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Alissa Crans

Loyola Marymount University, acrans@lmu.edu

Blake Mellor

Loyola Marymount University, blake.mellor@lmu.edu

Patrick Shanahan

Loyola Marymount University, patrick.shanahan@lmu.edu

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Finite n -quandles of torus and two-bridge links

Alissa S. Crans, Blake Mellor and Patrick D. Shanahan
*Loyola Marymount University, 1 LMU Drive,
Los Angeles, 90045 CA, USA*

Jim Hoste
*Pