_L)$ are unitarily equivalent. Here $Z(N)$ is just the center of N, which is the group $\mathbb{R}E$. In other words, for all $x \in N$, we have

$$\varrho(x) = U^{-1} \circ Ind_L^N(\chi_L)(x) \circ U, \tag{2.54}$$

where $U : \mathfrak{H} \overset{\approx}{\to} \mathcal{H}(l)$.

If, with Lion–Vergne (p. 13 of [34]), we simply write $W(l)$ instead of the more cumbersome expression $Ind_L^N(\chi_L)$, stressing dependence on l as per (2.49), then the preceding assertion yields immediately that for any Lagrangian planes l_1, l_2,

$$W(l_1)(x) = \mathsf{FT}_{1,2} \circ W(l_2)(x) \circ \mathsf{FT}_{2,1}, \tag{2.55}$$

with $\mathsf{FT}_{2,1}\colon \mathcal{H}(l_1) \to \mathcal{H}(l_2)$ a (partial) Fourier transform (of rather an abstract sort; see p. 30, ff., of [34]), so that $\mathsf{FT}_{1,2} = (\mathsf{FT}_{2,1})^{-1}$. This will figure prominently presently.

Furthermore, having introduced in Proposition 3 the form of the Stone–von Neumann Theorem we wish to use, we now come to its principal application in the present setting, namely, the (re)introduction and explication of the oscillator representation. Write $Sp(V; B)$ or, equivalently, $Sp(2n, \mathbb{R})$, with $\dim(V) = 2n$, for the isotropy group of the symplectic data (V, B):

$$\begin{aligned} Sp(V; B) &= Sp(2n, \mathbb{R}) \\ &= \{\sigma \in GL(2n, \mathbb{R}) \mid \forall x, y \in V,\ B(x^\sigma, y^\sigma) = B(x, y)\}. \end{aligned} \tag{2.56}$$

We immediately obtain the group action of $Sp(2n, \mathbb{R})$ on N *via*

$$\sigma : \exp(x + tE) \longmapsto \exp(x^\sigma + tE) \tag{2.57}$$

for which, obviously, $\sigma|_{Z(N)} = id_{Z(N)}$.

It follows from all the preceding that $(W(l), \mathcal{H}(l))$ is a unitary N-module, and, if we write $W^\sigma(l) : N \to \mathcal{U}(\mathcal{H}(l))$ for the map, or data, $W(l)(x^\sigma) : \mathcal{H}(l) \to \mathcal{H}(l)$, then one checks immediately that, $\forall \sigma \in Sp(2n, \mathbb{R})$,

$$W^\sigma(l)|_{Z(N)} = \chi_L, \tag{2.58}$$

so we can apply Proposition 3 to get

Corollary 1. *For any $\sigma \in Sp(2n, \mathbb{R})$, the unitary representations $W(l)$ and $W^\sigma(l)$ are intertwined:*

$$W(l)(x) = \mathsf{FT}_\sigma^{-1} \circ W^\sigma(l)(x) \circ \mathsf{FT}_\sigma, \tag{2.59}$$

where FT_σ maps $\mathcal{H}(l)$ isomorphically to itself, realizing $\mathcal{H}(l)$ first as the representation space for $W(l)$, then as the representation space for $W^\sigma(l)$.

Before we get to the *dénouement* of this section, namely, the exploitation of the preceding corollary to demonstrate that FT essentially determines a class in $H^2(Sp(2n, \mathbb{R}), \mathbb{C}_1^\times)$, we wish to note that we have here an illustration of how isomorphy can cover a lot of sins. Specifically, when it comes time to delineate the connection between the Weil(-Kubota) 2-cocycle, i.e. the cohomology class just mentioned, and the Kashiwara triple index, the data given above in terms of intertwining operators needs to be connected to the behavior of Lagrangian planes. Ultimately the net-effect will be that a special choice should be made for l, and therefore for $L = \exp(l \oplus \mathbb{R}E)$, and in this way a specific instance of $Ind_L^N(\chi_L) = W(l)$ is tagged. But this will only come to light after a number of maneuvers with, e.g., (2.59) and its ilk.

All this having been said, we can infer the following critical result, the aforementioned *dénouement*:

Proposition 4. *There exists a unimodular scalar* $c(\sigma_1, \sigma_2)$*, depending on* $\sigma_1, \sigma_2 \in Sp(2n, \mathbb{R})$*, such that*

$$\mathsf{FT}_{\sigma_1\sigma_2} = c(\sigma_1, \sigma_2)\mathsf{FT}_{\sigma_1} \circ \mathsf{FT}_{\sigma_2}. \tag{2.60}$$

Proof. It follows from Hilbert–Schmidt theory (cf. [34], p. 21, ff.) that a bounded unitary operator on $\mathcal{H}(l)$ that commutes with all the $W(l)(x)$ is a scalar. It then follows easily, e.g. from a manipulation of commutative diagrams conveying the relevant intertwinings, that $\mathsf{FT}_{\sigma_1} \circ \mathsf{FT}_{\sigma_2} \circ \mathsf{FT}_{\sigma_1\sigma_2}^{-1} \in \mathbb{C}^\times$. Lastly it follows directly from unitarity that $|\mathsf{FT}_{\sigma_1} \circ \mathsf{FT}_{\sigma_2} \circ \mathsf{FT}_{\sigma_1\sigma_2}^{-1}| = 1$. ■

Therefore we have a mapping

$$c : Sp(2n, \mathbb{R}) \times Sp(2n, \mathbb{R}) \to \mathbb{C}_1^\times \tag{2.61}$$

characterized by (2.60); it only remains for us to call upon the usual associativity arguments (cf. [26]) to establish that

$$c(\sigma_1\sigma_2, \sigma_3) = c(\sigma_1, \sigma_2\sigma_3)c(\sigma_2, \sigma_3) \tag{2.62}$$

in $\mathbb{C}_1^\times$, which implies, by definition, that $c \in H^2(Sp(2n, \mathbb{R}), \mathbb{C}_1^\times)$.

The thrust of Proposition 4 is that FT is a projective unitary representation with c as its associated factor set. It is this mapping that is commonly known as the Segal–Shale–Weil representation, or the oscillator representation, in keeping with the remarks in the Introduction. As we also mentioned there, we opt for the latter term in the present setting, i.e., that of the representation theory of the real Heisenberg Lie algebra $\mathfrak{N} = (\bigoplus_{i=1}^n \mathbb{R}P_i) \oplus (\bigoplus_{j=1}^n \mathbb{R}Q_j) \oplus \mathbb{R}E$ and its associated (Heisenberg) Lie group $\exp(\mathfrak{N}) = N$.

2.4. From the oscillator representation of the symplectic group to the Kashiwara triple index

Using (2.55) we note right off that if l is a Lagrangian plane in V and $\sigma \in Sp(V)$ then l^σ is also a Lagrangian plane, and it follows that we get

$$W(l) = \mathsf{FT}_{l^\sigma, l}^{-1} \circ W(l^\sigma) \circ \mathsf{FT}_{l^\sigma, l}, \tag{2.63}$$

where we have written $\mathsf{FT}_{l^\sigma, l}$ for the obvious transform mapping from $\mathcal{H}(l)$ to $\mathcal{H}(l^\sigma)$. Soon, this identity will serve us well in connection with the relation between the object of present concern, Kashiwara's triple index (cf. [29], p. 486), and the Weil–Kubota 2-cocycle c discussed in the preceding section.

However, the Kashiwara index $\tau(=\tau_\infty)$ is itself defined in terms of triples of Lagrangian planes. Specifically, given pairwise transverse Lagrangian planes l_1, l_2, l_3 in $V = (\bigoplus_{i=1}^n \mathbb{R}P_i) \oplus (\bigoplus_{j=1}^n \mathbb{R}Q_j)$, consider the quadratic form

$$\begin{aligned} q_B : l_1 \oplus l_2 \oplus l_3 \ni x_1 + x_2 + x_3 \\ \longmapsto B(x_1, x_2) + B(x_2, x_3) + B(x_3, x_1) \in \mathbb{R}. \end{aligned} \tag{2.64}$$

Recalling that a quadratic form's signature is the difference between the number of positive eigenvalues and the number of negative eigenvalues of the associated symmetric matrix (cf. e.g. [46], p. 704), we accordingly take the Kashiwara triple index to be

$$\tau_\infty(l_1, l_2, l_3) = \operatorname{sgn}(q_B) \tag{2.65}$$

in the present real case.

The preceding characterization of τ_∞, predicated on the presupposition that we have pairwise transversality among l_1, l_2, l_3, is a condition that Lion and Vergne proceed to remove by means of defining another quadratic form, namely, $B(p_{13}x, p_{31}x)$, acting on l_2, with the mappings p_{13}, p_{31} being the natural projections. This then allows:

Proposition 5. *Suppose* $l_1 \cap l_3 = (0)$*, i.e.,* l_1, l_3 *are transverse Lagrangian planes. Then*

$$\tau(l_1, l_2, l_3) = \operatorname{sgn}(l_2; B(p_{13}(\_), p_{31}(\_))). \tag{2.66}$$

In this connection, see also Proposition 14.

We can now list a number of properties of triples of Lagrangian planes and the attendant behavior of $\tau_\infty (=\tau,$ simply, in the remainder of this section) that are carefully worked out in [34], following Masaki Kashiwara; see the appendix to [29]. We say more about these things later; however, as far as detailed proofs go, the reader should consult [34].

Proposition 6. *For any triple of Lagrangian planes and any* $\sigma \in Sp(V)$ *we have that*

$$\tau(l_1^\sigma, l_2^\sigma, l_3^\sigma) = \tau(l_1, l_2, l_3). \tag{2.67}$$

Proposition 7. *For any quartet of Lagrangian planes* l_1, l_2, l_3, l_4 *we obtain*

$$\tau(l_1, l_2, l_3) = \tau(l_1, l_2, l_4) + \tau(l_2, l_3, l_4) + \tau(l_3, l_1, l_4), \tag{2.68}$$

which engenders a chain condition.

Proposition 8. *If* V_0 *is any isotropic subspace of* $(V; B)$*, which is to say that, by definition,* $B(V_0, V_0) = 0$*, then* $B|_{V_0^\perp/V_0}$ *is again a nondegenerate symplectic (i.e., skew-symmetric bilinear) form.*

Proposition 9. *If* l *is a Lagrangian plane in* V *and* V_0 *is isotropic as above, and if*

$$l^{V_0} = \left(l \cap V_0^\perp\right) + V_0 = (l + V_0) \cap V_0^\perp \subset V_0^\perp, \tag{2.69}$$

then $(l^{V_0})^\perp = (l^\perp)^{V_0}$*, and accordingly* l^{V_0}/V_0 *is a Lagrangian plane in* $V_0^\perp/V_0$*.*

Finally, as a result of the foregoing (cf. [34], p. 43 ff.), we get

Proposition 10. *If* $V_0 \subset (l_1 \cap l_2) + (l_2 \cap l_3) + (l_3 \cap l_1)$*, then*

$$\tau(l_1^{V_0}, l_2^{V_0}, l_3^{V_0}) = \tau(l_1, l_2, l_3). \tag{2.70}$$

These results illustrate some of the main properties enjoyed by the Kashiwara triple index, including its intrinsically cohomological nature (as per Proposition 7). We also include them at this stage as a prelude of sorts to our upcoming discussion in Section 3.3 regarding the relationship between this index τ, evidently acting on the set $\mathfrak{Lag}(V)^3 = \mathfrak{Lag}(V) \times \mathfrak{Lag}(V) \times \mathfrak{Lag}(V)$, of triples of Lagrangian planes, and the Maslov index μ of Arnol'd and Leray (and Souriau and de Gosson: cf. [1,33,53,8]), as explicated by Cappell, Lee, and Miller [7]. This relationship is critical for what we will do later *vis à vis* the business of bringing in phase integrals.

We note, in addition, that in [34] Lion and Vergne state these propositions without the earlier proviso in place that the Lagrangian planes involved should be pairwise transversal: regarding (2.64), for example, the expression $l_1 \oplus l_2 \oplus l_3$ should then be interpreted as a formal direct sum. This situation is addressed in [8] by de Gosson, whose object is to present these results with no restriction *qua* pairwise orthogonality whatsoever.

Next, for all $i, j \in \{1, 2, 3\}$ we have the relation

$$W(l_i) = \mathsf{FT}_{i,j} \circ W(l_j) \circ \mathsf{FT}_{j,i} \tag{2.71}$$

which follows immediately from (2.59), of course (suppressing x's). Thus, and again due to the fact that the relevant bounded unitary operator on a Hilbert space (in this case $\mathcal{H}(l_1)$) commutes with all the $W(l_i)(x)$ for all $x \in Sp(V)$, it follows (in large part by an induction on the dimension of V) that there is a unimodular scalar $a(l_1, l_2, l_3)$, i.e.

$$a : \mathfrak{Lag}(V)^3 \to \mathbb{C}_1^\times \tag{2.72}$$

such that

$$a(l_1, l_2, l_3) = e^{-\frac{i\pi}{4}\tau(l_1,l_2,l_3)}. \tag{2.73}$$

This puts us in the position to state a result that is of considerable importance to our enterprise, namely

Proposition 11. *There is a Lagrangian plane l_0 in V such that, for all $\sigma_1, \sigma_2 \in Sp(V)$,*

$$a(l_0, l_0^{\sigma_1}, l_0^{\sigma_2\sigma_1}) = c_{l_0}(\sigma_1, \sigma_2) = c(\sigma_1, \sigma_2), \tag{2.74}$$

i.e.,

$$c(\sigma_1, \sigma_2) = c_\infty(\sigma_1, \sigma_2) = a(l_1, l_2, l_3) = e^{-\frac{i\pi}{4}\tau(l_1,l_2,l_3)}. \tag{2.75}$$

In view of these protocols we stipulate that the Lagrangian plane l occurring in our definition (2.52) (see also (2.49)) of the real Schrödinger representation of the Heisenberg group $Heis(V; B) = N$ should be the object l_0 featured in the preceding proposition; thus, with $L_0 = \exp(l_0 + \mathbb{R}E)$, and $\chi_{L_0} : L_0 \to \mathcal{U}(\mathcal{H}(l_0))$,

$$Ind_{L_0}^N(\chi_{L_0}) =: W(l_0). \tag{2.76}$$

This material is part and parcel of §1.6 (pp. 47–63) of [34]. Since it is our objective to extend the characterization (2.75) of the general Weil–Kubota 2-cocycle in terms of Lagrangian planes, first to the indicated symplectic spaces $V_\mathfrak{p}$ defined over any relevant local fields (i.e., for $\mathfrak{p} \in \mathfrak{V}_k$), and then to pass to the according adèlization, we proceed to give an account of the proof of Proposition 11 later. At this stage, however, we should simply like to make a closing observation regarding the values taken by the Weil–Kubota 2-cocycle in light of (2.75): it is obvious that we have

$$c \in H^2(Sp(V; B), \mathbb{Z}_8). \tag{2.77}$$

On the other hand, as we shall also see later, it is in truth the case that

$$c \in H^2(Sp(V; B), \mathbb{Z}_2), \tag{2.78}$$

a fact proven by Weil himself in [63].

2.5. From the real case to the $\mathfrak{p}$-adic case

The situation considered by Weil in [61] is that of an algebraic number field k, i.e., a global field or, in Weil's suggestive language, an $\mathbb{A}$-field, so that the preceding considerations apply directly as long as k has at least one real embedding in $\mathbb{C}$ over $\mathbb{Q}$: we can realize $\mathbb{R}$ as k_∞ for $\infty = \infty_{\mathbb{R}} \in \mathfrak{V}_k$. For the sake of consistency we adopt this position *à forteriori* at this point and interpret Sections 2.3 and 2.4 in this light. With this convention in place we stipulate, then, that the local fields of current interest are the completions $k_\mathfrak{p}$ of k at the places $\mathfrak{p} \in \mathfrak{V}_k$, with, as we stipulated earlier, $\mathfrak{V}_k$ denoting the set of all places of k.

What follows next, crucial to our enterprise, is in effect a synopsis of the Appendix to Part I of [34, p. 104, ff.]. We refer to this as the definitive source for detailed proofs and further discussion. Also, in [34] the stipulation is made that we are not working in characteristic 2; this is of course no problem: each $k_\mathfrak{p}$ has characteristic 0. Additionally, before we get off the ground, it is proper to give credit to the originator of this generalization to arbitrary local fields: this was achieved by Patrice Perrin in his 1979 thesis [40] at Paris, and Perrin co-authored the aforementioned Appendix with Gérard Lion.

First, regarding the Heisenberg group in this general $\mathfrak{p}$-adic setting, fix a decomposition $V = E \oplus E^*$ to emphasize the duality between $E \approx \oplus_{i=1}^n k_\mathfrak{p} P_i$ and $E^* \approx \oplus_{j=1}^n k_\mathfrak{p} Q_j$ brought about through the services of the skew-symmetric ($\mathfrak{p}$-adic) bilinear form $B = B_\mathfrak{p}$ (all in keeping with (2.22)). There is an iota of housekeeping to be taken care of here in the sense that the symplectic structure on V in this form is actually given by means of the rule

$$B(x + f, y + g) = g(x) - f(y); \tag{2.79}$$

see also [23].

This said, we obtain, next, that

$$N = N_\mathfrak{p} = Heis(V_\mathfrak{p}; B_\mathfrak{p}) = V_\mathfrak{p} \times k_\mathfrak{p}. \tag{2.80}$$

(When there is no risk of ambiguity we shall occasionally omit the subscripts to $N_\mathfrak{p}$ and $V_\mathfrak{p}$: they will reappear and figure again when we get to adèlization.) Additionally, fix an additive character $\chi_\mathfrak{p}$ of $k_\mathfrak{p}$ (cf. [54]), pick a Lagrangian subspace $l \subset V$ (i.e., as before, l is by definition maximally isotropic with respect to B, or, what amounts to the same thing,

l is isotropic, i.e. $l = l^{\perp}$, and $\dim(l) = n = \frac{1}{2}\dim(V)$), set $L = l \times k_{\mathfrak{p}}$, and define

$$\mathcal{H}(l) = \Big\{\varphi : N \to \mathbb{C} \mid \forall x \in N, \forall y \in L, \varphi(xy) = (1 \otimes \chi_{\mathfrak{p}})(y)^{-1}\varphi(x) \text{ and } \int_{N/L} |\varphi|^2 < \infty\Big\} \tag{2.81}$$

(here the integral is taken with respect to Haar measure on N/L). Since $1 \otimes \chi_{\mathfrak{p}}$ is obviously a character of $L < N$, we get, parallel to (2.52),

$$W(l) := Ind_L^N(1 \otimes \chi_{\mathfrak{p}}) : N = Heis(V; B) \to \mathcal{U}(\mathcal{H}(l)) \tag{2.82}$$

as the attendant unitary ($\mathfrak{p}$-adic) Schrödinger representation of the Heisenberg group.

Next, by the standard functional analytic device of associating to any f in $Hom(E, k_{\mathfrak{p}}) = E^*$ the character $\chi_{\mathfrak{p}}(\langle f, \_\rangle)$, we obtain an identification of E^* with E's Pontryagin dual, so that there arises a pairing of Haar measures; indeed this obviously works for any $E \in \mathfrak{Vect}/k_{\mathfrak{p}}$, the category of $k_{\mathfrak{p}}$-vector spaces. Thus we get a general formalism of partial Fourier transforms *via*

$$\mathsf{FT} : f(x) \longmapsto \int_{k_{\mathfrak{p}}} \chi_{\mathfrak{p}}(-\langle x^*, x\rangle) f(x)dx =: \mathsf{FT}(f)(x^*), \tag{2.83}$$

a unitary operator.

Furthermore, it behooves us to recall briefly (cf. p. 3 of [64]) that if $\Phi : (E, dx) \xrightarrow{\sim} (F, dy)$ is an isomorphism of locally compact topological groups (e.g., the topological vector spaces we are concerned with), then the Haar modulus of Φ, written $|\Phi|$, is the unique positive real number for which, for any $f \in L^1(E)$,

$$\int_F f(\Phi^{-1}(y))dy = |\Phi| \int_E f(x)dy. \tag{2.84}$$

With these players in the game, we define, for any Lagrangian planes l_1, l_2

$$g_{l_2,l_1} : l_1 \ni x \mapsto [g_{l_2,l_1} : y \longmapsto B(x, y)] \in l_2^*. \tag{2.85}$$

Thus, abusing notation a bit, we get that $g_{l_2,l_1} : \frac{l_1}{l_1 \cap l_2} \to \left(\frac{l_2}{l_1 \cap l_2}\right)^*$. Under these circumstances it is shown in [34] that $\frac{l_2}{l_1 \cap l_2}$ admits a Haar measure $d\dot{x}$ such that $|g_{l_2,l_1}|^{\frac{1}{2}} d\dot{x}_2$ does not depend on the choice of Haar measure on $l_1 \cap l_2$ and we have.

Proposition 12.

$$\mathsf{FT}_{l_2,l_1} : \varphi(x) \longmapsto \int_{\frac{l_2}{l_1 \cap l_2}} \varphi(y(x_2, 0))|g_{l_2,l_1}|^{\frac{1}{2}} d\dot{x}_2, \tag{2.86}$$

where $x_2 = proj_{l_2}(x)$, extends to a unitary operator, again denoted by FT_{l_2,l_1}, from $\mathcal{H}(l_1)$ to $\mathcal{H}(l_2)$, intertwining the Schrödinger representations $W(l_1)$ and $W(l_2)$. Furthermore, $\mathsf{FT}_{l_2,l_1}^{-1} = \mathsf{FT}_{l_1,l_2}$.

(Compare Corollary 1, i.e., (2.59).)

Returning to the decomposition $V = E \oplus E^*$, where E is of course free to be any Lagrangian subspace of V, let Q be any symmetric bilinear form on E, define the mapping $s_Q : E \to E^*$ by the rule

$$s_Q : x \longmapsto [y \mapsto Q(x, y)], \tag{2.87}$$

and set

$$L_Q := \{x + s_Q x\}_{x \in E}, \tag{2.88}$$

evidently realizing a subspace of V in the above decomposition. Now L_Q is easily seen to be a Lagrangian plane in V, too, so that we can make sense of the mappings $\mathsf{FT}_{E,L_Q} \circ \mathsf{FT}_{L_Q,E^*}$ and FT_{E,E^*} as partial Fourier transforms acting between the Hilbert spaces $\mathcal{H}(E^*)$ and $\mathcal{H}(E)$ courtesy of Proposition 12. But then it follows readily, just by writing out a few obvious commutative diagrams, that the operator $\mathsf{FT}_{E,L_Q} \circ \mathsf{FT}_{L_Q,E^*} \circ \mathsf{FT}_{E,E^*}^{-1}$ is a unitary operator on $\mathcal{H}(E)$ which commutes with the Schrödinger representation $W(E)$. In the same manner as before, this implies that

$$\mathsf{FT}_{E,L_Q} \circ \mathsf{FT}_{L_Q,E^*} \equiv \mathsf{FT}_{E,E^*} (\mathrm{mod}^\times\, \mathbb{C}_1^\times). \tag{2.89}$$

Equivalently,

Proposition 13. *There exists a unimodular scalar $\gamma(Q)$ such that*

$$\mathsf{FT}_{E,L_Q} \circ \mathsf{FT}_{L_Q,E^*} = \gamma(Q) \cdot \mathsf{FT}_{E,E^*}. \tag{2.90}$$

Corollary 2. *γ is a character of the Witt group $\mathfrak{W}(k_\mathfrak{p})$.*

Proof. (2.83) and (2.90). ∎

Finally, we observe that the Kashiwara triple index $\tau(l_1, l_2, l_3)$, defined for real Lagrangians $l_i, i = 1, 2, 3$, in Section 2.4 (cf. (2.65)), is amenable to a closely related definition in the present more general $\mathfrak{p}$-adic context. The salient new feature is that we now take τ to be an element in $\mathfrak{W}(k_\mathfrak{p})$. Specifically, we again take

$$q_B(x_1 + x_2 + x_3) = B(x_1, x_2) + B(x_2, x_3) + B(x_3, x_1), \tag{2.91}$$

for $x_1 + x_2 + x_3 \in l_1 \oplus l_2 \oplus l_3$, now mapping into $k_\mathfrak{p}$, of course, and now simply set $\tau = q_B$. Under these circumstances many properties that hold in the real case go through essentially unchanged in the $\mathfrak{p}$-adic case; Propositions 6–8 carry over *verbatim*, for example. Beyond this, however, with $\tau \in \mathfrak{W}(k_\mathfrak{p})$ for all $\mathfrak{p} \in \mathfrak{V}_k$, we are now in a position to add a result to the mix that will eventually facilitate the critical transition from the present more or less geometrical material to the arithmetical results of Weil [63]. Indeed, in the current setting of the Witt group we obtain

Proposition 14. *Given three Lagrangian planes l_1, l_2, l_3, suppose $l_1 \cap l_3 = (0)$, i.e., l_1, l_3 are transverse Lagrangian planes. With p_{13} and p_{31} the usual projections, define the following quadratic form on l_2:*

$$r(x_2) = B(p_{13}x_2, p_{31}x_2). \tag{2.92}$$

Then

$$\tau(l_1, l_2, l_3) = (l_2; r) \in \mathfrak{W}(k_\mathfrak{p}). \tag{2.93}$$

Thus, the thrust of the preceding proposition (cf. [34], p. 109) is that for all places $\mathfrak{p}$ of k the story is the same: the Kashiwara triple index can be realized as an element of the Witt group of $k_\mathfrak{p}$, the difference between the archimedean and non-archimedean cases being the respective presence and absence of the signature map. It turns out that as far as we are concerned, i.e. in regard to the role played by the Kashiwara index in the definition of the local Weil(–Kubota) 2-cocycle, this distinction is neutralized by virtue of the action of local characters. We turn to this theme below; see Section 2.6.

In any event, with the hypotheses of Proposition 14 still in place, use B to identify l_3 and l_1^*; set $E = l_1$, $E^* = l_3$, and identify L_Q in (2.88) with l_2, which is to say that we set $L_Q = L_r$. Then we obtain from Proposition 13 that

$$\mathsf{FT}_{l_1,l_2} \circ \mathsf{FT}_{l_2,l_3} \circ \mathsf{FT}_{l_3,l_1} = \gamma(\tau(l_1, l_2, l_3)) \cdot id_{\mathcal{H}(l_1)}. \tag{2.94}$$

Next, we must expand the earlier remarks surrounding Propositions 4 and 11 to the present $\mathfrak{p}$-adic case, generalizing the multiplier $a(l_1, l_2, l_3)$ in the process, all in order to compare $\gamma(\tau)$ and c (as given by (2.75)), the crucially important 2-cocycle just mentioned.

So, fixing a Lagrangian plane l and therefore the associated Hilbert space $\mathcal{H}(l)$, we begin by defining a canonical projective representation (i.e. a Weil representation) of $Sp(V_\mathfrak{p}; B_\mathfrak{p}) \approx Sp(2n, k_\mathfrak{p})$ in $\mathcal{H}(l)$ as follows. First, for any $\sigma \in Sp(2n, k_\mathfrak{p})$, let

$$A : Sp(2n, k_\mathfrak{p}) \ni \varphi(x) \longmapsto \varphi(x^{\sigma^{-1}}) \in Mor(\mathcal{H}(l), \mathcal{H}(l^\sigma)); \tag{2.95}$$

then the desired mapping is

$$R_l : Sp(2n, k_\mathfrak{p}) \ni \sigma \mapsto \mathsf{FT}_{l,l^\sigma} \circ A(\sigma) \in \mathcal{U}(\mathcal{H}(l)). \tag{2.96}$$

With this definition in place we can argue as before (once more by means of a handful of obvious commutative diagrams) to get that, for any $\sigma, \sigma' \in Sp(2n, k_\mathfrak{p})$,

$$R_l(\sigma) \circ R_l(\sigma') \equiv R_l(\sigma\sigma')(\mathrm{mod}^\times\, \mathbb{C}_1^\times). \tag{2.97}$$

This implies that we have:

Proposition 15. *There exists a* 2*-cocycle* $c_{l,\mathfrak{p}} = c_\mathfrak{p} \in H^2(Sp(2n, k_\mathfrak{p}), \mathbb{C}_1^\times)$ *such that, for all* $\sigma, \sigma' \in Sp(2n, k_\mathfrak{p})$,

$$R_l(\sigma\sigma') \circ R_l(\sigma')^{-1} \circ R_l(\sigma)^{-1} = c_\mathfrak{p}(\sigma, \sigma') \cdot id_{\mathcal{H}(l)}. \tag{2.98}$$

At the same time, however, (2.94) yields that

$$\mathsf{FT}_{l,l^\sigma} \circ \mathsf{FT}_{l^\sigma,l^{\sigma'\sigma}} \circ \mathsf{FT}_{l^{\sigma'\sigma},l} = \gamma(\tau(l, l^\sigma, l^{\sigma'\sigma})) \cdot id_{\mathcal{H}(l)}, \tag{2.99}$$

and the characterization of R_l given by (2.96) therefore suggests a close relationship between $c_\mathfrak{p}(\sigma, \sigma')$ and $\gamma(\tau(l, l^\sigma, l^{\sigma'\sigma}))$. Indeed we get

Proposition 16. *For all* $\sigma, \sigma' \in Sp(2n, k_\mathfrak{p})$,

$$c_\mathfrak{p}(\sigma, \sigma') = \gamma(\tau(l, l^\sigma, l^{\sigma'\sigma}))^{-1}. \tag{2.100}$$

Proof. It follows directly from (2.96) that $R_l(\sigma)^{-1} = A(\sigma^{-1}) \circ \mathsf{FT}_{l^\sigma, l}$, for all σ, which implies, by means of an easy calculation starting from (2.99), that

$$\begin{aligned} &\gamma(\tau(l, l^\sigma, l^{\sigma'\sigma})) \cdot id_{\mathcal{H}(l)} \\ &\quad = R_l(\sigma) \circ A(\sigma^{-1}) \circ \mathsf{FT}_{l^\sigma, l^{\sigma'\sigma}} \circ A(\sigma'\sigma) \circ R_l(\sigma\sigma')^{-1}. \end{aligned} \tag{2.101}$$

Thus, since (from (2.98))

$$c_\mathfrak{p}(\sigma, \sigma')^{-1} \cdot id_{\mathcal{H}(l)} = R_l(\sigma) \circ R_l(\sigma') \circ R_l(\sigma\sigma')^{-1}, \tag{2.102}$$

it suffices to show that

$$A(\sigma^{-1}) \circ \mathsf{FT}_{l^\sigma, l^{\sigma'\sigma}} \circ A(\sigma'\sigma) = \mathsf{FT}_{l, l^{\sigma'}} \circ A(\sigma'), \tag{2.103}$$

or, equivalently,

$$\mathsf{FT}_{l, l^{\sigma'}} = A(\sigma^{-1}) \circ \mathsf{FT}_{l^\sigma, l^{\sigma'\sigma}} \circ A(\sigma), \tag{2.104}$$

given that $A(\sigma\sigma') \circ A(\sigma')^{-1} = A(\sigma)$ by (2.95). But by means of Proposition 12 the indicated (partial) Fourier transforms are of course rendered as Haar integrals and we get, respectively, that with $\varphi(x) \in \mathcal{H}(l^\sigma)$

$$\mathsf{FT}_{l, l^{\sigma'}}(\varphi(x)) = \int_{\frac{l}{l \cap l^{\sigma'}}} \varphi(x \cdot (x_2, 0)) |g_{l, l^{\sigma'}}| d\dot{x}_2 \tag{2.105}$$

and

$$\mathsf{FT}_{l^\sigma, l^{\sigma'\sigma}}(\varphi(x)) = \int_{\frac{l^\sigma}{l^\sigma \cap l^{\sigma'\sigma}}} \varphi(x \cdot (x_2, 0)) |g_{l^\sigma, l^{\sigma'\sigma}}| d\dot{x}_2. \tag{2.106}$$

The proof is completed by effecting the obvious changes of variable, $x \mapsto x^\sigma$ or $x \mapsto x^{\sigma^{-1}}$, in the respective integrals so as to obtain that

$$\mathsf{FT}_{l, l^{\sigma'}} \circ A(\sigma^{-1}) = A(\sigma^{-1}) \circ \mathsf{FT}_{l^\sigma, l^{\sigma'\sigma}}, \tag{2.107}$$

which is clearly all that is needed. ■

(This theorem appears without proof on p. 110 of [34]; it is evidently due to Patrice Perrin.)

Now, with Proposition 16 in place, we are in a position to discuss:

2.6. The double cover of $Sp(2n, k_\mathfrak{p})$; restriction to the lowest-dimensional case

In the lowest-dimensional case, $n = 1$, i.e. $Sp(2, k_\mathfrak{p}) = Sp(k_\mathfrak{p} \times k_\mathfrak{p}^*)$ (employing Pontryagin duality), it was André Weil who established [63] that the aforementioned projective representation actually takes values in $\{1, -1\}$, meaning that the attendant 2-cocycle $c_{\mathfrak{p}}$ defines a twisted group law on $Sp(k_{\mathfrak{p}} \times k_{\mathfrak{p}}^*) \times \{1, -1\}$ as follows:

$$(\sigma, \xi)(\sigma', \xi') = (\sigma\sigma', c_{\mathfrak{p}}(\sigma, \sigma')\xi\xi'), \tag{2.108}$$

where $\sigma, \sigma' \in Sp(k_{\mathfrak{p}} \times k_{\mathfrak{p}}^*)$ and $\xi, \xi' \in \{1, -1\}$. Committing the usual act of *abus de langage*, i.e. writing $\mathbb{Z}_2$ for the multiplicative group $\{1, -1\}$ (instead of the equally popular but multiplicative μ_2, a notation which we avoid, however, since μ is also standard notation for the Maslov index proper which will figure more and more in what follows), we define in this way the double cover $Sp(k_{\mathfrak{p}} \times k_{\mathfrak{p}}^*) \times_{c_{\mathfrak{p}}} \mathbb{Z}_2 =: \widetilde{Sp}(k_{\mathfrak{p}} \times k_{\mathfrak{p}}^*)$, meaning that we have the short exact sequence

$$1 \to \mathbb{Z}_2 \to \widetilde{Sp}(k_{\mathfrak{p}} \times k_{\mathfrak{p}}^*) \to Sp(k_{\mathfrak{p}} \times k_{\mathfrak{p}}^*) \to 1. \tag{2.109}$$

In point of fact, following both Weil [63] and Tomio Kubota [31], who opted to replace $Sp(k_{\mathfrak{p}} \times k_{\mathfrak{p}}^*)$ by its isomorph $SL(2, k_{\mathfrak{p}})$, the defining 2-cocycle $c_{\mathfrak{p}} \in H^2(Sp(k_{\mathfrak{p}} \times k_{\mathfrak{p}}^*), \mathbb{Z}_2)$ can be rendered in terms of local symbols, the Hilbert–Hasse symbol (at $\mathfrak{p}$) for Weil and the 2-Hilbert symbol for Kubota, and this provides a relatively easy route to adèlization. For instance, the fact that the 2-Hilbert symbol evaluated at a fixed pair of entries from the base field reduces to 1 for all but a finite number of $\mathfrak{p}$ immediately solves the problem of well-definition for the infinite product $\prod_{\mathfrak{p}} c_{\mathfrak{p}}$, where we now understand $c_{\mathfrak{p}}$ to mean the 2-cocycle used by Kubota [31] and this immediately gives us a formula for $c_{\mathbb{A}} \in H^2(Sp(2, k_{\mathbb{A}}), \mathbb{Z}_2)$. This will be of critical significance later: the fact that $c_{\mathbb{A}}$, equivalently $\widetilde{Sp}(k_{\mathfrak{p}} \times k_{\mathfrak{p}}^*)$, is split on the subgroup of rational points (being $Sp(2, k)$) is part and parcel of the product formula for the 2-Hilbert symbol, i.e. quadratic reciprocity for the number field k. (See also the appendix to [3].)

Lion–Vergne and Perrin, however, in dealing with the general case of $Sp(2n, k_{\mathfrak{p}})$, make the dependence of their construction of $c_{\mathfrak{p}}$ on quadratic forms more direct and explicit. Indeed, for the real case, treated at length in the first part of [34], it is in fact true that $\tau(l, l^{\sigma}, l^{\sigma'\sigma})$ is itself already a 2-cocycle, and this goes through essentially without change for arbitrary primes of $\mathcal{O}_k$, as brought out by Lion and Perrin in the aforementioned Appendix [40] to the first part of this book. The issue, quite simply, is to make a choice of additive character $\chi_{\mathfrak{p}}$ for each $\mathfrak{p}$ (in accord with what was done with (2.81)), leading to realizing $\tau(l, l^{\sigma}, l^{\sigma'\sigma})$ as an element of $H^2(Sp(2n, k_{\mathfrak{p}}), \mathfrak{W}(k_{\mathfrak{p}}))$, where, as before, $\mathfrak{W}(k_{\mathfrak{p}})$ is the Witt group for $k_{\mathfrak{p}}$. The real case is distinguished by an extra application of the signature map, *viz.* (2.65). However, in the indicated Appendix, all valuations $\mathfrak{p}$ are covered, so, if we jettison this signature map in the real case, we get, uniformly for all $\mathfrak{p} \in \mathfrak{V}_k$,

$$1 \to \mathfrak{W}(k_{\mathfrak{p}}) \to \widetilde{Sp}(2n, k_{\mathfrak{p}}) \to Sp(2n, k_{\mathfrak{p}}) \to 1, \tag{2.110}$$

which just says that $\widetilde{Sp}(2n, k_{\mathfrak{p}})$ is regarded as $Sp(2n, k_{\mathfrak{p}}) \times_{\tau(l, l^{\sigma}, l^{\sigma'\sigma})} \mathfrak{W}(k_{\mathfrak{p}})$, engendering the group law

$$(\sigma, q)(\sigma', q') = (\sigma\sigma', q + q' + \tau(l, l^{\sigma}, l^{\sigma'\sigma})). \tag{2.111}$$

Additionally it follows immediately from Propositions 15 and 16 that, with γ being a character of $\mathfrak{W}(k_{\mathfrak{p}})$, the mapping

$$(\sigma, q) \longmapsto \gamma(q) \cdot R_l(q) \tag{2.112}$$

gives a true (and unitary) representation of $\widetilde{Sp}(2n, k_{\mathfrak{p}})$, Weil's metaplectic group defined in [63]: we have obtained the (local) metaplectic representations.

For our ultimately arithmetical purposes, however, and as we already indicated at the start of Section 2.2, we now take the liberty to restrict ourselves to the case where $n = 1$ (for the time being), and we have that, for all $\mathfrak{p}$, $Sp(2n, k_{\mathfrak{p}})$ is just $Sp(2, k_{\mathfrak{p}}) = Sp(k_{\mathfrak{p}} \times k_{\mathfrak{p}}^*) \approx SL(2, k_{\mathfrak{p}})$.

2.7. Another realization of the double cover of the symplectic group

It behooves us at this point to add a different perspective to the mix, which we perhaps might characterize as closer in spirit to the geometric considerations that motivated Arnol'd, Leray, and Souriau in their original work on the Maslov index, in contrast to the later considerations on the part of, for example, Kashiwara and Lion, Vergne, and Perrin. The former were more closely concerned with quantum mechanics and its geometry, so that the according representation theory takes on a particular flavor, and the Arnol'd–Leray–Maslov index, as de Gosson terms it in [8], is accordingly characterized in terms of flows on the (real) Lagrangian Grassmannian. The Kashiwara triple index, however, which, as we have seen, is termed "Maslov index" by Lion and Vergne, is associated with a "static" triple of pairwise transverse Lagrangians. To be sure, the connections are there, as delineated very carefully by Cappell, Lee, and Miller in [7], but these two indices are not identical; see Section 3.1 for more on this matter.

We mentioned above, in the Introduction, that the works of Robbin and Salamon dating to the early 1990s [45,44] contain something of a parallel to Souriau's article [53] of 1975, although this is admittedly at best a sketchy appraisal (and it should be noted that Robbin and Salamon go on to consider Feynman path integrals properly so-called). What is of importance to our purposes, is that both of these works present explicit formulas tying the Arnol'd–Leray–Maslov index to a phase integral, and there are evocative similarities between these formulas; again, we say more about this in the third chapter of this paper. What we focus on at this point is that Robbin and Salamon present a development of the double cover of the symplectic group and the metaplectic representation in the context of unitary operators involving phase integrals, which by their own description devolves to that given by Leray in his classic, *Lagrangian Analysis and Quantum Mechanics* [33, pp. 19–20]. We now present a sketch of what Robbin and Salamon do to get at what is ultimately our $\widetilde{Sp}(2n, \mathbb{R})$.

Working over $\mathbb{R}$ it all starts with a Hamiltonian H on $\mathbb{R}^n$, which is to say that we are operating on the usual $2n$-dimensional real symplectic space. Thus, the Hamiltonian expresses the sum of kinetic and potential energy of a (quantum mechanical) system of n particles, and we are dealing with the familiar position and momentum coordinates; see Section 2.2. One can ascribe to H a so-called evolution system $\psi_{t_0}^{t_1}$, from initial time t_0 to final time t_1 (as per [45]: it is in fact a semigroup in the continuous parameter t, with

$t_0 \leq t \leq t_1$), which, as we shall see, is closely related to the time evolution of the quantum mechanical system under consideration and therefore to the 1-parameter Lie group of unitary operators arising in the Schrödinger picture of quantum mechanics. This evolution system is associated to a generating function $S(x_0, x_1)$, where at initial time t_0 we are at $z_0 = (x_0, y_0)$, while at final time t_1 we are at $z_1 = (x_1, y_1)$ (all on a so-called symplectic relation, i.e., by definition, a Lagrangian submanifold of the Cartesian product of a pair of symplectic manifolds tailored to support the formalism of the calculus of variations trained on quantum mechanics; see [45] for the details). This generating function in due course plays the role of a phase function in the sense of the physicists.

Now it is appropriate and natural [44] to introduce the Arnol'd–Leray–Maslov index as a mapping from the universal cover of $Sp(2n, \mathbb{R})$ to the set $n/2 + \mathbb{Z}$ in keeping with the characterizations presented by Arnol'd [1], Souriau [53], and of course Leray [33]. Since this approach allows one to regard this index as a function on the homotopy classes of paths in the Lagrangian Grassmannian manifold (cf. Teruji Thomas' pithy presentation in [55]), Robbin and Salamon can single out the Arnol'd–Leray–Maslov index $\mu(t_1, t_0; H)$ as the according index of the path given by the mapping of the time interval $[t_0, t_1]$ into the matrix group $Sp(2n, \mathbb{R})$ (regarded as a real manifold), given by $t \mapsto \Psi^t_{t_0}$, where the latter expression, i.e. $\Psi^t_{t_0}$, is the linear part of $\psi^t_{t_0}$. Additionally, write Υ for the upper right block of $\Psi^{t_1}_{t_0}$, so that properly $\Upsilon = \Upsilon(t_0, t_1)$ (Robbin and Salamon write B instead, but we avoid this, given that in this paper the notation B is reserved for the bilinear form imparting symplectic structure to V).

This puts us in the position to cite some relevant results by Robbin–Salamon (cf. p. 21 of [45]), which we do with the additional stipulation that Planck's constant should be set to 1.

Proposition 17. *If $EMp(2n, \mathbb{R})$ is the extended metaplectic group, i.e the group of unitary operators on $L^2(\mathbb{R}^n)$ of the form*

$$f(x) \mapsto \frac{1}{(2\pi)^{n/2}|\det \Upsilon|^{1/2}} e^{i\pi\mu(t_1,t_0;H)/2} \int_{\mathbb{R}^n} e^{iS(\xi,x)} f(x)dx, \tag{2.113}$$

where H is to be a time-dependent quadratic Hamiltonian, let $Mp(2n, \mathbb{R})$ be the subgroup of $EMp(2n, \mathbb{R})$ cut out by those operators of this type for which the Hamiltonian is also homogeneous. Additionally, let $ESp(\mathbb{R}^{2n} \times U(1))$ be the group of all diffeomorphisms of $\mathbb{R}^{2n} \times U(1)$ of the form

$$g^{t_1}_{t_0} : (z_0, u_0) \mapsto (\psi^{t_1}_{t_0}(z_0), u_0 e^{iS(x_1,x_0)}), \tag{2.114}$$

bringing the connection to the aforementioned 1-parameter group into the open. Then we obtain the following short exact sequences, the latter being a consequence of the former:

$$1 \to \mathbb{Z}_2 \to EMp(2n, \mathbb{R}) \to ESp(\mathbb{R}^{2n} \times U(1)) \to 1, \tag{2.115}$$

$$1 \to \mathbb{Z}_2 \to Mp(2n, \mathbb{R}) \to Sp(2n, \mathbb{R}) \to 1. \tag{2.116}$$

Furthermore, the mapping $EMp(2n, \mathbb{R}) \to ESp(\mathbb{R}^{2n} \times U(1))$ in (2.115) is given by the natural projection

$$g^{t_1}_{t_0} \mapsto \psi^{t_1}_{t_0}. \tag{2.117}$$

It is evident that the preceding provides another method whereby to construct the double cover of the real symplectic group, which is to say, another route to (2.78). However, it should be noted that one of the main themes in our analysis is obscured: the corresponding 2-cocycle, putatively Weil's and Lion–Vergne's $c \in H^2(Sp(2n, \mathbb{R}), \mathbb{Z}_2)$ (modulo coboundaries, of course), is not immediately forthcoming in this approach. Thus, even though the preceding proposition presents a direct relationship between $\mu(t_1, t_0; H)$ and the phase (in fact, oscillatory) integral $\int_{\mathbb{R}^n} e^{iS(\xi,x)} f(x)dx$, there is no explicit formulaic connection with $H^2(Sp(2n, \mathbb{R}), \mathbb{Z}_2)$ to be had, at least not at first sight: the connection with $Mp(2n, \mathbb{R})(\approx \widetilde{Sp}(2n, \mathbb{R}))$ is for now perhaps best described as extrinsic. This will soon lead us to the corresponding work of Souriau, and, indeed, we accordingly resume these considerations surrounding Robbin–Salamon's work, especially the relation (2.113), toward the end of our third chapter. We now go on to address the question of the adèlization of the earlier symplectic data over k.

2.8. Adèlization of the symplectic group $Sp(2, k)$ and its double cover

We note, first, that in what immediately follows we take the foundational material concerning non-archimedean analysis, dealing with local fields and rings of adèles, largely for granted, given that this material is standard fare for number theory (cf. [54,64] &c.). However, further down the line (Section 4.2, ff.) we will have occasion to consider the question of quantum mechanics over non-archimedean fields, and at that point we will have to recall a number of basic facts and propositions especially about the attendant character theory. Accordingly some of the things we discuss in the present section(s) will be given a good deal more air-play below, in Chapter 4.

Let $\mathfrak{p}$ be any valuation of k, archimedean or not. The local objects we have garnered so far include the following which are of particular interest in the sequel. First of all, the algebraic as well as topological setting is of course provided by the local field $k_\mathfrak{p}$ equipped with its maximal order, or ring or integers, $\mathcal{O}_\mathfrak{p}$, with its unique maximal ideal $\mathfrak{p}\mathcal{O}_\mathfrak{p}$; thus,

$$\mathcal{O}_\mathfrak{p} = \{x_\mathfrak{p} \in k_\mathfrak{p} \mid \|x_\mathfrak{p}\|_\mathfrak{p} \leq 1\}, \tag{2.118}$$

$$\mathfrak{p}\mathcal{O}_\mathfrak{p} = \{x_\mathfrak{p} \in \mathcal{O}_\mathfrak{p} \mid \|x_\mathfrak{p}\|_\mathfrak{p} < 1\}. \tag{2.119}$$

Next, the symplectic group $Sp(2, k_\mathfrak{p})$, featured so prominently in the preceding sections, can be realized as an algebraic group scheme by regarding $SL(2)$ as a subscheme of $GL(2)$, which is in turn realized as the locus defined by the polynomial $\det(g)t - 1 = 0$, where one takes $g \in \mathbb{M}_{2\times 2}$ and t a coordinate for the affine line $\mathcal{A}$. In this way $Sp(2) \approx SL(2) \subset GL(2) \hookrightarrow \mathcal{A}^5$. Now $Sp(2, k_\mathfrak{p})$ is nothing else than the preceding subscheme taken over the field $k_\mathfrak{p}$, in other words the $k_\mathfrak{p}$-valued points. As we shall see momentarily, this algebraic geometric formulation of the local symplectic groups facilitates an elegant compact phrasing of the all-important adélic symplectic group.

As we already indicated at the close of the preceding section, the Weil 2-cocycle data, $\{c_\mathfrak{p}\}_\mathfrak{p}$, of such importance to our enterprise, is exceptionally amenable to adèlization in the form given to it by Tomio Kubota in [31], which is to say, in terms of the local 2-Hilbert symbol. Specifically this is due to the fact that this symbol always reduces to 1 for almost every valuation so that we can legitimately write down the infinite product $\prod_\mathfrak{p} c_\mathfrak{p}$: in the present lowest dimensional case we are done. (We reiterate this below.)

Finally, with (2.99) in place, i.e.

$$\mathsf{FT}^{(\mathfrak{p})}_{l,l^{\sigma}} \circ \mathsf{FT}^{(\mathfrak{p})}_{l^{\sigma},l^{\sigma'\sigma}} \circ \mathsf{FT}^{(\mathfrak{p})}_{l^{\sigma'\sigma},l} = \gamma_{\mathfrak{p}}(\tau_{\mathfrak{p}}(l, l^{\sigma}, l^{\sigma'\sigma})) \cdot id_{\mathcal{H}(l)}, \tag{2.120}$$

where the superscripts have been added so as to indicate that we have local Fourier transforms to deal with here (and, but for the unwieldiness of what would result, we should really also replace every l by $l^{(\mathfrak{p})}$ and every σ by $\sigma_{\mathfrak{p}}$, but enough is enough). It is evident that since $\gamma_{\mathfrak{p}}$ is a character of the Witt–Grothendieck group $\mathfrak{W}(k_{\mathfrak{p}})$ we need to look at what it means to adèlize the group theoretic data $\{\mathfrak{W}(k_{\mathfrak{p}})\}_{\mathfrak{p}}$, but this is actually a very straightforward matter. Finally, since, as we just noted, the $\mathsf{FT}^{(\mathfrak{p})}$ are local Fourier transforms, the question of adèlic Fourier transforms arises, and it turns out this is addressed in some recent work by Branko Dragovich [17].

On to some of the details of adèlization, then. Starting off with the base field itself, we have that, by definition,

$$k_{\mathbb{A}} := \{(x_{\mathfrak{p}})_{\mathfrak{p}} =: x_{\mathbb{A}} \mid x_{\mathfrak{p}} \in k_{\mathfrak{p}}, \forall \mathfrak{p} \text{ and } x_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}},\ a.e.\mathfrak{p}\}, \tag{2.121}$$

or, equivalently,

$$k_{\mathbb{A}} := \{(x_{\mathfrak{p}})_{\mathfrak{p}} \mid \|x_{\mathfrak{p}}\|_{\mathfrak{p}} \leq 1, a.e.\mathfrak{p}\}. \tag{2.122}$$

From a topological perspective $k_{\mathbb{A}}$ is realized as the topological product $\prod' k_{\mathfrak{p}}$ with respect to the collection of maximal orders $\mathcal{O}_{\mathfrak{p}} \subset k_{\mathfrak{p}}$.

Continuing in this topological vein, taking into account our earlier remarks regarding $Sp(2n)$ as an algebraic subscheme of $GL(2)$, it is now all but automatic to render the adèlization of the data $\{Sp(2, k_{\mathfrak{p}})\}_{\mathfrak{p}}$ as

$$Sp(2, k)_{\mathbb{A}} = \prod_{\mathfrak{p}}{}' Sp(2, k_{\mathfrak{p}}), \tag{2.123}$$

which is to say,

$$\begin{aligned} Sp(2, k)_{\mathbb{A}} &= Sp(2, k_{\mathbb{A}}) \\ &= \{(\sigma_{\mathfrak{p}})_{p} =: \sigma_{\mathbb{A}} \mid \sigma_{\mathfrak{p}} \in Sp(2, k_{\mathfrak{p}}), \forall \mathfrak{p},\ \text{and}\ \sigma_{\mathfrak{p}} \in k_{\mathfrak{p}}^{4},\ a.e.\mathfrak{p}\}. \end{aligned} \tag{2.124}$$

When it comes to local second cohomology, or, rather, the adèlization thereof, we may obviously simply reiterate that

$$c_{\mathbb{A}} := \prod_{\mathfrak{p}} c_{\mathfrak{p}} \in H^2(Sp(2, k)_{\mathbb{A}}, \mathbb{Z}_2) \tag{2.125}$$

following Kubota [31], as we already indicated.

Accordingly, in terms of short exact sequences realizing the corresponding double covers of the local as well as adèlic symplectic groups, in other words the local and adèlic metaplectic groups (in the sense of Weil, not Kazhdan–Patterson [30]), the data $\{c_{\mathfrak{p}}\}_{\mathfrak{p}}$ gives rise to

$$1 \to \mathbb{Z}_2 \to \widetilde{Sp}(k_{\mathfrak{p}} \times k_{\mathfrak{p}}^*) \to Sp(k_{\mathfrak{p}} \times k_{\mathfrak{p}}^*) \to 1 \tag{2.126}$$

where

$$\widetilde{Sp}(k_{\mathfrak{p}} \times k_{\mathfrak{p}}^*) = Sp(k_{\mathfrak{p}} \times k_{\mathfrak{p}}^*) \times_{c_{\mathfrak{p}}} \mathbb{Z}_2, \tag{2.127}$$

so that we get the corresponding adèlization

$$1 \to \mathbb{Z}_2 \to \widetilde{Sp}(k_{\mathbb{A}}) \to Sp(k_{\mathbb{A}}) \to 1, \tag{2.128}$$

and this in turn means that

$$\widetilde{Sp}(k_{\mathbb{A}}) = Sp(k_{\mathbb{A}}) \times_{c_{\mathbb{A}}} \mathbb{Z}_2. \tag{2.129}$$

With this object in place, and with the accompanying adèlic 2-cocycle $c_{\mathbb{A}}$, we have essentially finished the task of adèlizing the first major player in our projected connection between arithmetic data in the form of Weil's metaplectic group and data from physics in the form of phase integrals (or, more precisely, oscillatory integrals); it falls to us down the line to address the adèlization of the other players in the game, specifically the Maslov cocycles and, of course, the indicated integrals. We address these matters in Sections 4.2 and 5.2.

2.9. Quadratic reciprocity

In the immediately preceding sections we have trained our focus on the double cover of the (simplest) symplectic group over the localizations of the ground field, k, at all of the places $\mathfrak{p}$ as well as over the associated ring of k-adèles, $k_{\mathbb{A}}$, and the proper algebraic location of these double covers, locally as well as adèlically, is the set of short exact sequences given by (2.126) and (2.128). The according group laws on the $\widetilde{Sp}(k_{\mathfrak{p}} \times k_{\mathfrak{p}}^*)$ (i.e. the $\widetilde{Sp}(2, k_{\mathfrak{p}})$) and on $\widetilde{Sp}(k_{\mathbb{A}})$ are twisted by, respectively, the 2-cocycles $c_{\mathfrak{p}}$, with $\mathfrak{p} \in \mathfrak{V}_k$, and $c_{\mathbb{A}}$, with (2.125) in effect. We also mentioned above that $c_{\mathfrak{p}}$ was given explicitly in terms of the 2-Hilbert symbol by Tomio Kubota in [31]: he made this move in the context of a simplification of sorts *vis à vis* Weil's original presentation of this material consisting in replacing the symplectic group $\widetilde{Sp}(k_{\mathfrak{p}} \times k_{\mathfrak{p}}^*)$ by the special linear group $SL(2, k_{\mathfrak{p}})$, given their natural isomorphism. Weil had of course developed the entire theory of these double covers (in [63]) as part of his explication of Carl Ludwig Siegel's vaunted analytic theory of quadratic forms (cf., e.g. [51]) in terms of unitary group representation theory, or, in Weil's own description, abstract Fourier analysis. In Weil's presentation, quadratic reciprocity now appears as a consequence of exploiting the behavior of $\widetilde{Sp}(k_{\mathbb{A}})$ with respect to its so-called rational points, i.e. $Sp(k)$, the emerging form of this arithmetical law being that of Hilbert–Hasse (what with quadratic forms taking center stage).

Specifically, Weil notes in [63] that what he has done is to rephrase the classical Fourier-analytic proof of quadratic reciprocity for any number field k given by Erich Hecke in [27] in representation theoretic terms by letting the pivotal role of Hecke's ϑ-functional equations be taken over by an invariance of what is now called the Weil Θ-functional. The parallels are as follows.

A Hecke ϑ-function obeys a functional equation by virtue of much the same Fourier-analytic maneuvers Riemann himself developed for his second proof of the functional equation for the ζ-function, and with this functional equation in place, it is just a matter of passing to ϑ-constants (*Thetanullwerte*) to get a relation between Gauss sums. Since Gauss sums transform nicely with respect to the Legendre symbol, the aforementioned relation soon translates to nothing less than quadratic reciprocity for k in the form of Gauss–Euler. By comparison, Weil's approach centers on the fact that through the services of the adèlic Weil representation (one adèlizes (2.96)) one can define a natural action of $Sp(k_{\mathbb{A}})$ on a particular functional, the earlier Weil Θ-functional, and observe that the restricted action of the rational points, $Sp(k)$, leaves the Θ-functional invariant. This implies that, for instance, (2.128), which is not in itself split exact, does split on the rational points, and in view of (2.125) this is sufficient to yield, in Weil's hands, 2-Hilbert–Hasse reciprocity, and, in Kubota's hands, 2-Hilbert reciprocity: we obtain the required product formulas (over $\mathfrak{V}_k$) for the indicated local symbols. It only remains to note the commonplace fact that all forms of quadratic reciprocity are equivalent. (These themes are discussed at great length in [3]; see especially p. 105.)

In view of the splitting of (2.128) on $Sp(k)$, or, more evocatively for our purposes, the fact that

$$c_{\mathbb{A}}|_{Sp(k)\times Sp(k)} \equiv 1, \tag{2.130}$$

which, as we have just seen, is part and parcel of quadratic reciprocity for k, we now make the observation that what lies ahead for us, namely, the task of tying the Weil–Kubota 2-cocycle data $\{c_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathfrak{V}_k}, c_{\mathbb{A}} = \prod_{\mathfrak{p}} c_{\mathfrak{p}}$, to path (and oscillatory) integrals, will have to exhibit this splitting behavior in a new guise, as a property of such integrals in a generalized (adèlic) quantum mechanical context. In other words, when all is said and done, the main thrust of the present work is not only to render these Weil–Kubota 2-cocycles directly in terms of building blocks of quantum mechanics in the style of functional phase integrals, but to reveal connections with number theory by translating (2.130) into adèlic quantum mechanical language.

3. The Maslov index

3.1. The Kashiwara triple index

Going back to Sections 2.4 and 2.5 we recall that, working over $\mathbb{R}$, if $V = \bigoplus_{i=1}^n \mathbb{R}P_i \oplus (\bigoplus_{j=1}^n \mathbb{R}Q_j)$ is a symplectic space with respect to the skew-symmetric bilinear form B, and if q_B is the quadratic form

$$q_B : x_1 + x_2 + x_3 \mapsto B(x_1, x_2) + B(x_2, x_3) + B(x_3, x_1), \tag{3.1}$$

then the Kashiwara triple index is given by the signature of q_B (cf. (2.65)). Furthermore, when we are working over the local field $k_{\mathfrak{p}}$, $\mathfrak{p} \in \mathfrak{V}_k$, we simply take (2.64) as the definition of the Kashiwara index, leaving the signature mapping out of the picture (cf. Section 2.5) so that we get an element of the Witt group of $k_{\mathfrak{p}}$. We also saw in Section 2.5 that the latter convention, which evidently also covers the real case, makes for a uniform means whereby to tie all the local Kashiwara index data to $H^2(Sp(2n, k_{\mathfrak{p}}), \mathbb{C}_1^{\times})$. Specifically, with

$$\tau : \mathfrak{Lag}(2n, k_{\mathfrak{p}}) \to \mathfrak{W}(k_{\mathfrak{p}}) \tag{3.2}$$

the Kashiwara index acting on the triples of Lagrangian planes in $V \approx k_{\mathfrak{p}}^{2n}$, and given the character

$$\gamma : \mathfrak{W}(k_{\mathfrak{p}}) \to \mathbb{C}_1^{\times}, \tag{3.3}$$

then, following Proposition 16, the object $\gamma(\tau(l, l^{\sigma}, l^{\sigma'\sigma}))$, with $l \in \mathfrak{Lag}(2n, k_{\mathfrak{p}})$ and $\sigma, \sigma' \in Sp(2n, k_{\mathfrak{p}})$, defines a 2-cocycle of $Sp(2n, k_{\mathfrak{p}})$ with values in $\mathbb{C}_1^{\times}$. Indeed, we have seen above, in (2.100) that $\gamma(\tau(l, l^{\sigma}, l^{\sigma'\sigma}))^{-1}$ agrees with $c_{\mathfrak{p}}$, Weil's defining 2-cocycle for his (local) metaplectic group.

We also indicated above that the definition of the Kashiwara index as the signature of q_B is evidently Kashiwara's own (cf. the Appendix of [29] and p. 162 of [7]), and there is some terminological confusion in the game because in Lion–Vergne [34] this object is simply termed the Maslov index, and to be fair, this license is already extended on p. 487 of [29]. But this just thickens the plot since the literature actually sports at least a half dozen carriers of this name. Evidently the first Maslov index indeed goes back to V.P. Maslov [35], but the thread was soon picked up by V.I. Arnol'd [1]. A connected contribution was made by C.T.C. Wall [62], but soon thereafter the focus shifted to France with the compact paper [53] by J.-M. Souriau, followed by the seminal work by J. Leray [33], carrying the historical note: "In Moscow in 1967 … Arnol'd asked me [Leray] my thoughts on Maslov's work …The present book is an answer to that question". For latter considerations, having to do with what M. de Gosson (cf. e.g. [8]) now aptly calls the Arnol'd–Leray–Maslov index, a nomenclature we have already embraced above, the setting is manifestly that of symplectic geometry and analysis, with quantum mechanics the driving force in the background. The ambient space is therefore real and, as we already suggested in Section 2.7, the central idea is that the index should be thought of as attached to homotopy classes of paths in the Lagrangian Grassmannian manifold $\mathfrak{Lag}(V) = \mathfrak{Lag}(2n, \mathbb{R})$. It is possible, therefore, to characterize the Maslov index of Kashiwara and Lion–Vergne, acting on triples of Lagrangians in V, as a more static affair, so to speak, and we single these two respective indices out as representatives of two classes into which the various so-called Maslov indices can be collected.

It turns out, however, that we find in Souriau's aforementioned paper of 1975 an exceedingly suggestive formula tying the Arnol'd–Maslov–Leray index μ (or, in Souriau's preferred notation, m) to an oscillatory integral, and so, in keeping with our goals stated in the Introduction, the first order of business must be to inquire after the relationship between, on the one hand, μ and m, and, on the other, τ; *à propos*, μ and m are not identical either, but they only differ by a factor of 2, about which we say more presently. Of course, the greater objective is to exploit this pending relationship between the Arnol'd–Leray–Maslov index and the Kashiwara(–Maslov) index to get, rather easily, a relationship between Weil's c, defining the group law on $\widetilde{Sp}(2n, \mathbb{R}) \approx Mp(2n, \mathbb{R})$, and an oscillatory integral; after this is done (in Section 3.4), the questions of getting $\mathfrak{p}$-adic counterparts to this relation and subsequently adèlizing it are to be addressed: see Section 4.2, ff. But first the question of μ, m, and τ needs to be settled, and for this we turn to the important paper [7] by S.E. Cappell, R. Lee, and E.Y. Miller.

3.2. An axiomatic characterization of the Maslov index

For our purposes, the first relevant result of [7] is that what the authors refer to as Maslov indices proper (and there are six such candidates) are in fact one, appearances notwithstanding. Furthermore, it is possible to write down a list of axioms which completely characterize such a μ and, as regards getting identifications between different flavors of Maslov indices, it is accordingly just a matter of checking that each of these indices satisfies these (six) axioms.

We stay with the same setting as before: V is a symplectic space over $\mathbb{R}$ and $\mathfrak{Lag}(V) = \mathfrak{Lag}(2n, \mathbb{R})$ is the accompanying Lagrangian Grassmannian manifold. With Cappell–Lee–Miller let

$$\begin{aligned} &P([a,b]; \mathfrak{Lag}(V)^2) \\ &\quad = \{f : [a,b] \to \mathfrak{Lag}(V)^2 \mid f \text{ is continuous and piece-wise differentiable}\}, \end{aligned} \tag{3.4}$$

so that we can write $f(t) = (l_1(t), l_2(t))$, with $a \leq t \leq b$. Equip $P([a,b]; \mathfrak{Lag}(V)^2)$ with the piece-wise smooth topology. The focus falls on integer-valued mappings

$$\mu : P([a,b]; \mathfrak{Lag}(V)^2) \to \mathbb{Z}. \tag{3.5}$$

As just indicated, Cappell–Lee–Miller present the following six properties for a Maslov index μ to obey which we may present as axioms in view of the upcoming Proposition 19:

Axiom 1. Let $\psi : [a,b] \to [ka+l, kb+l]$, for fixed $k > 0$, $l \geq 0$ in $\mathbb{R}$, so that composition with ψ takes any path $f \in P([a,b]; \mathfrak{Lag}(V)^2)$ to a path $f \circ \psi \in P([a,b]; \mathfrak{Lag}(V)^2)$. Then $\mu(f \circ \psi) = \mu(f)$.

Axiom 2. Let $f(s) : t \mapsto (l_1(s)(t), l_2(s)(t))$, for $0 \leq s \leq 1$ and $a \leq t \leq b$, and f varying continuously with s. Suppose, too, that $\forall s \in [0,1]$ we have that $l_1(s)(a) = l_1(s)(b)$ and $l_2(s)(a) = l_2(s)(b)$, so that we are dealing with a family of paths in $\mathfrak{Lag}(V)^2$ all having the same starting points and end points. Then $\mu(f(0)) = \mu(f(1))$.

Axiom 3. If $f \in P([a,b]; \mathfrak{Lag}(V)^2)$ and $a < c < b$ then $\mu(f) = \mu(f|_{[a,c]}) + \mu(f|_{[c,b]})$.

Axiom 4. For W also a symplectic space, define $P([a,b]; \mathfrak{Lag}(V \oplus W)^2)$ in the natural manner: if $f(t) = (l_1(t), l_2(t)) \in P([a,b]; \mathfrak{Lag}(V)^2)$ and $g(t) = (l_1'(t), l_2'(t)) \in P([a,b]; \mathfrak{Lag}(W)^2)$, then $f \oplus g \in P([a,b]; \mathfrak{Lag}(V \oplus W)^2)$ by means of the rule $f \oplus g(t) = (l_1(t) \oplus l_1'(t), l_2(t) \oplus l_2'(t))$. Then $\mu(f \oplus g) = \mu(f) + \mu(g)$.

Axiom 5. Let $\varphi_t : V \xrightarrow{\sim} V$, i.e. $\varphi_t \in Sp(V)$, varying continuously and piece-wise smoothly with $t \in [a, b]$. Write φ_* for the pull-back: if $f = (l_1(t), l_2(t)) \in P([a, b]; \mathfrak{Lag}(V)^2)$ then $\varphi_*(f) \in P([a, b]; \mathfrak{Lag}(V)^2)$ is given by $\varphi_*(f)(t) = (\varphi_t(l_1(t)), \varphi_t(l_2(t)))$. Then $\mu(\varphi_*(f)) = \mu(f)$.

Axiom 6. Impart to $\mathbb{C} \approx \mathbb{R}^2$ the canonical symplectic structure $\{,\}$ given by $\{z_1, z_2\} = \{(x_1, y_1), (x_2, y_2)\} = x_1y_2 - y_1x_2 = -\mathrm{Im}(z_1\bar{z_2}) = \mathrm{Re}(iz_1\bar{z_2})$. Consider the path $f_0 \in P([-\pi/4, \pi/4], (\mathbb{C}, \{,\}))$ defined by $f_0(t) = (\mathbb{R} \cdot 1, \mathbb{R} \cdot e^{it})$. Then taking $\mu : P([-\pi/4, \pi/4]; \mathfrak{Lag}(V)^2) \to \mathbb{Z}$, get that $\mu(f_0) = 1$, $\mu(f_0|_{[-\pi/4,0]}) = 0$, and $\mu(f_0|_{[0,\pi/4]}) = 1$.

And now we get, as promised,

Proposition 18. *There exists one and only one mapping* $\mu : P([a, b]; \mathfrak{Lag}(V)^2) \to \mathbb{Z}$ *satisfying Axioms 1–6 for all real symplectic spaces* (V, B)*. Furthermore, if* $\xi : P([a, b]; \mathfrak{Lag}(V)^2) \to \mathbb{Z}$ *satisfies Axioms 1–5 for all* (V, B)*, then there exist fixed integers* A, B *such that, with* $f(t) = (l_1(t), l_2(t)) \in P([a, b]; \mathfrak{Lag}(V)^2)$*, we have that* $\xi(f) = (A + B)\mu(f) + B(\dim(l_1(a) \cap l_2(a)) - \dim(l_1(b) \cap l_2(b)))$.

Proof. Section 4 of [7]. ■

Proposition 18 clearly justifies the terminology *the* Maslov index.

With this regime in place, it follows from several theorems presented by Cappell–Lee–Miller in [7] (see especially their p. 172) that if $\pi : \widetilde{\mathfrak{Lag}}(V) \to \mathfrak{Lag}(V)$ denotes the natural projection from its universal covering space to the Lagrangian Grassmannian of V, then

$$\tau(\pi(\tilde{l}_1), \pi(\tilde{l}_2), \pi(\tilde{l}_3)) = 2\{m(\tilde{l}_1, \tilde{l}_2) + m(\tilde{l}_2, \tilde{l}_3) + m(\tilde{l}_3, \tilde{l}_1)\} \tag{3.6}$$

where, generally, $\tilde{l}$ is above l, i.e. $\pi(\tilde{l}) = l$, so that we can also write, simply,

$$\tau(l_1, l_2, l_3) = 2\{m(\tilde{l}_1, \tilde{l}_2) + m(\tilde{l}_2, \tilde{l}_3) + m(\tilde{l}_3, \tilde{l}_1)\}. \tag{3.7}$$

A quicker presentation of this relation can be found in de Gosson [8]: see his p. 2, p. 5.

Note that in view of (2.75) this immediately implies that upon setting $l_1 = l_0, l_2 = l_0^{\sigma_1}, l_3 = l_0^{\sigma_2\sigma_1}$ we obtain the following important relation:

$$c(\sigma_1, \sigma_2) = e^{i\frac{\pi}{2}\{m(\tilde{l}_0, \tilde{l}_0^{\sigma_1}) + m(\tilde{l}_0^{\sigma_1}, \tilde{l}_0^{\sigma_2\sigma_1}) + m(\tilde{l}_0^{\sigma_2\sigma_1}, \tilde{l}_0)\}}. \tag{3.8}$$

It is this relation which we will use to great advantage later. Before doing so, however, we take some time out to discuss at greater length how the all-but *sub rosa* maneuvers with Maslov indices fit into this development.

3.3. The different appearances of the Maslov index

In their careful analysis of the proliferation of indices laying claim to the name "Maslov index", Cappell, Lee and Miller distinguish four classes of such claimants, $\mu_{geo,1}$, $\mu_{geo,2}$, $\mu_{anal,1}$, and $\mu_{anal,2}$; they take $\mu_{geo,1}$ as the paradigm, as given by Guillemin and Sternberg in [25]: in their vernacular, $\mu_{geo,1} = \mu_{proper}$. Regarding this index they note that "[t]he idea is to count with signs and multiplicities the number of times that [, for a 'proper path' $f(t) = (l_1(t), l_2(t))$, $a \leq t \leq b$, one gets $l_1(t) \cap l_2(t) \neq (0)$]". This characterization hearkens back to the original ideas of Arnol'd in [1].

The next rendering of μ given by the authors, i.e. $\mu_{geo,2}$, is an ostensibly more intricate affair in the sense that the prevailing setting requires a complex structure on V (yielding a particular rendering of B, of course), as well as a subsequent identification of $\mathfrak{Lag}(V)$ with a unitary (factor) group: this is in fact also the setting in which Leray [33] and Souriau [58] operate and it accordingly behooves us to look at this presentation of μ more closely. The complex structure on V is given *via* J, with $J^2 = -id_V$, preserving B, so that, with $\langle \_, \_ \rangle$ the associated Hermitian inner product on V, we have

$$B(x, y) = -\operatorname{Im}\langle x, y \rangle \tag{3.9}$$

$$\langle x, y \rangle = -B(Jx, y) - iB(x, y). \tag{3.10}$$

Under these circumstances any Lagrangian plane in V, i.e. $l \in \mathfrak{Lag}(V)$, can be realized as

$$l = \bigoplus_{j=1}^{n} \mathbb{R} \cdot e_j, \tag{3.11}$$

where $\{e_j\}_{j=1}^n$ is an orthonormal basis of the n-dimensional $\mathbb{C}$-vector space (V, J). Obtain immediately that, upon fixing an l and a basis $\{e_j\}$ for l, the natural morphism (depending on l)

$$U(n) \ni A \mapsto \bigoplus_{j=1}^{n} \mathbb{R} \cdot Ae_j \in \mathfrak{Lag}(V) \tag{3.12}$$

of, e.g., varieties, is a surjection with kernel $O(n)$ so that we get the identification

$$\mathfrak{Lag}(V) \approx U(n)/O(n), \tag{3.13}$$

from which it follows that $\mathfrak{Lag}(V)$ can be regarded as a subobject of the unitary group. Subsequently Cappell, Lee and Miller show that with multiplication by $\exp(J\vartheta)$ always being an automorphism of $\mathfrak{Lag}(V)$, given any pair of Lagrangian planes, l_1, l_2, there exists $0 < \varepsilon < \pi$ such that if $0 < |\vartheta| < \pi$, then l_1 and $e^{J\vartheta} l_2$ are transverse: $l_1 \cap e^{J\vartheta} l_2 = (0)$. Additionally, since real orthogonal matrices have determinant ± 1, the mapping $\det^2 : U(n) \to \mathbb{C}_1^\times$ factors through $U(n)/O(n)$, whence we get that

$$\det{}^2 : \mathfrak{Lag}(V) \ni Al \mapsto (\det A)^2 \in \mathbb{C}_1^\times. \tag{3.14}$$

For any choice of basis $\{e_j\}_{j=1}^n$ the wedge product $e_1 \wedge \cdots \wedge e_n \in \bigwedge_{\mathbb{C}}^n V$ has norm 1 (relative to the norm induced by $\langle \_, \_ \rangle$), and so a change of orthonormal basis merely engenders introducing a factor of ± 1, which is of course obliterated by squaring. Accordingly we get that $\det^2$ is no longer dependent on the choice of $l =$

$\oplus_{j=1}^{n} \mathbb{R} \cdot e_j$:

$$\det{}^2 : \mathfrak{Lag}(V) \to S^1\left[\left(\bigwedge_{\mathbb{C}}^{n} V\right)^{\otimes 2}\right], \tag{3.15}$$

where (as per [7]) the object $S^1[(\bigwedge_{\mathbb{C}}^n V)^{\otimes 2}]$ is the unit circle ($S^1 \approx \mathbb{C}_1^{\times}$, after all) in $(\bigwedge_{\mathbb{C}}^n V)^{\otimes 2} = (\bigwedge_{\mathbb{C}}^n V) \otimes (\bigwedge_{\mathbb{C}}^n V)$. Now, if $f(t) = (l_1(t), l_2(t))$, $a \le t \le b$, is a path in $P([a,b]; \mathfrak{Lag}(V)^2)$, choose $0 < \varepsilon < \pi$ such that for all $0 < |\vartheta| < \varepsilon$ we have $l_1(\vartheta) \cap e^{J\vartheta} l_2(\vartheta) = (0) = l_1(b) \cap e^{J\vartheta} l_2(b)$. Then, following Guillemin–Sternberg [25], we can also pick $0 < \vartheta' < \varepsilon'/n$ such that there exist two paths of Lagrangians $\gamma_{left}(t)$ (resp. $\gamma_{right}(t)$), for $a-1 \le t \le a$ (resp. $b \le t \le b+1$), such that $\gamma_{left}(t)$ (resp. $\gamma_{right}(t)$) is transverse to $l_1(a)$ (resp. $l_2(b)$) for $t \in [a-1, a]$ (resp. $t \in [b, b+1]$) and $\gamma_{left}(a-1) = e^{-J\vartheta'} l_1(a)$, $\gamma_{left}(a) = e^{-J\vartheta'} l_2(a)$, $\gamma_{right}(b) = e^{-J\vartheta'} l_2(b)$, and $\gamma_{right}(b+1) = e^{-J\vartheta'} l_1(b)$. With this machinery in place, we can compare the following two (composite) paths of Lagrangians:

$$\Gamma := l_1(a) \circ \{l_1(t)\}_{t=a}^{t=b} \circ l_1(b), \tag{3.16}$$

$$\Gamma' := \gamma_{left}(t) \circ \{e^{-J\vartheta'} l_2(t)\}_{t=a}^{t=b} \circ \gamma_{right}(t) \tag{3.17}$$

by going over to their respective images

$$\bar{\Gamma} := \{(t, \det{}^2 \Gamma(t))\}_{a-1 \le t \le b+1}, \tag{3.18}$$

$$\bar{\Gamma}' = \{(t, \det{}^2 \Gamma'(t))\}_{a-1 \le t \le b+1} \tag{3.19}$$

in the cylinder $[a-1, b+1] \times S^1$, where we have identified S^1 with $S^1[(\bigwedge_{\mathbb{C}}^n V)]$ in the natural way. This done, Cappell–Lee–Miller provide that

$$\mu(f) = \mu_{geo,2}(f) = \mu_{geo,2}(l_1(t), l_2(t)) = \#(\bar{\Gamma} \cap \bar{\Gamma}'). \tag{3.20}$$

As noted earlier, this characterization of μ is representative of the quasi-combinatorial approach consonant with Axiom 6.

The other flavor of Maslov index Cappell–Lee–Miller consider is what they term "analytic", which is to say that in addition to the preceding renderings of $\mu_{geo,1}$ and $\mu_{geo,2}$ they present us with $\mu_{anal,1}$, and $\mu_{anal,2}$. For our purposes, however, it suffices to consider $\mu_{anal,1}$. Write $D(l_1, l_2)$ for the differential operator $-J\frac{d}{dt}$ acting on the class of functions $\varphi : [0,1] \to (V, J)$ such that $\varphi(0) \in l_1$ and $\varphi(1) \in l_2$. It is in fact the case [7] that as an operator mapping the Sobolev completion of the set of φ of class C^∞ satisfying the given boundary conditions into the L^2-completion $C^\infty(V^{[0,1]})$, the mapping $D(l_1, l_2)$ is self-adjoint with kernel $\mathbb{R} \cap (l_1 \cap l_2)$; furthermore, we get the eta-invariant of Atiyah–Singer–Patodi by first meromorphically continuing

$$\eta(s) := \sum_{\substack{\lambda \ne 0 \\ \lambda \in spec D(l_1, l_2)}} \frac{\operatorname{sgn}(\lambda)}{|\lambda|^s} \tag{3.21}$$

and then setting $\eta(0) =: \eta(D(l_1, l_2))$. Subsequently, returning to the above morphism $\det^2$ and again identifying S^1 with $S^1[(\bigwedge_{\mathbb{C}}^n V)]$, define the Keller–Arnol'd–Maslov form to be the 1-form $\omega := (\det^2)^*(\frac{d\vartheta}{2\pi})$, obtained by pulling the normalizing 1-form $\frac{d\vartheta}{2\pi}$ on S^1 back *via* $\det^2$. With $l_{1,2} : [a.b] \to \mathfrak{Lag}(V)$ and $\det^2 : \mathfrak{Lag}(V) \to S^1$ so that $\omega \in \mathfrak{Lag}(V)$, we can pull back again to get $l_{1,2}^*(\omega) \in \bigwedge^1([a, b])$, and we are now in a position to define

$$\mu(f) = \mu_{anal,1}(f) := \frac{1}{2}\{\eta(D(l_1(b), l_2(b))) - \eta(D(l_1(a), l_2(a)))\}$$
$$+ \left\{\int_a^b l_2^*(\omega) - l_1^*(\omega)\right\} + \frac{1}{2}\{\dim(l_1(a) \cap l_2(a)) - \dim(l_1(b) \cap l_2(b))\}. \tag{3.22}$$

For completeness, as regards $\mu_{anal,2}(f)$, the last index of the quartet considered in [7], we actually return to the first-mentioned theme of characterizing μ more or less combinatorially. Specifically, for f as above, the authors demonstrate that there exists $\varepsilon > 0$, dominated by the absolute values of all the non-zero eigenvalues of both $\eta(D(l_1(a), l_2(a)))$ and $\eta(D(l_1(b), l_2(b)))$, so that the self-adjoint operator $D(l_1(t), l_2(t)) - \varepsilon \cdot id$ has no nonzero eigenvalues at $t = a, b$. And then $\mu(f) = \mu_{anal,2}(f)$ is defined as the spectral flow of this operator on $[a, b]$; thus, the computation of $\mu_{anal,2}(f)$ entails that one "counts with signs ([i.e.]+1 for increasing value, −1 for decreasing) and multiplicities the number of eigenvalues of $D(l_1(t), l_2(t))$ crossing the line $\lambda = \varepsilon$" ([7, p. 158]).

We reiterate that one of the main thrusts of this work by Cappell, Lee, and Miller is that these four renderings of μ cut out the same object since each of them in point of fact satisfies the six axioms given in Section 3.2, and one needs only to cite Proposition 18 in this connection. Thus, any of these four definitions may be used to characterize the Maslov index.

We proceed now to take a look at how the index considered by Souriau fits into this taxonomy, with his index largely agreeing with the one discussed by Leray in [33] (in reply to a query raised by Arnol'd, as mentioned above), and thereafter we tie the earlier relation (3.8) between the Weil 2-cocycle c and the Maslov index m, favored by Souriau, in with the latter's marvelous rendering of the time evolution of a certain quantum mechanical system involving this same m.

3.4. Souriau's Maslov index, and Leray's, and the formulas of Souriau and Robbin–Salamon

In his 1975 paper, "Construction explicite de l'indice de Maslov. Applications", [53], Jean-Marie Souriau presents the following disarmingly simple characterization of a Maslov index, evidently due originally to Jean Leray (cf. p. 74 of [33]) and which in Souriau's notation is written m rather than μ.

Again, let $\widetilde{\mathfrak{Lag}}(V)$ denote the universal cover of the topological space $\mathfrak{Lag}(V)$ (recalling that $V \approx \mathbb{R}^{2n}$), and realize a point of $\widetilde{\mathfrak{Lag}}(V)$ as a pair $\tilde{l} = (A_l, \vartheta)$ where A_l is a unitary matrix (see immediately below) and $\vartheta \in \mathbb{R}$ is characterized by $e^{i\vartheta} = \det(A_l)$. Then, with

$\tilde{l}_1 = (A_{l_1}, \vartheta_1)$ and $\tilde{l}_2 = (A_{l_2}, \vartheta_2)$ in $\widetilde{\mathfrak{Lag}}(V)$,

$$m(\tilde{l}_1, \tilde{l}_2) := \frac{1}{2\pi}\{\vartheta_1 - \vartheta_2 + iTr(Log(-A_{l_1}A_{l_2}^{-1}))\}, \tag{3.23}$$

where, by definition ([33], p. 126)

$$Log(A) = \int_{\infty}^{0}\{(sI - A)^{-1} - (sI - I)^{-1}\}ds, \tag{3.24}$$

for A a square matrix.

The identification of $\mathfrak{Lag}(V)$ with $U(n)/O(n)$ as above (cf. (3.13)), i.e. the characterization of $\mathfrak{Lag}(V)$ as a quotient of the unitary group, can be replaced by an actual embedding of $\mathfrak{Lag}(V)$ in $U(n)$ in accord with Arnol'd's approach in [1]. Specifically, once more (as in (3.11)) writing any $l \in \mathfrak{Lag}(V)$ as $l = \bigoplus_{j=1}^{n} \mathbb{R} \cdot e_j$, with $\{e_j\}$ orthonormal, we can write $A_l = (e_1 \vdots e_2 \vdots \cdots \vdots e_n) \in U(n)$ and thereby obtain an unambiguous presentation of $\widetilde{\mathfrak{Lag}}(V)$ in $U(n)$. We can certainly cover the (latter) unitary group by $\tilde{U}(n) = \{(A, \vartheta) \mid A \in U(n), \vartheta \in \mathbb{R}$, and $\det(A) = e^{i\vartheta}\}$, a Lie group, so that we get the following exact sequence

$$1 \to \{(id, 2k\pi)\}_{k\in\mathbb{Z}} \to \tilde{U}(n) \to U(n) \to 1, \tag{3.25}$$

where the surjection is just projection onto the first coordinate. Under these circumstances the universal cover of $\mathfrak{Lag}(V)$ can be rendered as a diffeomorphic image of $\tilde{U}(n)/SO(n)$, namely,

$$\widetilde{\mathfrak{Lag}}(V) = \Bigg\{\tilde{l} = (A_l, \vartheta) \mid l = \bigoplus_{j=1}^{n} \mathbb{R} \cdot e_j \in \mathfrak{Lag}(V),\ A_l = (e_1 \vdots e_2 \vdots \cdots \vdots e_n) \in U(n),$$
$$\text{and } \vartheta \in \mathbb{R}, \text{ with } \det(A_l) = e^{i\vartheta}\Bigg\}. \tag{3.26}$$

Here we have used the fact that as an algebraic (i.e. topological) group $\tilde{U}(n)/SO(n)$ is simply connected to get the universality of this cover. One also notes that

$$\pi_1(\mathfrak{Lag}(V)) \approx \mathbb{Z}, \tag{3.27}$$

an observation going back to Arnol'd [1].

Returning to Souriau [53], working with a Lagrangian formalism for a simple quantum mechanical system in which the potential energy is given by a positive quadratic form $\frac{M}{2}\sum_{k=1}^{n}\omega_k^2 q_k^2$, with $\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{R}^n$, the ω_k stand for the oscillator's proper periods, and M naturally means mass, he demonstrated that if a_τ is the diagonal matrix $(e^{-i\omega_k\tau}\delta_{kl})_{1\le k,l\le n}$ with τ standing for (the passage of) time, then the value

$$m(u_t, u_{t+\tau}) = \frac{1}{2\pi}\{2(\omega_1 + \cdots + \omega_n) + iTrace(Log(-a_{2\tau}))\} \tag{3.28}$$

realizes a manifestation of the Maslov index, with $u_t, u_{t+\tau}$ being suitable Lagrangian planes. For our purposes the most salient result in Souriau's article is that, in point of fact,

$$m(u_t, u_{t+\tau}) = \frac{n}{2} + \sum_{k=1}^{n} \left\lfloor \frac{\omega_k \tau}{\pi} \right\rfloor, \tag{3.29}$$

where $\lfloor - \rfloor$ denotes the greatest integer (or floor) function.

With this formalism laid out, Souriau goes on to prove that

$$\psi_{\tau+t}(\mathbf{q}) = \left[\prod_{j=1}^{n} \left\{ \frac{\omega_j}{2\pi} |\csc(\omega_j \tau)| \right\} \right]^{1/2} e^{i\frac{\pi}{2} m_{Sou}(u_t, u_{\tau+t})}$$
$$\times \int_{\mathbb{R}^n} \psi_t(\mathbf{q}') e^{\frac{i}{2}\mathfrak{S}(\boldsymbol{\omega}, \mathbf{q}, \mathbf{q}', \tau)} d\mathbf{q}', \tag{3.30}$$

where $\mathbf{q}$ and $\mathbf{q}'$ are position vectors in an n-dimensional quantum mechanical phase space with ψ_t denoting a state at time t, where $\boldsymbol{\omega} = (\omega_k)_{1 \le k \le n}$, and where, finally,

$$\mathfrak{S}(\boldsymbol{\omega}, \mathbf{q}, \mathbf{q}', \tau) = \sum_{j=1}^{n} \omega_j \csc(\omega_j \tau) \cdot \{2q_j q_j' - (q_j^2 + q_j'^2)\cos(\omega_j \tau)\}. \tag{3.31}$$

We observe that the integral in this equation has the form of an oscillatory integral: its relationship to quantum mechanics hinges on the interpretation of $\mathfrak{S}(\boldsymbol{\omega}, \mathbf{q}, \mathbf{q}', \tau)$.

Additionally, it turns out that in the presence of de Gosson's observation (p. 80 of [8]), to the effect that, up to a factor of 2, the Souriau index agrees with the Arnol'd–Leray–Maslov index, we may bring the following more recent analysis to bear on the matter. In [45] and [44] Joel Robbin and Dietmar Salamon present the relation

$$\mathcal{U}(t, t_0; H)(f(x)) = \frac{e^{i\pi\mu(t,t_0;H)}}{(2\pi)^{n/2}|\det B|^{1/2}} \int_{\mathbb{R}^n} e^{iS(x,x_0)} f(x_0) dx_0, \tag{3.32}$$

where $\mathcal{U}(t, t_0; H)$ is a unitary (quantum mechanical) evolution operator in the presence of a Hamiltonian H on an n-dimensional real phase space, and S is a phase function. Furthermore, as already hinted, $\mu(t, t_0; H)$ is indeed the Maslov index in a form tailored to fit the present situation. Robbin and Salamon identify (or stipulate the existence of) a symplectomorphism $\psi_{t_0}^t$, conveying the change of state from time t_0 to time t and belonging to the Hamiltonian formalism governed by H, with the property that the matrix B is the upper right hand (time-dependent) block of $\psi_{t_0}^t$ in its matrix form. We note, again, that we have set Planck's constant equal to 1; Robbin and Salamon deal with the more general situation. Moreover, comparing (3.32) and (3.30) it is evident that if we can make the identification $S(x, x_0) \sim \frac{1}{2}\mathfrak{S}(\boldsymbol{\omega}, \mathbf{q}, \mathbf{q}', \tau)$, the corresponding results are equivalent and we can say without equivocation that, in the present real case, they capture a direct connection between the formalism of Maslov indices (and thus the archimedean Weil 2-cocycle) and the yoga of oscillatory integrals: $\int_{\mathbb{R}^n} e^{iS(x,x_0)} f(x_0) dx_0$ is a prime example of the latter.

The presence of the factor $e^{i\pi\mu(t,t_0;H)}$ in (3.32) together with an integral of the right form—witness the kinship between $e^{i\pi\mu(t,t_0;H)} \int_{\mathbb{R}^n} e^{iS(x,x_0)} f(x_0) dx_0$ and $e^{i\frac{\pi}{2} m_{Sou}(u_t, u_{\tau+t})} \int_{\mathbb{R}^n} \psi_t(\mathbf{q}') e^{\frac{i}{2}\mathfrak{S}(\boldsymbol{\omega}, \mathbf{q}, \mathbf{q}', \tau)} d\mathbf{q}'$—suggest, from a somewhat different angle, that by means of the proper identifications we could let (3.32) do the same bidding as (3.30) in what we are about to do, namely, to get from Weil's 2-cocycle for the double cover of $Sp(2n, \mathbb{R})$ to oscillatory integrals of the type found early on in Feynman's version of quantum mechanics. In view of the explicit nature of Souriau's treatment of the matter, presented at length above, we first develop this connection in terms of (3.30) and proceed to craft an overt connection between a phase integral and Weil's c_∞, so that we can in due course provide non-archimedean counterparts and proceed to adèlization. Nonetheless, it is useful for us to observe that in view of the heuristic identification $S(x, x_0) \sim \mathfrak{S}(\boldsymbol{\omega}, \mathbf{q}, \mathbf{q}', \tau)$, Robbin–Salamon's unitary operator $\mathcal{U}(t, t_0; H)$, having the shape of oscillatory integration against a kernel $e^{iS(x,x_0)}$ modulo a factor of some arithmetical significance (as we shall see later), should correspond to the evolution of $\psi_t(\mathbf{q})$ (to $\psi_{\tau+t}(\mathbf{q})$) in Souriau's formulation. In any event, these time evolutionary behaviors require that at $t = 0$, say, we set $u_t = u_0 = \tilde{l}$ while at τ_σ we set $u_{t+\tau} = u_{\tau_\sigma} = \tilde{l}^\sigma$ whence we get, writing $\omega^0_{\sigma,j}$ in place of ω_j in order to bring out the dependency of this parameter on $\tilde{l} \in \widetilde{\mathfrak{Lag}}(V)$ and $\sigma \in Sp(V)$,

$$e^{-i\frac{\pi}{2}m(\tilde{l},\tilde{l}^\sigma)} = \frac{\left[\prod_{j=1}^{n}\left\{\frac{\omega^0_{\sigma,j}}{2\pi}|\csc(\omega^0_{\sigma,j}\tau_\sigma)|\right\}\right]^{1/2} \int_{\mathbb{R}^n} \psi_0(\mathbf{q}')e^{\frac{i}{2}\mathfrak{S}(\boldsymbol{\omega}^0_\sigma,\mathbf{q},\mathbf{q}',\tau_\sigma)}d\mathbf{q}'}{\psi_{\tau_\sigma}(\mathbf{q})}. \tag{3.33}$$

This takes care of the transit from $t = 0, u_t = u_0 = \tilde{l}$ to $t = \tau_\sigma, u_{t+\tau} = u_{\tau_\sigma} = \tilde{l}^\sigma$, effecting the state evolution from $\psi_0(\mathbf{q})$ to $\psi_{\tau_\sigma}(\mathbf{q})$. Proceeding similarly, but taking $u_0 = \tilde{l}^\sigma$ and $u_{\tau_{\sigma'\sigma}} = \tilde{l}_0^{\sigma'\sigma}$ from $t = 0$ to $t = \tau_{\sigma'\sigma}$, effecting a state evolution from $\varphi_0(\mathbf{q})$ to $\varphi_{\tau_{\sigma'\sigma}}(\mathbf{q})$, we obtain

$$e^{-i\frac{\pi}{2}m(\tilde{l},\, \tilde{l}_0^{\sigma'\sigma})} = \frac{\left[\prod_{j=1}^{n}\left\{\frac{\omega^\sigma_{\sigma'\sigma,j}}{2\pi}|\csc(\omega^\sigma_{\sigma'\sigma,j}\tau_{\sigma'\sigma})|\right\}\right]^{1/2} \int_{\mathbb{R}^n} \varphi_0(\mathbf{q}')e^{\frac{i}{2}\mathfrak{S}(\omega^\sigma_{\sigma'\sigma,j},\mathbf{q},\mathbf{q}',\tau_{\sigma'\sigma})}d\mathbf{q}'}{\varphi_{\tau_{\sigma'\sigma}}(\mathbf{q})}, \tag{3.34}$$

and then, taking $u_0 = \tilde{l}_0^{\sigma'\sigma}$ and $u_\tau = \tilde{l}$ from $t = 0$ to $t = \tau$, effecting a state evolution from $\xi_0(\mathbf{q})$ to $\xi_\tau(\mathbf{q})$, we obtain

$$e^{-i\frac{\pi}{2}m(\tilde{l}_0^{\sigma'\sigma},\tilde{l})} = \frac{\left[\prod_{j=1}^{n}\left\{\frac{\omega^{\sigma'\sigma}_{0,j}}{2\pi}|\csc(\omega^{\sigma'\sigma}_{0,j}\tau)|\right\}\right]^{1/2} \int_{\mathbb{R}^n} \xi_0(\mathbf{q}')e^{\frac{i}{2}\mathfrak{S}(\omega^{\sigma'\sigma}_{0,j},\mathbf{q},\mathbf{q}',\tau)}d\mathbf{q}'}{\xi_\tau(\mathbf{q})}. \tag{3.35}$$

Substituting these relations into (3.8) then quickly yields the following result, of critical importance to our enterprises:

Proposition 19. *With* $\mathbf{q}, \mathbf{q}', \mathbf{r}, \mathbf{r}', \mathbf{s}, \mathbf{s}'$ *ranging over* $\mathbb{R}^n$ *we obtain that*

$$c(\sigma,\sigma') = \frac{\left[\prod_{j=1}^{n}\left\{\frac{\omega^0_{\sigma,j}\omega^{\sigma}_{\sigma'\sigma,j}\omega^{\sigma'\sigma}_{0,j}}{8\pi^3}|\csc(\omega^0_{\sigma,j}\tau_\sigma)\csc(\omega^{\sigma}_{\sigma'\sigma,j}\tau_{\sigma'\sigma})\csc(\omega^{\sigma'\sigma}_{0,j}\tau)|\right\}\right]^{1/2}}{\psi_{\tau_\sigma}(\mathbf{q})\varphi_{\tau_{\sigma'\sigma}}(\mathbf{r})\xi_\tau(\mathbf{s})}$$
$$\times \int_{\mathbb{R}^{3n}} \psi_0(\mathbf{q}')\varphi_0(\mathbf{r}')\xi_0(\mathbf{s}')$$
$$\times e^{\frac{i}{2}\{\mathfrak{S}(\boldsymbol{\omega}^0_\sigma,\mathbf{q},\mathbf{q}',\tau_\sigma)+\mathfrak{S}(\omega^{\sigma}_{\sigma'\sigma,j},\mathbf{r},\mathbf{r}',\tau_{\sigma'\sigma})+\mathfrak{S}(\omega^{\sigma'\sigma}_{0,j},\mathbf{s},\mathbf{s}',\tau)\}}d\mathbf{q}'d\mathbf{r}'d\mathbf{s}'. \tag{3.36}$$

The proof is a direct consequence of the foregoing. ■

Furthermore, given the structural agreement between Souriau's (3.30) and Robbin–Salamon's (3.32) discussed earlier, the preceding result can be restated in terms of the latter formalism as follows, with the *caveat* that we have now written the Maslov indices in the form favored by Cappell–Lee–Miller and incorporated the according proportionality:

Corollary 3. $c(\sigma,\sigma')\mathcal{U}(t,t_0;H)^{0,\sigma}(f(\mathbf{x}))\mathcal{U}(t,t_0;H)^{\sigma,\sigma'\sigma}(g(\mathbf{y}))\mathcal{U}(t,t_0;H)^{\sigma'\sigma,0}(h(\mathbf{z})) = \frac{1}{(8\pi^3)^{n/2}|\det(B_{0,\sigma}B_{\sigma,\sigma'\sigma}B_{\sigma'\sigma,0})|^{\frac{1}{2}}}\int_{\mathbb{R}^{3n}} e^{i\{S(\mathbf{x},\mathbf{x}')+S(\mathbf{y},\mathbf{y}')+S(\mathbf{z},\mathbf{z}')\}}f(\mathbf{x}')g(\mathbf{y}')h(\mathbf{z}')d\mathbf{x}'d\mathbf{y}'d\mathbf{z}'$, *where we take as our Maslov index Cappell–Lee–Miller's* $\mu(f) = \mu_{geo,2}(f) = \mu_{geo,2}(l_1(t),l_2(t))$ *as per* (3.20), *and, with the obvious indexing in place as regards* (3.33)–(3.35), *we set, first,* $\mu(t_0,t_1)^{0,\sigma}_{RS} = \mu(\pi(\tilde{l}^\sigma),\pi(\overset{\sim 0}{\gamma}_\sigma))$, *with* $\tilde{l} = \tilde{l}$, $\tilde{l} = \tilde{l}^\sigma$, *second,* $\mu(t_0,t_1)^{\sigma,\sigma'\sigma}_{RS} = \mu(\pi(\tilde{l}^{\sigma'\sigma}),\pi(\overset{\sim\sigma}{\gamma}_{\sigma'\sigma}))$, *with* $\tilde{l} = \tilde{l}^\sigma$, $\tilde{l} = \tilde{l}^{\sigma'\sigma}$, *and finally, third,* $\mu(t_0,t_1)^{\sigma'\sigma,0}_{RS} = \mu(\pi(\tilde{l}),\pi(\overset{\sim\sigma'\sigma}{\gamma}_0))$, *with* $\tilde{l} = \tilde{l}^{\sigma'\sigma}$, $\tilde{l} = \tilde{l}$; *in other words, we have, first,* $\mathcal{U}(t,t_0;H)^{0,\sigma}(f(\mathbf{x})) = \frac{e^{i\pi\mu(t_0,t_1)^{0,\sigma}_{RS}}}{(2\pi)^{n/2}|\det B_{0,\sigma}|^{1/2}}\int_{\mathbb{R}^n} e^{iS(\mathbf{x},\mathbf{x}')}f(\mathbf{x}')d\mathbf{x}'$, *second,* $\mathcal{U}(t,t_0;H)^{\sigma,\sigma'\sigma}(g(\mathbf{y})) = \frac{e^{i\pi\mu(t_0,t_1)^{\sigma,\sigma'\sigma}_{RS}}}{(2\pi)^{n/2}|\det B_{\sigma,\sigma'\sigma}|^{1/2}}\int_{\mathbb{R}^n} e^{iS(\mathbf{y},\mathbf{y}')}g(\mathbf{y}')d\mathbf{y}'$, *and, third,* $\mathcal{U}(t,t_0;H)^{\sigma'\sigma,0}(h(\mathbf{z})) = \frac{e^{i\pi\mu(t_0,t_1)^{\sigma'\sigma,0}_{RS}}}{(2\pi)^{n/2}|\det B_{\sigma'\sigma,0}|^{1/2}}\int_{\mathbb{R}^n} e^{iS(\mathbf{z},\mathbf{z}')}h(\mathbf{z}')d\mathbf{z}'$.

The notation in Proposition 19 is manifestly cumbersome, and the same can be said about Corollary 3, if to a lesser extent. Indeed, the equality that forms the thrust of Corollary 3 is substantially more amenable to the generalization processes we will propose in order to lift this connection between Weil–Kubota data and oscillatory integrals to the indicated localizations and adèlizations of k rather than just $\mathbb{Q}$. Here we also note that the respective multipliers, i.e. $[\prod_{j=1}^{n}\{\frac{\omega^0_{\sigma,j}\omega^{\sigma}_{\sigma'\sigma,j}\omega^{\sigma'\sigma}_{0,j}}{8\pi^3}|\csc(\omega^0_{\sigma,j}\tau_\sigma)\csc(\omega^{\sigma}_{\sigma'\sigma,j}\tau_{\sigma'\sigma})\csc(\omega^{\sigma'\sigma}_{0,j}\tau)|\}]^{1/2}$ and $\frac{1}{(8\pi^3)^{n/2}|\det(B_{0,\sigma}B_{\sigma,\sigma'\sigma}B_{\sigma'\sigma,0})|^{\frac{1}{2}}}$, are of evidently well-equipped with inner symmetries that *ab initio* have to be consonant with the fact that $c(\sigma,\sigma')\in H^2(Sp(2,\mathbb{R}),\mathbb{Z}_2)$. We train our focus on the latter of these two expressions, and taking all the conventions above into account, abbreviate it to $\mathcal{F}_{\infty_\mathbb{R}}(c_{\infty_\mathbb{R}}(\sigma,\sigma'))$.

Thus we get, compactly, that for all $\sigma, \sigma' \in Sp(2, \mathbb{R})$,

$$c_{\infty_{\mathbb{R}}}(\sigma, \sigma')\mathcal{U}(t, t_0; H)^{0,\sigma}(f(\mathbf{x}))\mathcal{U}(t, t_0; H)^{\sigma,\sigma'\sigma}(g(\mathbf{y}))\mathcal{U}(t, t_0; H)^{\sigma'\sigma,0}(h(\mathbf{z}))$$
$$= \mathcal{F}_{\infty_{\mathbb{R}}}(c_{\infty_{\mathbb{R}}}(\sigma, \sigma')) \int_{\mathbb{R}^{3n}} e^{i\{S_{\infty_{\mathbb{R}}}(\mathbf{x},\mathbf{x}')+S_{\infty_{\mathbb{R}}}(\mathbf{y},\mathbf{y}')+S_{\infty_{\mathbb{R}}}(\mathbf{z},\mathbf{z}')\}}$$
$$\times f(\mathbf{x}')g(\mathbf{y}')h(\mathbf{z}')d\mathbf{x}'d\mathbf{y}'d\mathbf{z}', \tag{3.37}$$

where we have made the dependency on the completion of $\mathbb{Q}$ to $\mathbb{R}$ (as the only real local field over $\mathbb{Q}$) explicit. With the foregoing simplifications in place, and seeing that the indicated generalization of Proposition 19 and Corollary 3 are from now on the main order of business (which we will refer to as effecting, successively, p-adicization and $\mathbb{Q}$-adèlization, followed by $\mathfrak{p}$-adicization and k-adèlization), we bring these maneuvers to fruition in the next chapter. The focus will fall on (3.37).

4. Phase and oscillatory integrals in quantum mechanics: heading toward Feynman

4.1. Brief preliminaries concerning Feynman's method

The integrals we have encountered in the preceding, particularly in Proposition 19, Corollary 3, and (3.37), are of a shape that ultimately points in the direction of Feynman's vaunted path integrals. They are certainly both oscillatory and phase integrals and, as such, can in principle already be tied to quantum mechanics; in fact the cited work [45] by Robbin–Salamon also makes this point explicitly. So we are now leaning in the direction of Feynman's version of quantum physics.

There are many sources available dealing with Richard Feynman's path-integral method in quantum mechanics, quantum electrodynamics, and quantum field theory. Regarding the latter, recent activity concerned with, e.g., string theory, has spawned a number of texts on the subject which of course count Feynman's approach as foundational, for instance, [66,15], and [18]. The method's origin is found in Feynman's 1942 doctoral thesis [20], wherein he credits P.A.M. Dirac for some of the fundamental ideas behind this new approach to quantum mechanics. Dirac's paper [12] is particularly relevant in this connection. Regarding Feynman's approach as such, we also mention [21], his later text written together with A.P. Higgs.

It is useful for us to repeat at this stage that, as is so often the case, there is an intrinsic difference in style and presentation of all this material, separating physicists from mathematicians. For example, the aforementioned treatments of quantum field theory [15,18], by, respectively, Dolgachev and Etinghof, come more heavily equipped with mathematical rigor and formalism than the influential text [66] by Zee, which is aimed at budding physicists *per se*. Additionally, and particularly relevant to our purposes, there is a text in print that perhaps best illustrates the (tenuous) ecumenism extant between physicists and mathematicians working at the frontier of what quantum field theory has wrought, *viz.* the two-volume compendium [9], *Quantum Fields and Strings: A Course for Mathematicians*: with the physics presented here expressly tailored for mathematicians, it is poised to serve our purposes. We explicitly single out the lectures, "Introduction to Quantum Field Theory", by Ludwig Faddeev, especially the first lecture [19] on the "Basis of quantum mechanics and canonical quantization in Hilbert space" ([19], pp. 515–522).

Properly speaking, the backdrop for this development of quantum mechanics is Weyl quantization with the focus placed on the time evolution operator for a simple quantum mechanical system. To wit, if H is the operator corresponding to the Hamiltonian (total energy) of the system, then it follows from Stone's theorem [42] (p. 335) that we are dealing with a 1-parameter group of unitary operators, e^{-iHt}, where, again, we have taken Planck's constant $\hbar$ to be 1. Accordingly, if $A(t)$ is an observable for the system, then the Schrödinger equation governing the dynamics is, in Dirac's language [13],

$$\frac{dA}{dt} = \{H, A(t)\} = i(HA(t) - A(t)H), \tag{4.1}$$

and the time evolution of the according dynamical system is given by the formalism

$$U(t) = e^{-iHt} \tag{4.2}$$

$$A(t) = U^{-1}(t)A(0)U(t). \tag{4.3}$$

The phase integral approach to this situation engenders providing an integral expression for U, giving a kernel for realizing the effect of U as an operator on an L^2 state function f through the services of the usual functional analysis convolution scheme: we are after all in the conventional quantum mechanical setting in which a system's states are identified with unit vectors in a suitable Hilbert space and the attendant observables arise as (generally densely defined) self-adjoint operators [42,13]. In Faddeev's aforementioned treatment, the space of paths is equipped with a Liouville measure; see [19] and also [15].

The upshot is that we are almost always dealing, at least in somewhat rough terms, with integrating kernels of the form

$$U = \int e^{i\mathfrak{S}}, \tag{4.4}$$

suppressing the measure and its putative measure space, and obtain subsequently that the action of U on a square integrable f takes the form

$$U(f) = \int e^{i\mathfrak{S}} f. \tag{4.5}$$

The all-important object $\mathfrak{S}$ is of course the action attending the given quantum mechanical system and is itself generally expressed as a linear functional in its integral guise, at least when we are working in the classical context (courtesy of something like the Riesz representation theorem). For our purposes it should be noted that our earlier relations (3.36) and (3.37) evidently fit into this scheme rather well, and kindred observations are present in Souriau [53] and Robbin–Salamon [45,44]; we observe, too, that this type of integral formalism is consonant with Hörmander's development of oscillatory integrals in [28].

We might characterize this situation in tactical terms as a first approximation to developing a formalism of quantum mechanics along the lines favored by Feynman, but we should mention that Feynman famously goes on to develop his version of not just quantum mechanics but also quantum electrodynamics in terms of integrals taken over all paths in a certain region of space–time. This is the physicists' well-known "sum over all histories" technique which contains, in general, a very nasty mathematical pitfall: the method begs the question of how properly to define a measure on such a space of paths. This question, as such, remains open, except for a few cases such as those pertaining to the Feynman–Kac formula [52,24].

In any event, a relationship between the real symplectic group's double cover and quantum mechanical operators given as oscillatory integrals has been delineated in Proposition 19, and we now turn to the matter of adèlizing this data. This entails first and foremost that we write down the corresponding counterparts for every place of the underlying field k, be it archimedean or non-archimedean, and then glue this cumulative data together to get a single relation of the same form as what is given in Proposition 19 or (3.36). Since we have already addressed the questions of non-archimedean Weil 2-cocycles and Maslov indices above, in Sections 2.5–2.6, and even the adèlization of the former data in Section 2.8, it now falls to us to develop quantum mechanics from these perspectives.

4.2. Regarding non-archimedean quantum mechanics and adèlic quantum mechanics: some background

In his 2006 lecture, "Has God made the quantum world p-adic?" [57], V.S. Varadarajan starts by quoting Dirac's introduction to his 1931 paper [11] proposing magnetic monopoles, to the effect that "[n]on-euclidean geometry and noncommutative algebra, which were at one time considered to be purely fictions of the mind and pastimes of logical thinkers, have now been found to be very necessary for the description of the general facts of the physical world . . . [T]his process of increasing abstraction will continue . . . and advance in physics [will] be associated with a continual modification of the axioms at the base of the mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation". In their important work [59], V.S. Vladimirov, I.V. Volovich, and E.I. Zelenov observe that because "the Planck length is the smallest distance that can in principle be measured . . . a suggestion emerges to abandon the archimedean axiom at small distances". Subsequently, in the authors' introductory remarks to the chapter "p-Adic Quantum Theories", we read that "[i]nvestigation of p-adic quantum mechanics is of great interest even independent of possible new physical applications because it can lead to better understanding of [the] usual quantum theory". Vladimirov, Volovich, and Zelenov then go on to state something particularly apposite for our objectives, namely that they "hope also that the investigation of p-adic quantum mechanics and field theory will be useful in pure mathematical researches in number theory, representation theory, and p-adic analysis".

Thus, there are certainly various precedents in place in mathematical physics for considering both archimedean and non-archimedean quantum mechanics, corresponding respectively to a model of the material universe in which the rational numbers are completed either with respect to the ordinary absolute value or with respect to a p-adic valuation. Beyond this, as we shall see, there is a precedent as well for considering archimedean and non-archimedean quantum mechanics together: adèlic quantum mechanics.

The physicists' focus in this regard is on the case where the base field is $\mathbb{Q}$ because, after all, the direct measurements that are the life's blood of physics are by their very nature rational numbers. The idea that discreteness rules at the Planck scale therefore suggests the imperative that the non-archimedean completions to be considered are just the $\mathbb{Q}_p$ with p a rational prime, and beyond this, i.e. when one adèlizes, the paradigmatic ring of $\mathbb{Q}$-adèles $\mathbb{Q}_{\mathbb{A}}$. In keeping with what we have done above, however, we take a more general position in that we work with an arbitrary base field k (with $(k : \mathbb{Q}) = d$, say) and have as our completions the local fields $k_{\mathfrak{p}}$ and obtain the ring $k_{\mathbb{A}}$ as the according adèlization, all as before.

However, it is proper for us at this stage to go into greater detail regarding the attendant character theory, and we shall pursue this next. We adopt, by and large, the original conventions going back to Tate's 1950 thesis [54], in which $\mathfrak{p}$ is taken to range over all valuations, infinite (archimedean) as well as finite (non-archimedean). Of course, the question of direct physical interpretation in this more general case is a tenuous one at best.

Turning to the perspective assumed by Vladimirov, Volovich, and Zelenov, we first encounter the following adaptation, presented in the paper [58] by the first two authors (cf. p. 661, [58]): "the usual Schrödinger representation cannot be used for [the] construction of p-adic quantum mechanics with complex wave functions. We …use a formalism of p-adic quantum mechanics with complex wave functions in the Hilbert space $L^2(\mathbb{Q}_p)$. It is quite remarkable that the Weyl representation can be used not only in [the] usual quantum mechanics but also in the p-adic case. p-Adic quantum mechanics does not possess a Hamiltonian and we propose to work directly with a unitary group of time translations". Thus, it is evident and fortunate that the developments of the preceding pages surrounding the Weil representation, as well as the Maslov index and what derives from it, are consonant with the present approach to quantum mechanics in the manner of Hermann Weyl (and the authors of [58] and [59] indeed note that the seminal reference in this connection is Weyl's famous monograph [65]).

We can recapitulate the salient points as follows. As far as physics is concerned, unitary representation theory is the order of the day, and within this representation-theoretic context the emergence of 1-parameter subgroups of the over-arching unitary groups resides at the very heart of the foundational architecture. The latter feature is particularly in evidence when it comes to formulating the connection between the (Arnol'd-Leray-)Maslov index and phase integrals as presented above in Section 3.4, and equally so in connection with Robbin–Salamon's work. In general everything is pervaded by "unitary group[s] of time translations".

Accordingly, with the ideology of Weyl quantization in place, we can argue that, even for an arbitrary number field k, it is legitimate to consider the collective $\mathfrak{p}$-adic data surrounding both the Weil 2-cocycle for the local metaplectic group (the double cover of the symplectic group) and the Maslov index (*gratia* Lion–Vergne and Perrin) in parallel with a corresponding collection of $\mathfrak{p}$-adic phase integrals (all for $\mathfrak{p} \leq \infty$, so to speak). This then sets the stage for the move that ultimately constitutes the *raison d'être* for all these machinations with archimedean as well non-archimedean valuations: the k-adèlization of the relevant parts both of the arithmetical themes developed above surrounding the Weil

2-cocycle and quantum mechanics in keeping with Weyl quantization, and (ultimately Feynman's) path integral formulations.

With these objectives in mind, we need to note that, generally, in order to adèlize the data provided by a collection of equations of the right form, a certain inner compatibility has to be in place: the collection is indexed on all the places of the base field and the hypotheses for adèlization as set forth by Tate [54] or Weil [63] are met. Given that k-adèlization of local data on the arithmetical side of things, i.e. the indicated maneuvers surrounding $c_{\mathbb{A}}$, is already soundly in place, it follows *à forteriori* that if the various analytic expressions on the other (physics) side of (3.37) are amenable to both $\mathfrak{p}$-adicization and k-adèlization, any questions about the inner compatibility of the data on this quantum mechanics side of the divide, so to speak, are largely moot. Nonetheless, in the next section, we address a good amount of this material directly, specifically regarding the phase and oscillatory integrals (and unitary operators) populating the earlier 1-parameter family of such; we also return to this theme at the end of Section 4.6.

Apparently the first work focused on this theme to appear on the scene is the 2004 work [17] by Branko Dragovich. In his Introduction to this article, harking back to both the aforementioned work by Vladimirov, Volovich, and Zelenov and work by (e.g.) P.G.O. Freund, E. Witten, and P. Ruelle, the author presents the appraisal that "[s]ince 1987, the application of p-adic numbers has been of interest in string theory, quantum mechanics, and some other areas of mathematical physics", and then, getting down to specifics, he goes on to state that "in [his] formulation of p-adic quantum mechanics [he follows] the Vladimirov–Volovich approach [and] [t]his approach is generalized to adèlic quantum mechanics".

To convey what is going on, therefore, with both non-archimedean and adèlic quantum mechanics, we now proceed by sketching Dragovich's approach to the adèlic case, which includes the protocols for the non-archimedean case as presented by Vladimirov, Volovich, and Zelenov. The background for adèles *qua* adèles is of course still Tate's thesis [54].

4.3. Local fields and adèle rings: preparation for quantum mechanics

First, following [17] and [59] we take $\mathbb{Q}$ as our base field, noting that presently these maneuvers will apply to facilitate generalization to our case of interest, an arbitrary number field k. The rationale for this *intermezzo* is that adèlic quantum mechanics is still quite novel and relatively little known outside its circle of devotées, and it is accordingly useful to have a model at our disposal of what happens in the simplest case of $\mathbb{Q}$ itself. We take some trouble at this stage to delineate rather explicitly what we covered more discursively in the earlier chapters of this article, in the more general context of the local fields $k_{\mathfrak{p}}$, because we now have occasion to make rather pointed comparisons between quantum mechanics over $\mathbb{R}$ and quantum mechanics over, first, $\mathbb{Q}_p$, and subsequently $\mathbb{Q}_{\mathbb{A}}$. Because these presentations are by their very nature rife with formulas amenable to calculations of the type physicists find meaningful, the viewpoint assumed in regard to non-archimedean analysis is necessarily more prosaic than ours in the earlier sections, focused as they were on, ultimately, second degree cohomology and structural results. With the task before us being to convey this still quite novel non-archimedean quantum mechanics in terms amenable to the prospect of interpreting our results, Proposition 19 and Corollary 3, more specifically (3.37) (which indeed encapsulates its two foregoing results) in the according non-archimedean and adèlic terms, it is clearly necessary to develop the relevant non-archimedean and adèlic quantum mechanics in as effective a fashion as possible.

Thus, if $a_{\mathbb{A}} = (a_\infty; a_p)_{p\ prime} = (a_\infty; a_2, a_3, a_5, \ldots, a_p, \ldots)$ is an adèle, i.e. $a_{\mathbb{A}} \in \mathbb{Q}_{\mathbb{A}}$, so that $a_\infty \in \mathbb{R}$ and $a_p \in \mathbb{Z}_p$ which is to say that $\|a_p\|_p \leq 1$, a.e. p (meaning for all but a finite number of primes, p), then the idèles, comprising the set (and multiplicative group) $\mathbb{Q}_{\mathbb{A}}^\times \subset \mathbb{Q}_{\mathbb{A}}$, are cut out by the requirement $a_\infty \neq 0$, $a_p \neq 0$, $\forall p$, and $\|a_p\|_p = 1$, a.e. p. A principal adèle (resp. idèle) is any element of $\mathbb{Q}_{\mathbb{A}}$ (resp. $\mathbb{Q}_{\mathbb{A}}^\times$) for which $a_\infty = r = a_p$, $\forall p$, with $r \in \mathbb{Q}$ (resp. $r \in \mathbb{Q}^\times$, i.e., $r \neq 0$). We recall (cf. [62]) that the adèles obtain as a restricted direct product (manifesting itself in the requirement given above) and comprise a topological ring; the subset of idèles is a multiplicative group. Furthermore, we are of course dealing with complete normed spaces with, *qua* adèles,

$$\|a_{\mathbb{A}}\| = \|a_\infty\|_\infty \prod_p \|a_p\|_p = |a_\infty| \cdot \prod_p \|a_p\|_p, \tag{4.6}$$

and we obtain as a consequence of the fundamental theorem of arithmetic that if (r) is a principal idèle ($r \in \mathbb{Q}^\times$), then

$$\|(r)\| = 1. \tag{4.7}$$

Furthermore, as far as the indicated topologies are concerned, given that $\mathbb{R}$ and the $\mathbb{Q}_p$ are locally compact abelian groups and, as just mentioned, the adèles are a restricted direct (topological) product, we obtain a pair of Haar measures on $\mathbb{Q}_{\mathbb{A}}$ and $\mathbb{Q}_{\mathbb{A}}^\times$ as follows. Writing $x_{\mathbb{A}} = (x_\infty; x_p)_p$, and with Lebesgue measure on $\mathbb{R}$ (resp. $\mathbb{Q}_{\mathbb{A}}$) normalized *via* $\int_{|x_\infty|\leq 1} dx_\infty = 2$ (resp. $\int_{\|x_p\|_p \leq 1} dx_p = 1$), first, the product measure

$$dx_{\mathbb{A}} = dx_\infty dx_2 dx_3 \cdots dx_p \cdots \tag{4.8}$$

gives a measure on $\mathbb{Q}_{\mathbb{A}}$. Second, if we stipulate that

$$d^\times x_\infty = \frac{dx_\infty}{|x_\infty|} \quad \text{and} \quad d^\times x_p = \frac{1}{1-\frac{1}{p}} \cdot \frac{dx_p}{\|x_p\|_p} \tag{4.9}$$

then we get a measure on $\mathbb{Q}_{\mathbb{A}}^\times$ by setting

$$d^\times x_{\mathbb{A}} = d^\times x_\infty d^\times x_2 d^\times x_3 \cdots d^\times x_p \cdots . \tag{4.10}$$

Next, it is of course the case that any p-adic number can be written (formally) as an expansion

$$a_p = \pm \sum_{k \geq k_0} \alpha_k p^k \tag{4.11}$$

where, for each k, $\alpha_k \in \mathbb{Z}/p\mathbb{Z}$, which is to say that $\alpha_k \in \{0, 1, \ldots, p-1\}$, so that we get, in particular, that $\alpha_k \in \mathbb{Z}_p$ if and only if $k_0 \geq 0$. Under these circumstances the so-called fractional part, $\{a_p\}$, of the p-adic number a_p is defined to be the finite sum

$\sum_{0>k\geq k_0}\alpha_k p^k$. Then we fix an additive character on $\mathbb{Q}_{\mathbb{A}}$ as the mapping

$$\chi_{\mathbb{A}} : x_{\mathbb{A}} = (x_\infty; x_p)_p \mapsto e^{-2\pi i x_\infty}\prod_p e^{2\pi i\{x_p\}}. \tag{4.12}$$

Evidently the individual (local) factors of $\chi_{\mathbb{A}}$ are the respective canonical local factors, namely, $\chi_\infty(x_\infty) = e^{-2\pi i x_\infty}$ and $\chi_p = e^{2\pi i\{x_p\}}$, for all p. Thus, (4.12) can be rewritten in the compact from

$$\chi_{\mathbb{A}}(x_{\mathbb{A}}) = \chi_{\mathbb{A}}((x_\infty; x_p)_p) = \chi_\infty(x_\infty)\prod_p \chi_p(x_p). \tag{4.13}$$

Because, under all circumstances, $x_p \in \mathbb{Z}_p$, a.e. p, meaning that $\{x_p\} = 0$ a.e. p, the product is well-defined.

As far as the idèles are concerned, an idèlic character looks like

$$\pi_{\mathbb{A},s}(x_{\mathbb{A}}) = |x_\infty|^s\prod_p \|x_p\|_p^s = \|x_{\mathbb{A}}\|^s \tag{4.14}$$

for a fixed $s \in \mathbb{C}$ (cf. [54]). Again, this character is well-defined because when evaluated at a given idèle, its local factors always reduce to 1 almost everywhere. On principal adèles and idèles the respective characters (4.13) and (4.14) reduce to unity.

With this back-ground material in place, we can now begin to make a path toward the adèlic functional analysis required for adèlic quantum mechanics. First off, by definition, an elementary function on $\mathbb{Q}_{\mathbb{A}}$ is an object

$$\varphi_{\mathbb{A}}(x_{\mathbb{A}}) = \varphi_{\mathbb{A}}((x_\infty; x_p)_p) = \varphi_\infty(x_\infty)\prod_p \varphi_p(x_p), \tag{4.15}$$

where φ_∞ and the φ_p are complex-valued Schwartz–Bruhat functions on $\mathbb{R}$ and the $\mathbb{Q}_p$ respectively, and then the Schwartz–Bruhat class $\mathcal{S}(\mathbb{Q}_{\mathbb{A}})$ for the ring of adèles is the linear closure (i.e. the set of all finite linear combinations) of these adèlic elementary functions. The upshot of this is that $\varphi_\infty \in C^\infty(\mathbb{R})$ and, for any $n \geq 0$, $|x_\infty|^n\varphi_\infty(x_\infty) \to 0$ as $|x_\infty| \to \infty$; φ_p has compact support in $\mathbb{Q}_p$ and there exists a natural number ν (depending on φ_p) such that if $\|y_p\|_p \leq p^{-\nu}$ then $\varphi_p(x_p + y_p) = \varphi_p(x_p)$ (in other words, φ_p is locally constant); and finally, for almost every p we have that $\varphi_p(x_p) = 1$ if $0 \leq \|x_p\|_p \leq 1$, while $\varphi_p(x_p) = 0$ if $\|x_p\|_p > 1$. So $\mathcal{S}(\mathbb{Q}_{\mathbb{A}})$ is the vector space of all finite linear combinations of such $\varphi_{\mathbb{A}}$.

Given $\mathcal{S}(\mathbb{Q}_{\mathbb{A}})$ one defines two important transforms, the Fourier transform and the Mellin transform, as follows: for the Fourier transform, $\hat{\varphi}_{\mathbb{A}}(\xi_{\mathbb{A}})$, of $\varphi_{\mathbb{A}}(x_{\mathbb{A}})$ we have the formula

$$\hat{\varphi}_{\mathbb{A}}(\xi_{\mathbb{A}}) = \int_{\mathbb{Q}_{\mathbb{A}}}\varphi_{\mathbb{A}}(x_{\mathbb{A}})\chi_{\mathbb{A}}(x_{\mathbb{A}})dx_{\mathbb{A}}, \tag{4.16}$$

and for the Mellin transform, $\Phi_{\mathbb{A}}(s)$, of $\varphi_{\mathbb{A}}(x_{\mathbb{A}})$, we have, with $s \in \mathbb{C}$ and $\mathrm{Re}(\dot{s}) > 1$, the formula

$$\Phi_{\mathbb{A}}(s) = \int_{\mathbb{Q}_{\mathbb{A}}^\times}\varphi_{\mathbb{A}}(x_{\mathbb{A}})\pi_{\mathbb{A},s}(x_{\mathbb{A}})d^\times x_{\mathbb{A}} = \int_{\mathbb{Q}_{\mathbb{A}}^\times}\varphi_{\mathbb{A}}(x_{\mathbb{A}})\|x_{\mathbb{A}}\|^s d^\times x_{\mathbb{A}}. \tag{4.17}$$

These two players are famously situated at the heart of Tate's thesis [54], devoted as it is to "generalizing the notion of ζ-function and simplifying the proof of the analytic continuation and functional equation for it, by making fuller use of analysis in the spaces of valuation vectors [a.k.a. adèles] and idèles ...", and accordingly providing, for example, that $\Phi_{\mathbb{A}}(s)$ is amenable to analytic continuation to $\mathbb{C}$ except for singularities at $s = 0, 1$, where there are simple poles with respective residues $-\varphi_{\mathbb{A}}((0))$ and $\hat{\varphi}_{\mathbb{A}}((0))$; additionally we have the functional equation $\Phi_{\mathbb{A}}(s) = \hat{\Phi}_{\mathbb{A}}(1-s)$.

Returning to functional analysis, however, one next defines a number of Hilbert spaces. Working locally, we have of course $L^2(\mathbb{R})$ equipped with the inner product $\langle f, g\rangle = \int_{\mathbb{R}} \bar{f}(x_\infty) g(x_\infty) dx_\infty$ and its attendant norm, and, for all primes p, $L^2(\mathbb{Q}_p)$ equipped with $\langle f, g\rangle_p = \int_{\mathbb{Q}_p} \bar{f}(x_p) g(x_p) dx_p$ and its norm. Then, adèlically, we set

$$L^2(\mathbb{Q}_{\mathbb{A}}) = \left\{ \varphi_{\mathbb{A}} : \mathbb{Q}_{\mathbb{A}} \to \mathbb{C} \mid \int_{\mathbb{Q}_{\mathbb{A}}} \bar{\varphi}(x_{\mathbb{A}}) \varphi_{\mathbb{A}}(x_{\mathbb{A}}) dx_{\mathbb{A}} < \infty \right\}. \tag{4.18}$$

In other words, $L^2(\mathbb{Q}_{\mathbb{A}})$ obtains as a complete inner product space relative to the norm

$$\langle \varphi_{\mathbb{A}}, \psi_{\mathbb{A}} \rangle = \int_{\mathbb{Q}_{\mathbb{A}}} \bar{\varphi}(x_{\mathbb{A}}) \psi_{\mathbb{A}}(x_{\mathbb{A}}) dx_{\mathbb{A}}, \tag{4.19}$$

so that $\|\varphi_{\mathbb{A}}\|_2^2 = \langle \varphi_{\mathbb{A}}, \varphi_{\mathbb{A}} \rangle$, and $L^2(\mathbb{Q}_{\mathbb{A}})$ is characterized, as always, by the rule that $\varphi_{\mathbb{A}} \in L^2(\mathbb{Q}_{\mathbb{A}})$ if and only if $\|\varphi_{\mathbb{A}}\|_2 < \infty$.

4.4. Quantum mechanics over $\mathbb{Q}_p$ and $\mathbb{Q}_{\mathbb{A}}$

At this point in the discussion we are evidently in a position (or very nearly so) to mimic in the indicated local and adèlic contexts what amounts to von Neumann's functional analytic rendering of the quantum mechanics of, primarily, Heisenberg and Schrödinger, as presented in the classic monograph [61]. But before we turn to this, a few remarks are in order about something already briefly alluded to earlier (in Section 2.2), namely, the matter of what quantization model is proper for the non-archimedean or adèlic cases. In their article [58], Vladimirov and Volovich note that "standard quantum mechanics starts with a representation of ... the Heisenberg commutation relation $[\hat{q}, \hat{p}] = i$ in ... $L^2(\mathbb{R})$ [where] ... the operators $\hat{q}$ and $\hat{p}$ are realized by multiplication and differentiation respectively. However, in ... p-adic quantum mechanics we have $x \in \mathbb{Q}_p$ and $\psi(x) \in \mathbb{C}$, and therefore the operator $\psi(x) \mapsto x\psi(x)$ of multiplication by x has no meaning. Fortunately ... there is the Weyl representation ...". Thus, as we have already indicated above in a slightly different connection, the proper perspective on the entire affair of developing adèlic quantum mechanics also centers on Weyl quantization in a very natural fashion given that adèles' coordinates are largely p-adic.

In Dragovich's presentation, following Weyl's lead in that the work is done by 1-parameter groups of unitary operators on a Hilbert space of states, the local spaces $L^2(\mathbb{R})$ and $L^2(\mathbb{Q}_p)$, $\forall p$, are assigned evolution operators $U_\infty(t_\infty)$ and $U_p(t_p)$, $\forall p$, respectively, and one is immediately able to impart physically meaningful eigenbases to $L^2(\mathbb{R})$ and the $L^2(\mathbb{Q}_p)$, taking quantum mechanical requirements into account. Specifically, with the same provisos in place as above, one sets

$$U_{\mathbb{A}}(t_{\mathbb{A}}) := U_\infty(t_\infty)\prod_p U_p(t_p), \tag{4.20}$$

so that eigenbases for $L^2(\mathbb{R})$ and the $L^2(\mathbb{Q}_p)$ relative to the operators U_∞ and the U_p give rise to an eigenbasis for $L^2(\mathbb{Q}_{\mathbb{A}})$ relative to $U_{\mathbb{A}}(t_{\mathbb{A}})$. One observes, with Dragovich, that in this way membership in the various local as well as adèlic Schwartz–Bruhat classes is automatically taken care of.

Next, before we get to Weyl quantization proper, we lay out a few standard conventions concerning what, in contrast to adèlic quantum mechanics, we might call local quantum mechanics. In the paradigmatic real case we do have at our disposal the non-relativistic classical Hamiltonian

$$H = \frac{1}{2m}\kappa^2 + \frac{m\omega^2}{2}q^2, \tag{4.21}$$

in the notation of [17] (which we continue to follow as closely as possible), with q being position, κ momentum (seeing that p and k, which are Dragovich's choices, are already spoken for), m being mass, and ω the frequency for the harmonic oscillator. Let $z = \binom{q}{\kappa}$, let $T_t = \begin{pmatrix} \cos\omega t & \frac{1}{m\omega}\sin\omega t \\ -m\omega\sin\omega t & \cos\omega t \end{pmatrix}$. One obtains immediately that $T_t T_{t'} = T_{t+t'}$, and, if B is the (familiar, if repackaged) skew-symmetric bilinear form

$$B(z,z') = B\left(\begin{pmatrix} q \\ \kappa \end{pmatrix}, \begin{pmatrix} q' \\ \kappa' \end{pmatrix}\right) = -\kappa q' + q\kappa' \tag{4.22}$$

on the given phase space, then

$$B(T_t(z), T_t(z')) = B(z,z'), \tag{4.23}$$

and we actually find ourselves on very familiar ground.

We note, also, that in the real setting (and in standard notation) the harmonic oscillator is of course given by the Schrödinger wave equation:

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar}\left(E - \frac{m\omega^2}{2}x^2\right)\psi = 0, \tag{4.24}$$

which, in a normalized form, provides, as far as solutions go, an orthonormal basis $\{\psi_n(x)\}_{n\geq 0}$ of $L^2(\mathbb{R})$ with each basis element being a scalar multiple of the product of the Gaussian density $e^{-\pi x^2}$ and a Hermite polynomial. In this arrangement, which is something of an amalgamation of the Heisenberg and Schrödinger "pictures" of quantum mechanics, as Dirac described these formalisms (cf. [13]), the eigenfunctions $\psi_n(x)$ are indeed the fundamental data of quantization itself, and therefore, given that Weyl quantization must yield a physically equivalent description of the indicated harmonic oscillator, these eigenfunctions have to agree with what one would obtain under this régime.

When it comes to the interpretation of the immediately preceding objects, in the p-adic case one obviously takes the local quantities and variables to be p-adic, noting that regarding p-adic versions of $\cos \omega t$ and $\sin \omega t$ one has to restrict attention to domains cut out by the inequalities $\|\omega t\|_p \leq \frac{1}{p}$ if $p \neq 2$ and $\|\omega t\|_2 \leq \frac{1}{4}$.

Now we proceed to Weyl quantization and sacrifice any putative Hamiltonian formalism. Over $\mathbb{R}$ this entails the data $(L^2(\mathbb{R}), W(z), U(t))$, where W is related to a unitary representation of the Heisenberg(–Weyl) group in $L^2(\mathbb{R})$ and $U(t)$ is, as already indicated, a unitary representation of the attendant evolution operator on the same space. Specifically, if we present the Heisenberg–Weyl group in question as the set of pairs (z, a), with z as above and $a \in \mathbb{R}$ or $\mathbb{Q}_p$, respectively, subject to the group law $(z, a)(z', a') = (z + z', a + a' + \frac{1}{2}B(z, z'))$, then the Weyl representation is given by

$$(z, a) \mapsto \chi(a)W(z), \tag{4.25}$$

where $\chi = \chi_\infty$ or χ_p, respectively.

It is useful for us to take note of the fact that we evidently have a point of close contact here with one of our principal earlier themes, *viz.* the unitary representation theory of the Heisenberg group as discussed in Section 2.3, ff. Indeed, in Dragovich's present notation the Weyl quantization relation is rendered as

$$W(z)W(z') = \chi_\infty\left(\frac{1}{2}B(z, z')\right)W(z + z'), \tag{4.26}$$

in which we recognize nothing less than our earlier (2.12). Additionally, the action of the $W(z)$ on the eigenfunctions $\psi_n(x) \in L^2(\mathbb{R})$ is given by the relation

$$W(z)(\psi_n(x)) = \chi_\infty\left(\frac{\kappa q}{2} + \kappa x\right)\psi_n(x + q). \tag{4.27}$$

Furthermore, regarding the data $U(t)$, which, as we said, is part and parcel of a 1-parameter group of unitary operators (since $U(t)U(t') = U(t + t')$), we have that

$$U(t)(\psi)(x) = \int_{\mathbb{R}} K_t(x, y)\psi(y)dy \tag{4.28}$$

for $\psi \in L^2(\mathbb{R})$ and K_t the usual kernel for the harmonic oscillator:

$$K_t(x, y) = \frac{1}{\sqrt{2}}\{1 - i \cdot sign(2\sin t)\}\frac{e^{2\pi i(\frac{x^2+y^2}{2\tan t} + \frac{xy}{\sin t})}}{\sqrt{|\sin t|}}. \tag{4.29}$$

Moreover,

$$U(t)W(z) = W(T_t(z)), \tag{4.30}$$

in which we might recognize some other telling parallels with what we presented in Section 2.3. We do not pursue this at present, however.

Going on to the p-adic case, and following [17,58] and [59], in this setting the fundamental quantum mechanical data is given by a triple $(L^2(\mathbb{Q}_p), W_p(z_p), U_p(t_p))$, where the obvious changes have been made: in our present non-archimedean functional analytic setting W_p is again connected to a unitary representation of the according p-adic Heisenberg(–Weyl) group, and U_p yields p-adic time evolution. This means that we get explicit relations

$$W_p(z_p)W_p(z'_p) = \chi_p\left(\frac{1}{2}B_p(z_p, z'_p)\right) W_p(z_p + z'_p), \tag{4.31}$$

$$W_p(z)(\psi^{(p)}(x_p)) = \chi_p\left(\frac{\kappa q}{2} + \kappa x_p\right) \psi^{(p)}(x_p + q), \tag{4.32}$$

$$U_p(t_p)(\psi^{(p)}(x_p)) = \int_{\mathbb{Q}_p} K_{t_p}(x_p, y_p)\psi^{(p)}(y_p)dy_p, \tag{4.33}$$

along the same lines as in the real case, including the intertwining of the indicated unitary representation of the p-adic Heisenberg group and the time-evolution operator, namely,

$$U_p(t_p)W_p(z_p) = W_p(T_{t_p}(z_p))U_p(t_p). \tag{4.34}$$

The expression for the p-adic integrating kernel involves some novel maneuvers, but happily its "shape" is amenable to that of (4.28):

$$K_{t_p}(x_p, y_p) = \begin{cases} \delta_p(x_p - y_p), & \text{if } t_p = 0 \\ \lambda_p(2t_p)\|t_p\|_p^{-\frac{1}{2}}\chi_p\left(\dfrac{x_p y_p}{\sin t_p} - \dfrac{x_p^2 + y_p^2}{2\tan t_p}\right), & \text{if } t_p \neq 0 \end{cases} \tag{4.35}$$

for δ_p the p-adic Dirac delta function. Here the function λ_p is given by the following definitions involving the Legendre symbol (a circumstance we will have occasion to return to later):

$$\lambda_p(a_p) = \begin{cases} 1, & \text{if } p \neq 2, \text{ and } 2|k_0 \text{ in (4.11)} \\ \left(\dfrac{\alpha_{k_0}}{p}\right), & \text{if } p \neq 2, 2 \nmid k_0 \text{ in (4.11) and } p \equiv 1 \ (\mathrm{mod}\ .4) \\ i\left(\dfrac{\alpha_{k_0}}{p}\right), & \text{if } p \neq 2, 2 \nmid k_0 \text{ in (4.11) and } p \equiv 3 \ (\mathrm{mod}\ .4) \\ \dfrac{1}{\sqrt{2}}\{1 + (-1)^{\alpha_{k_0+1}} i\}, & \text{if } p = 2 \text{ and } 2|k_0 \text{ in (4.11)} \\ \dfrac{1}{\sqrt{2}}(-1)^{\alpha_{k_0+1}+\alpha_{k_0+2}}\{1 + (-1)^{\alpha_{k_0+1}} i\}, & \text{if } p = 2 \text{ and } 2 \nmid k_0 \text{ in (4.11).} \end{cases} \tag{4.36}$$

Additionally it bears repeating that the time-evolution operators engender a 1-parameter group in both the real and p-adic cases, i.e. in both the archimedean and non-archimedean settings, which is the main thrust of the proposition that

$$U_\infty(t_\infty + t'_\infty) = U_\infty(t_\infty)U_\infty(t'_\infty), \qquad U_p(t_p + t'_p) = U_p(t_p)U_p(t'_p), \tag{4.37}$$

where we have written $U_\infty(t_\infty)$ for our earlier $U(t)$. Furthermore, as regards the integrating kernels (4.29) and (4.35) we have

$$
\begin{aligned}
K_{t_\infty+t'_\infty}(x_\infty, y_\infty) &= \int_{\mathbb{R}} K_{t_\infty}(x_\infty, z_\infty) K_{t'_\infty}(z_\infty, y_\infty) dy_\infty, \\
K_{t_\infty+t'_\infty}(x_p, y_p) &= \int_{\mathbb{Q}_p} K_{t_\infty}(x_p, z_p) K_{t'_\infty}(z_p, y_p) dy_p
\end{aligned}
\tag{4.38}
$$

where we have taken care to place the individual valuations on the same notational footing in view of the upcoming task of adèlizing much of this data.

Thus, with local Weyl quantization given as above, as the archimedean data $(L^2(\mathbb{R}), W_\infty(z_\infty), U_\infty(t_\infty))$ and the non-archimedean data $(L^2(\mathbb{Q}_p), W_p(z_p), U_p(t_p))$ for all p, where

$$
(z_\infty, a_\infty) \longmapsto \chi_\infty(a_\infty) W_\infty(z_\infty); \qquad (z_p, a_p) \longmapsto \chi_p(a_p) W_p(z_p), \quad \forall p, \tag{4.39}
$$

provide the various local Weyl representations, and

$$
\begin{aligned}
U_\infty(t_\infty)(\psi)(x_\infty) &= \int_{\mathbb{R}} K_{t_\infty}(x_\infty, y_\infty)\psi(y_\infty) dy_\infty; \\
U_p(t_p)(\psi)(x_p) &= \int_{\mathbb{Q}_p} K_{t_p}(x_p, y_p)\psi(y_p) dy_p, \quad \forall p,
\end{aligned}
\tag{4.40}
$$

define the indicated 1-parameter groups of unitary operators characterizing, respectively, archimedean and non-archimedean time-evolution, with $\psi \in L^2(\mathbb{R})$ or $L^2(\mathbb{Q}_p)$, we can see our way clear to what needs to be done in formulating the required results over $k_\mathfrak{p}$, with $\mathfrak{p}$ running over $\mathfrak{V}_k$, i.e., to develop the local quantum mechanics $(L^2(k_v), W_v(z_v), U_v(t_v))$. We address this in Section 4.5.

Next we turn to the adèlic triple $(L^2(\mathbb{Q}_\mathbb{A}), W_\mathbb{A}(z_\mathbb{A}), U_\mathbb{A}(t_\mathbb{A}))$, where $L^2(\mathbb{Q}_\mathbb{A})$ is given by (4.18) and $U_\mathbb{A}(t_\mathbb{A})$ by (4.20). Following suit, we have that

$$
W_\mathbb{A}(z_\mathbb{A}) = W_\mathbb{A}((z_\infty; z_p)_p) = W_\infty(z_\infty) \prod_p W_p(z_p), \tag{4.41}
$$

and, for the adèlic Weyl representation,

$$
\begin{aligned}
&(z_\mathbb{A}, a_\mathbb{A}) = ((z_\infty; z_p)_p, (a_\infty; a_p)_p) \\
&\quad \longmapsto \chi_\infty(a_\infty) W_\infty(z_\infty) \prod_p \chi_p(a_p) W_p(z_p),
\end{aligned}
\tag{4.42}
$$

while we have as the adèlic counterpart to (4.26) and (4.31) the identity

$$
W_\mathbb{A}(z_\mathbb{A}) W_\mathbb{A}(z'_\mathbb{A}) = \chi_\mathbb{A}\left(\frac{1}{2} B(z_\mathbb{A}, z'_\mathbb{A})\right) W_\mathbb{A}(z_\mathbb{A} + z'_\mathbb{A}). \tag{4.43}
$$

Regarding (4.27) and (4.32), the adèlic counterpart reads

$$
W_\mathbb{A}(z_\mathbb{A})(\psi_\mathbb{A}(x_\mathbb{A})) = \chi_\mathbb{A}\left(\frac{\kappa_\mathbb{A} q_\mathbb{A}}{2} + \kappa_\mathbb{A} x_\mathbb{A}\right) \psi_\mathbb{A}(x_\mathbb{A} + q_\mathbb{A}), \tag{4.44}
$$

and the local data (4.28), (4.33) leads to

$$U_{\mathbb{A}}(t_{\mathbb{A}})(\psi_{\mathbb{A}}(x_{\mathbb{A}})) = \int_{\mathbb{Q}_{\mathbb{A}}} K_{t_{\mathbb{A}}}(x_{\mathbb{A}}, y_{\mathbb{A}})\psi_{\mathbb{A}}(y_{\mathbb{A}})dy_{\mathbb{A}}, \tag{4.45}$$

with $\psi_{\mathbb{A}} \in L^2(\mathbb{Q}_{\mathbb{A}})$. Naturally,

$$K_{t_{\mathbb{A}}}(x_{\mathbb{A}}, y_{\mathbb{A}}) = K_{t_\infty}(x_\infty, y_\infty)\prod_p K_{t_p}(x_p, y_p) \tag{4.46}$$

and, as already mentioned above,

$$U_{\mathbb{A}}(t_{\mathbb{A}}) = U_\infty(t_\infty)\prod_p U_p(t_p).$$

In view of convergence questions it should be noted that these formulas have to be interpreted in the sense of distributions. Additionally, with the local data given by (4.38) in place, it should be the case that

$$K_{t_{\mathbb{A}}+t'_{\mathbb{A}}}(x_{\mathbb{A}}, y_{\mathbb{A}}) = \int_{\mathbb{Q}_{\mathbb{A}}} K_{t_{\mathbb{A}}}(x_{\mathbb{A}}, z_{\mathbb{A}})K_{t'_{\mathbb{A}}}(z_{\mathbb{A}}, y_{\mathbb{A}})dz_{\mathbb{A}}, \tag{4.47}$$

and this is indeed so, provided the right side is not regarded naïvely as a product of the respective right sides of (4.38). Dragovich notes simply (p. 14 of [17]) that this "would be inconsistent with the adèlic approach", and the upshot is that one must require that, due to the fact that for any adèle its coordinates live in a $\mathbb{Z}_p$ for all but a finitely many p, and accordingly have p-adic norm ≤ 1, the corresponding non-archimedean local contributions to (4.47) reduce to

$$\int_{\|z_p\|_p\leq 1} K_{t_p}(x_p, z_p)K_{t'_p}(z_p, y_p)dz_p = K_{t_p+t'_p}(x_p, y_p). \tag{4.48}$$

This maneuver is standard; see e.g. [54,64].

Presently we will require a path integral formulation of quantum mechanics in relation to local and adèlic versions of (3.37) for the number field k, as we have already abundantly indicated above.

If the very subject of non-archimedean quantum mechanics is novel, the sub-case of adèlic quantum mechanics is even more so, and it is reasonable to say a few words about it now. Indeed, we take the liberty of developing quantum mechanics over $\mathbb{Q}_{\mathbb{A}}$ at this stage, to the point of being able to formulate what should subsequently occur for $k_{\mathbb{A}}$. According to Section 4.1, phase integrals, which unfortunately occur in the aforementioned results in an obviously rather involved form, are fundamentally tied to the time evolution of the quantum mechanical systems they purport to describe, so it is very important to do justice to the operators $U_{\mathbb{A}}(t_{\mathbb{A}})$. In the present context of the adèlic harmonic oscillator, the stipulation concerning adèlic time is that we should have $t_\infty \in \mathbb{R}$, $\|t_2\|_2 \leq \frac{1}{4}$, and for $p > 2$, $\|t_p\|_p \leq \frac{1}{p}$. In keeping with what happens locally, this collective data cuts out an additive subgroup of $\mathbb{Q}_{\mathbb{A}}$. Moreover, we will have occasion later to examine the adèlic counterparts to (3.37) from the perspective of the interplay between adèlic generalizations of the operators $U(t, t_0; H)$ occurring there, these being the putative instances of the present $U_\infty(t_\infty)$, and the corresponding players contributing to the development of the Weil–Kubota 2-cocycles $c_\mathfrak{p}$ and $c_\mathbb{A}$. As we have taken pains to disclose in Section 2, the local 2-cocycles' behavior is intimately connected to the representation-theoretic dividends derived from the Stone–von Neumann theorem, and it is therefore proper to note (also for completeness) that the data (4.30), (4.34) adèlizes to the relation

$$U_\mathbb{A}(t_\mathbb{A})W_\mathbb{A}(z_\mathbb{A}) = W_\mathbb{A}(T_{t_\mathbb{A}}(z_\mathbb{A}))U_\mathbb{A}(t_\mathbb{A}). \tag{4.49}$$

We can complete the description of adèlic quantum mechanics in this exemplar by mentioning that, to be sure, the eigenfunctions of the adèlic harmonic oscillator satisfy the right equations when compared with what happens in the real case (*viz.* (4.21): the adèlization takes place in the standard manner). As these eigenfunctions' appearance in the non-archimedean case is rather involved and we have no need for them in our present considerations, we take the liberty merely to cite their occurrence in [17], Sections 3, 4, covering respectively the local and adèlic cases. The main point is that, also adèlically, the time evolution operators, which are of course at the heart of Weyl quantization (where they are defined by means of integrating kernels as per (4.28), (4.33), and (4.45)), should be compatible with the eigenvalue problem for the Schrödinger wave equation, when the latter makes sense. Dragovich takes great pains in [17] to demonstrate that this is truly the case.

The upshot is that we indeed have in the foregoing a prototype for adèlic quantum mechanics, specifically the adèlization of local quantum mechanics in the presentation of Hermann Weyl (cf. [65]), with the archimedean case corresponding to what is currently accepted in physics as descriptive of physical reality at the Planck scale. Procedurally, as we have just seen, one starts with $\mathbb{R}$ as only one option for completing $\mathbb{Q}$ (where to be sure actual physical measurements take their values) relative to a valuation, and stipulates that the non-archimedean valuations should be given equal time, and then one develops local quantum mechanics for all valuations of $\mathbb{Q}$. Thereafter one systematically adèlizes the given data to get quantum mechanics over $\mathbb{Q}_\mathbb{A}$. As far as our objectives go, then, the next order of business is dictated by what we have done in Sections 2 and 3: we proceed, as promised, to delineate a generalization of quantum mechanics to $k_\mathbb{A}$, where, as above, k is an algebraic number field, i.e. a finite extension of $\mathbb{Q}$.

In any event, under the present circumstances, we have, at least in principle, sufficient material in place with which to effect an extension of (3.37). First, given that we are now working with Weyl quantization and without a Hamiltonian as such (Vladimirov, Volovich, and Zelenov do use *ersatz* non-archimedean Hamiltonians as "heuristics"— cf. p. 208 of [59]), we recast (3.37) as

$$\begin{aligned} &c_{\infty_\mathbb{R}}(\sigma,\sigma')\mathcal{U}(t_{\infty_\mathbb{R}})^{0,\sigma}(f(\mathbf{x}))\mathcal{U}(t_{\infty_\mathbb{R}})^{\sigma,\sigma'\sigma}(g(\mathbf{y}))\mathcal{U}(t_{\infty_\mathbb{R}})^{\sigma'\sigma,0}(h(\mathbf{z})) \\ &\quad = \mathcal{F}_{\infty_\mathbb{R}}(c_{\infty_\mathbb{R}}(\sigma,\sigma'))\int_{\mathbb{R}^{3n}} e^{i\{S_{\infty_\mathbb{R}}(\mathbf{x},\mathbf{x}')+S_{\infty_\mathbb{R}}(\mathbf{y},\mathbf{y}')+S_{\infty_\mathbb{R}}(\mathbf{z},\mathbf{z}')\}} \\ &\qquad \times f(\mathbf{x}')g(\mathbf{y}')h(\mathbf{z}')d\mathbf{x}'d\mathbf{y}'d\mathbf{z}', \end{aligned} \tag{4.50}$$

also changing the notation for the time-dependency of the given unitary operators in the obvious way to get a match with the according notations in the present section. Now

p-adicization entails the stipulation

$$c_p(\sigma,\sigma')\mathcal{U}(t_p)^{0,\sigma}(f(\mathbf{x}))\mathcal{U}(t_p)^{\sigma,\sigma'\sigma}(g(\mathbf{y}))\mathcal{U}(t_p)^{\sigma'\sigma,0}(h(\mathbf{z}))$$
$$= \mathcal{F}_p(c_p(\sigma,\sigma'))\int_{\mathbb{Q}_p^{3n}} e^{i\{S_p(\mathbf{x},\mathbf{x}')+S_p(\mathbf{y},\mathbf{y}')+S_p(\mathbf{z},\mathbf{z}')\}} f(\mathbf{x}')g(\mathbf{y}')h(\mathbf{z}')d\mathbf{x}'d\mathbf{y}'d\mathbf{z}' \quad (4.51)$$

for all $\sigma,\sigma' \in Sp(2n,\mathbb{Q}_p)$, and $\mathbb{Q}$-adèlization entails the stipulation that

$$c_{\mathbb{A}}(\sigma,\sigma')\mathcal{U}(t_{\mathbb{A}})^{0,\sigma}(f(\mathbf{x}))\mathcal{U}(t_{\mathbb{A}})^{\sigma,\sigma'\sigma}(g(\mathbf{y}))\mathcal{U}(t_{\mathbb{A}})^{\sigma'\sigma,0}(h(\mathbf{z}))$$
$$= \mathcal{F}_{\mathbb{A}}(c_{\mathbb{A}}(\sigma,\sigma'))\int_{\mathbb{Q}_{\mathbb{A}}^{3n}} e^{i\{S_{\mathbb{A}}(\mathbf{x},\mathbf{x}')+S_{\mathbb{A}}(\mathbf{y},\mathbf{y}')+S_{\mathbb{A}}(\mathbf{z},\mathbf{z}')\}} f(\mathbf{x}')g(\mathbf{y}')h(\mathbf{z}')d\mathbf{x}'d\mathbf{y}'d\mathbf{z}' \quad (4.52)$$

for all $\sigma,\sigma' \in Sp(2n,\mathbb{Q}_{\mathbb{A}})$. Of course in each of the two immediately preceding cases it is understood that the variables **x, y, z,** and $\mathbf{x}'$, $\mathbf{y}'$, $\mathbf{z}'$, as well as the functions f, g, and h, are to be regarded as, respectively, p-adic and k-adèlic objects, with the earlier adèlization conventions and formulas in place, for example, $\mathbf{x}_{\mathbb{A}} = (x_\infty; x_p)_p$, $dx_{\mathbb{A}} = dx_\infty dx_2 dx_3 \cdots dx_p \cdots$, and similarly for **y, z,** $\mathbf{x}'$, $\mathbf{y}'$, $\mathbf{z}'$, and $t_{\mathbb{A}}$, and then $U_{\mathbb{A}}(t_{\mathbb{A}}) = U_\infty(t_\infty)\prod_p U_p(t_p)$, with similar product formulas defining all the other players in (4.52), specifically $c_{\mathbb{A}}$, $\mathcal{F}_{\mathbb{A}}$, and the $S_{\mathbb{A}}$. There is indeed a lot of preparation hiding in the shadows, largely topological and measure-theoretic: see the preceding section, as well as earlier remarks in the present section.

4.5. Quantum mechanics over k_ν and $k_{\mathbb{A}}$

Let $(k:\mathbb{Q}) = d = r_1 + 2r_2$, meaning that we have r_1 real $\mathbb{Q}$-embeddings of k in $\mathbb{C}$, and $2r_2$ complex ones, arranged in complex conjugate pairs. Thus, there are correspondingly r_1 real valuations on k, $2r_2$ complex ones, exhausting the set of archimedean places of k, and as far as non-archimedean places go we obviously have a bijection between these and the primes (or prime ideals) $\mathfrak{p}$ of O_k. The adèle ring $k_{\mathbb{A}}$, of k, featured so prominently in the earlier sections of this paper, is realized as a restricted direct product of the local fields obtained by Cauchy-completing k with respect to these valuations, taken with respect to the local rings $O_{\mathfrak{p}} \subset k_{\mathfrak{p}}$ for $\mathfrak{p} < \infty$. This amounts to the following: continuing to write $\mathfrak{V}_k$ for the set of all valuations of k, but now writing $\nu \in \mathfrak{V}_k$, archimedean or not, we have r_1 (resp. $2r_2$) instances of $\nu = \infty_{\mathbb{R}}$ (resp. $\nu \in \infty_{\mathbb{C}}$) and a countable infinity of instances where $\nu = \mathfrak{p}$; consequently a k-adèle, i.e. an element of $k_{\mathbb{A}}$, looks like

$$a_{\mathbb{A}} = (a_{\infty_{\mathbb{R}}}; a_{\infty_{\mathbb{C}}}; a_{\mathfrak{p}})_{\mathfrak{p}} = (a_\nu)_{\nu\in\mathfrak{V}_k} \quad (4.53)$$

subject to the requirements that there are r_1 (resp. $2r_2$) real entries (resp. complex entries) in the subvector $a_{\infty_{\mathbb{R}}}$ (resp. $a_{\infty_{\mathbb{C}}}$), and $\forall\mathfrak{p}$, $a_{\mathfrak{p}} \in k_{\mathfrak{p}}$, with $a_{\mathfrak{p}} \in O_{\mathfrak{p}}$ a.e. $\mathfrak{p}$. Again, one cuts out the multiplicative group of k-idèles and, inside this set, the principal k-idèles, in the obvious manner.

Furthermore,

$$\|a_{\mathbb{A}}\| = \prod_{\nu\in\mathfrak{V}_k} \|a_\nu\|_\nu, \quad (4.54)$$

but it is important to note that if ν is complex then $\|a_\nu\|_\nu = |a_\nu|^2 = a_\nu\overline{a_\nu}$, and for $\mathfrak{p} < \infty$ we have that $\|a_\mathfrak{p}\|_\mathfrak{p} = \{N_{k/\mathbb{Q}}(\mathfrak{p})\}^{-\varepsilon}$, where $a_\mathfrak{p} \in \mathfrak{p}^\varepsilon O_\mathfrak{p}$ but $a_\mathfrak{p} \notin \mathfrak{p}^{\varepsilon+1} O_\mathfrak{p}$.

Next, with $x_\mathbb{A} = (x_\nu)_\nu$ (in keeping with (4.53)), stipulate that each $dx_{\infty_\mathbb{R}}$ is Lebesgue measure on $\mathbb{R}$, each $dx_{\infty_\mathbb{C}}$ is twice Lebesgue measure, and $dx_\mathfrak{p}$ is normalized Haar measure on the locally compact abelian group $k_\mathfrak{p}$ such that

$$\int_{O_\mathfrak{p}} dx_\mathfrak{p} = \frac{1}{\sqrt{N_{k_\mathfrak{p}/\mathbb{Q}_p}(\mathfrak{d}_\mathfrak{p})}}, \tag{4.55}$$

with $\mathfrak{d}_\mathfrak{p}$ being the discriminant of $k_\mathfrak{p}$ and $p \in \mathbb{Z}$ with $\mathfrak{p}|p$. Then Haar measure on the topological ring $k_\mathbb{A}$ is given by

$$dx_\mathbb{A} = \prod_{\nu \in \mathfrak{V}_k} dx_\nu = \prod dx_{\infty_\mathbb{R}} \prod dx_{\infty_\mathbb{C}} \prod dx_\mathfrak{p}. \tag{4.56}$$

Additionally, we obtain Haar measure on $k_\mathbb{A}^\times$, the multiplicative group of k-idèles, by setting $dx_\infty^\times = \frac{dx_\infty}{\|x_\infty\|}$ in both the real and complex cases,

$$dx_\mathfrak{p}^\times = \frac{1}{1 - \frac{1}{N_{k/\mathbb{Q}}(\mathfrak{p})}} \cdot \frac{dx_\mathfrak{p}}{\|x_\mathfrak{p}\|} \tag{4.57}$$

in the non-archimedean cases, and then again defining

$$dx_\mathbb{A}^\times = \prod_{\nu \in \mathfrak{V}_k} dx_\nu^\times. \tag{4.58}$$

As far as characters are concerned, first, for each non-archimedean $k_\mathfrak{p}$ pick $p \in \mathbb{Z}$ such that $\mathfrak{p}|p$ and determine, to start with, $\lambda_p : \mathbb{Q}_p \to \mathbb{R}$ by the recipe, for $x_p \in \mathbb{Q}_p$, pick $a, n \in \mathbb{Z}$ such that $p^a x_p - n \in p^a\mathbb{Z}$, and set $\lambda_p(x_p) = \frac{n}{p^a}$; subsequently define $\Lambda_\mathfrak{p}(x_\mathfrak{p}) = \lambda_p(Tr_{k_p/\mathbb{Q}_p}(x_\mathfrak{p}))$ for all $x_\mathfrak{p} \in k_\mathfrak{p}$. With these conventions in place, define

$$\chi_\mathbb{A}((x_\nu)_\nu) = \prod_{\nu|\infty} \chi_\nu(x_\nu) \prod_{\mathfrak{p}<\infty} e^{2\pi i \Lambda_\mathfrak{p}(x_\mathfrak{p})}, \tag{4.59}$$

taking care of the additive case.

Turning to the multiplicative case, recall from Tate's thesis (specifically § 2.3 of [54]) that for any valuation $\nu \in \mathfrak{V}_k$ any continuous multiplicative mapping $k_\nu \to \mathbb{C}$ is, by definition, a quasi-character, while the subclass of such mappings that map into $\mathbb{C}_1^\times$ are the conventional characters. Quasi-characters of k are unramified provided they are of the form $x_\nu \to \|x_\nu\|_\nu^s$ for some fixed $s \in \mathbb{C}$. Generally, which is to say, including the ramified case in our discussion, a ν-adic (multiplicative) quasi-character is a mapping $k_\nu^\times \to \mathbb{C}$ of the form $x_\nu \longmapsto \|x_\nu\|_\nu^s c_\nu(x_\nu')$ for c_ν a character of the kernel U_ν of the mapping $x_\nu \longmapsto \|x_\nu\|_\nu$ and x_ν' given by the rule $x_\nu = x_\nu'\varrho$, $x_\nu' \in U_\nu$, $\varrho > 0$, if $\nu|\infty$, and by the rule $x_\nu = x_\nu' r^\varepsilon$ where, again, $x_\nu' \in U_\nu$, but now $\|r\|_\nu = 1$ (and ε is of no consequence for our purposes), if $\nu < \infty$. With the multiplicative characters taken care of in this manner, we may forego any real coverage of idèlic characters *per se* given that we have no need for them in regard to our goal of k-adèlizing the counterparts to (4.50) and (4.51). And this finally brings us back to our question of what a generalized quantum mechanics of this type should look like: precisely what does it consist in?

In point of fact, the answer to this question has already been foreshadowed a number of times in the foregoing discussion: in a non-archimedean setting, including an adèlic one, the right move is to postulate a quantum mechanics as a triple of the form "(Hilbert (phase) space; Weyl quantization (a family of unitary operators); time-evolution operator)", as was done in the preceding section for the paradigm of $k = \mathbb{Q}$. We are therefore led to our k-adèlic quantum mechanics as the data $(L^2(k_{\mathbb{A}}), W_{\mathbb{A}}(z_{\mathbb{A}}), U_{\mathbb{A}}(t_{\mathbb{A}}))$, modulo pending generalizations of (4.41), (4.45) and (4.47). Beyond this, with our goal being to bring a path integral formalism into play, it is important to note that it is the evolution operator that readily introduces the desired integrals into the game.

Specifically, reverting briefly to the case of $\mathbb{Q}_{\mathbb{A}}$, we have from (4.47) and (4.33) that $U_{\mathbb{A}} = U_\infty \prod_p U_p$ where, suppressing a number of p's,

$$U_p(t)\psi(x) = \int_{\mathbb{Q}_p} K_t(x, y)\psi(y)dy \tag{4.60}$$

and, additionally, in the current more general context than that of the earlier harmonic oscillator, it is indicated that we present $K_t(x, y)$ as

$$K_t(x, y) = \int_{\mathsf{P}} \chi\left(\frac{1}{h}\int_0^t L(q, \dot{q})dt\right)\prod_t dq(t), \tag{4.61}$$

following p. 207 of [59]. Here, $L(q, \dot{q})$ is the attendant Lagrangian, and in accord with [58], $h \in \mathbb{Q}$, while $q, t \in \mathbb{Q}_p$; additionally, and perhaps most significantly, P is the set of classical p-adic trajectories going between $x = q(1)$ and $y = q(0)$, noting that $0 \leq t \leq 1$, and it is here that we expressly encounter a Feynman integral properly so-called.

If we compare this presentation of the integrating kernel to how things are done in quantum field theory, taking note of the fact (cf. [15]) that quantum mechanics can be realized as a so-called $(0 + 1)$-QFT, it is clear that (4.61) can be recast in the evocative form

$$K_t(x, y) = \int_{\mathsf{P}} e^{\frac{i}{h}S(\gamma(t))}D[\gamma(t)], \tag{4.62}$$

where

$$S(\gamma(t)) = \int_0^1 L(\gamma(t), \dot{\gamma}(t))dt \tag{4.63}$$

realizes the critically important action functional for the system. Here $\gamma(t)$ is a path from x to y, i.e. an element of P, and $D[\gamma(t)]$ is the much-embattled phantom measure involved in Feynman's path integral formulation of quantum mechanics. With this presentation of K_t in place, however, we need only compare what we have here with our earlier (4.4) and (4.5) to see that we are on the threshold of connecting the developments of the first parts of this paper to Feynman's formulation.

So, at this point there are two specific tasks before us, namely to explicate the data $(L^2(k_v), W_v(z_v), U_v(t_v))$ for all $v \in \mathfrak{V}_k$ (resp. $(L^2(k_{\mathbb{A}}), W_{\mathbb{A}}(z_{\mathbb{A}}), U_{\mathbb{A}}(t_{\mathbb{A}}))$) by generalizing

(4.41), (4.42) and (4.47) to the case where k_ν is the base field (resp. $k_{\mathbb{A}}$ is the base ring (or topological space)) and to carry this extension over so as to yield counterparts to (4.60) and therefore to (4.58) with k_ν taking the place of $\mathbb{Q}_p$ (resp. $k_{\mathbb{A}}$ taking the place of $\mathbb{Q}_{\mathbb{A}}$). Thereafter what remains is the task of tying these generalizations into (4.4), (4.5), and of course (3.37).

As we now set out to do this we should like to warn that, specifically in the local (as opposed to adèlic) situation, we leave a few things undone, given that our coverage is already rather baroque. In point of fact, concerning the local integrating kernels K_t as per (4.35) and (4.36), it is clear already from the example given by Dragovich regarding what happens in the relatively elementary case of the simple harmonic oscillator that explicit renderings of these expressions are hard to come by. Interestingly, as they involve the Legendre symbol when working over $\mathbb{Q}_p$ (cf. (4.36)), one expects to encounter the extended Legendre symbol, defined over the ring of integers of k_ν, in our more general case. We leave this as an exercise however, and stipulate that such integrating kernels are available over k_ν, and just note that we can render them in a form amenable to the present non-archimedean path integral formalism conveyed by (4.62) and (4.63). After all, even in classical quantum mechanics this is the case, lest Feynman's entire approach were compromised: an integrating kernel is always given in the form of (4.4).

Now, turning our attention specifically to our central result (3.37) and its proposed appearance(s) when we go to the arbitrary number field k and effect, first, $\mathfrak{p}$- or more precisely ν-adicization (with ν running over $\mathfrak{V}_k$) and, finally, k-adèlization, we evidently get very similar results to (4.51) and (4.52). In the first place, locally we obtain

$$\begin{aligned} &c_\nu(\sigma,\sigma')\mathcal{U}(t_\nu)^{0,\sigma}(f(\mathbf{x}))\mathcal{U}(t_\nu)^{\sigma,\sigma'\sigma}(g(\mathbf{y}))\mathcal{U}(t_\nu)^{\sigma'\sigma,0}(h(\mathbf{z}))\\ &\quad= \mathcal{F}_\nu(c_\nu(\sigma,\sigma'))\int_{k_\nu^{3n}} e^{i\{S_\nu(\mathbf{x},\mathbf{x}')+S_\nu(\mathbf{y},\mathbf{y}')+S_\nu(\mathbf{z},\mathbf{z}')\}} f(\mathbf{x}')g(\mathbf{y}')h(\mathbf{z}')d\mathbf{x}'d\mathbf{y}'d\mathbf{z}' \end{aligned} \tag{4.64}$$

where $\sigma,\sigma' \in Sp(2n,k_\nu)$, and the indicated independent variables (or integration variables) range over k_ν (as a local field equipped with a Haar measure—see above). Subsequently, we get, k-adèlically,

$$\begin{aligned} &c_{\mathbb{A}}(\sigma,\sigma')\mathcal{U}(t_{\mathbb{A}})^{0,\sigma}(f(\mathbf{x}))\mathcal{U}(t_\nu)^{\sigma,\sigma'\sigma}(g(\mathbf{y}))\mathcal{U}(t_{\mathbb{A}})^{\sigma'\sigma,0}(h(\mathbf{z}))\\ &\quad= \mathcal{F}_{\mathbb{A}}(c_{\mathbb{A}}(\sigma,\sigma'))\int_{k_{\mathbb{A}}^{3n}} e^{i\{S_{\mathbb{A}}(\mathbf{x},\mathbf{x}')+S_{\mathbb{A}}(\mathbf{y},\mathbf{y}')+S_{\mathbb{A}}(\mathbf{z},\mathbf{z}')\}} f(\mathbf{x}')g(\mathbf{y}')h(\mathbf{z}')d\mathbf{x}'d\mathbf{y}'d\mathbf{z}', \end{aligned} \tag{4.65}$$

where we have that $\sigma,\sigma' \in Sp(2n,k_{\mathbb{A}})$ and the integration variables are adèlic, ranging over the topological ring $k_{\mathbb{A}}$, equipped with an adèlic Haar measure (as per (4.6)).

4.6. One more *caveat* regarding adèlization

There are a few final remarks in order as regards the processes of adèlization carried out at the end of Section 4.4, culminating in (4.52), and at the end of Section 4.5, culminating in (4.65). In each of these two cases we set about adèlizing local data of the form $c_? = \mathcal{F}_? \int \chi_? f_?$ where *ab initio* we can replace ? consistently by some $\nu \in \mathfrak{V}_k$, starting off in Section 4.4 with $k = \mathbb{Q}$ In the paradigm case of $\nu = \infty_{\mathbb{R}}$, with which the entire first part of this article is of course concerned, we are guaranteed that (3.37) holds and that the left hand side data, $c_?$, or more properly $\{c_\nu\}_{\nu \in \mathfrak{V}_\mathbb{Q}}$, adèlizes into $c_\mathbb{A}$, which is nothing more than the product, properly understood, of all the c_ν. We have argued in Section 4.4 that the right hand side data should adèlize in itself, given that we have compatible non-archimedean quantum mechanics in place, including both the local and adèlic cases. Thus, it is certainly possible to build an adèlic right hand side, which we might for the sake of the present discussion denote most succinctly by $(\mathcal{F} \int \chi f)_\mathbb{A}$. The question that needs to be addressed is how we can justify that $c_\mathbb{A}$ and $(\mathcal{F} \int \chi f)_\mathbb{A}$ coincide. The same construction applies in the more general case, for an arbitrary k, of course, and the same question is raised.

In sketching a resolution, we are guided by something of a parallel with what takes place in the theory of sheaves, i.e. the procedure of sheafification. The idea is that at each intermediate stage, so to speak, i.e. for ? replaced by each $\nu \in \mathfrak{V}_k$, we have at least in principle (given the present anything-but-explicit status of non-archimedean quantum mechanics over k) a sensible identification $c_\nu = \mathcal{F}_\nu \int \chi_\nu f_\nu$, and this carries with it that the requirements that are already in place to guarantee that $c_\mathbb{A}$ is well-defined apply also to the data $\{\mathcal{F}_\nu \int \chi_\nu f_\nu\}_{\nu \in \mathfrak{V}_k}$, so that we certainly get that $c_\mathbb{A} = \prod_\nu c_\nu = \prod_\nu \mathcal{F}_\nu \int \chi_\nu f_\nu$.

But it is not transparent that $\prod_\nu \mathcal{F}_\nu \int \chi_\nu f_\nu$ and $(\mathcal{F} \int \chi f)_\mathbb{A}$ coincide. In order to decide whether in fact they do, which would then completely legitimize (4.52) and (4.65), it is necessary to carry out the detailed work of explicitly constructing the local factors going into the $\mathcal{F}_\nu$, χ_ν, and f_ν, with $\nu \in \mathfrak{V}_k$. This would take us quickly to the heavy labor of generalizing certain of the earlier results by Souriau (Section 3.4) and even Leray, and crafting explicit versions of k-local and k-adèlic quantum mechanics: tasks of such great scope that we postpone them until later publications.

For now we present (4.52) and (4.65) in the indicated form, modulo future amendments which, conjecturally, we claim should at worst amount to the introduction of compatible adjustment factors. In other words, we propose that $\prod_\nu \mathcal{F}_\nu \int \chi_\nu f_\nu$ and $(\mathcal{F} \int \chi f)_\mathbb{A}$ differ only by an innocuous product of local factors, meaning that indeed, suitably understood, we can write $c_\mathbb{A} = (\mathcal{F} \int \chi f)_\mathbb{A}$.

5. Quadratic reciprocity revisited

5.1. Quadratic reciprocity in the language of Feynman integrals

We saw in Section 2.9 that the law of quadratic reciprocity for the algebraic number field k redounds to (2.130), i.e. the statement that

$$c_\mathbb{A}|_{Sp(k)\times Sp(k)} \equiv 1,$$

and we have from (4.65) that with $(\sigma, \sigma') \in Sp(2, k_\mathbb{A}) \times Sp(2, k_\mathbb{A})$, which is to say that we set $n = 1$,

$$\begin{aligned} &c_\mathbb{A}(\sigma, \sigma')\mathcal{U}(t_\mathbb{A})^{0,\sigma}(f(\mathbf{x}))\mathcal{U}(t_\mathbb{A})^{\sigma,\sigma'\sigma}(g(\mathbf{y}))\mathcal{U}(t_\mathbb{A})^{\sigma'\sigma,0}(h(\mathbf{z})) \\ &\quad = \mathcal{F}_\mathbb{A}(c_\mathbb{A}(\sigma, \sigma')) \int_{k_\mathbb{A}^3} e^{i\{S_\mathbb{A}(\mathbf{x},\mathbf{x}')+S_\mathbb{A}(\mathbf{y},\mathbf{y}')+S_\mathbb{A}(\mathbf{z},\mathbf{z}')\}} f(\mathbf{x}')g(\mathbf{y}')h(\mathbf{z}')d\mathbf{x}'d\mathbf{y}'d\mathbf{z}'. \end{aligned}$$

Additionally, we note that $Sp(k) = Sp(2, k) < Sp(2, k_\mathbb{A})$. And these are all the ingredients we require for the culmination:

Theorem 5.1. *The law of quadratic reciprocity for the number field k is equivalent to the fact that for all $(\sigma, \sigma') \in Sp(2, k) \times Sp(2, k)$ we have that*

$$\begin{aligned} &\mathcal{U}(t_{\mathbb{A}})^{0,\sigma}(f(\mathbf{x}))\mathcal{U}(t_{\mathbb{A}})^{\sigma,\sigma'\sigma}(g(\mathbf{y}))\mathcal{U}(t_{\mathbb{A}})^{\sigma'\sigma,0}(h(\mathbf{z})) \\ &\quad = \mathcal{F}_{\mathbb{A}}(c_{\mathbb{A}}(\sigma, \sigma')|_{Sp(2,k)\times Sp(2,k)} \equiv 1) \int_{k_{\mathbb{A}}^3} e^{i\{S_{\mathbb{A}}(\mathbf{x},\mathbf{x}')+S_{\mathbb{A}}(\mathbf{y},\mathbf{y}')+S_{\mathbb{A}}(\mathbf{z},\mathbf{z}')\}} \\ &\qquad \times f(\mathbf{x}')g(\mathbf{y}')h(\mathbf{z}')d\mathbf{x}'d\mathbf{y}'d\mathbf{z}', \end{aligned} \tag{5.1}$$

where all the earlier provisos and conventions are in place and we have taken an obvious notational license regarding the factor $\mathcal{F}_{\mathbb{A}}$. ■

5.2. Certain consequences

The philosophical thrust of (5.1) is that it engenders in its very structure the product formula for the 2-Hilbert symbol for k, at the same time that the given integral is an oscillatory integral, more specifically a proto-Feynman integral as yet free of the trap of an improperly defined region of integration, and the data present the effects of three unitary operators on functions f, g, h in the according Hilbert space. From the perspective of physics, noting in particular that f, g, and h are, as it were, free variables, meaning that (5.1) holds for all choices of f, g, and h modulo the relatively mild provisos in Section 3.4, the most remarkable feature of this relation is that the factor $\mathcal{F}_{\mathbb{A}}(c_{\mathbb{A}}(\sigma, \sigma'))$ represents a number of inner symmetries of the situation *qua* physics because of the defining relations of second group cohomology, the locale for $c_{\mathbb{A}}$ (cf. [26], for example).

Beyond this we should reiterate that the fact that the unitary operators on the left side of (5.1) deal with time evolution of quantum mechanical systems is reflected in, or balanced by, the fact that there are (adèlized) Maslov indices present in $\mathcal{F}_{\mathbb{A}}(c_{\mathbb{A}}(\sigma, \sigma'))$. Also, as was already made clear in the context of Robbin–Salamon's relation (3.32), as well as Souriau's essentially equivalent relation (3.31), these Maslov indices are intrinsically tied to these time evolutions. This just underscores the importance of $\mathcal{F}_{\mathbb{A}}(c_{\mathbb{A}}(\sigma, \sigma'))$ in the physical scheme of things.

From the complementary perspective of number theory, it can be observed from the outset that true to form in the area of functional equations in analytic number theory, the factor $\mathcal{F}_{\mathbb{A}}(c_{\mathbb{A}}(\sigma, \sigma'))$ is bound to carry all the secrets of these relations. It is appropriate therefore to emphasize that $\mathcal{F}_{\mathbb{A}}(c_{\mathbb{A}}(\sigma, \sigma'))$ in and of itself ought to be the focus of future investigations.

And then, where to go next, once $\mathcal{F}_{\mathbb{A}}(c_{\mathbb{A}}(\sigma, \sigma'))$ has been explicated? There are two interlinked possibilities that come to mind right away. The first, coming from analytic number theory, stems from the observation that ultimately (5.1) is connected to the proof of quadratic reciprocity for k by Fourier analytic means, *gratia* Weil and Kubota, and the question is immediately raised how one would generalize (5.1) to the setting of higher reciprocity laws, e.g., n-Hilbert reciprocity. For this more general case Kubota demonstrated [32] that such laws are equivalent to having n-fold covers of an adèlic symplectic group again split on the rational points, but unlike in the quadratic case, there is as yet no independent derivation of the latter by (generalized) Fourier analytic means, and so the analytic proof of higher reciprocity remains an open problem (see e.g., [32,30], and §7 of [3]). The relation (5.1) is emphatically analytic, of course, even as we have taken matters in the direction of oscillatory integrals, and it stands to reason that generalizing it to n-fold covers of symplectic groups would yield some insight on the aforementioned open question, seeing that the presence of unitary operators is suggestive: these are after all the central players in what Weil would perhaps refer to as abstract Fourier analysis, bordering as it does on unitary group representation theory.

The other avenue to pursue is subsequently connected with a question that is in actuality already present at this stage, *viz.* what is the physical meaning of relations like (5.1) (and now its projected generalization to n-Hilbert reciprocity) within the context of adèlic quantum mechanics? This is a very broad question, of course, that should already be asked for (4.52), with $k_{\mathbb{A}} = \mathbb{Q}_{\mathbb{A}}$, where it partly redounds to whatever rationale would be given by Dragovich, as well as by Vladimirov, Volovich, and Zelenov, for introducing non-archimedean methods into quantum mechanics in the first place. Thus, this line of thought is certainly tenuous and controversial as far as relevance to actual natural processes goes: there is a lot of philosophy to be done. For our part, therefore, we take our refuge in mathematics and offer these results as fundamentally number theoretic considerations.

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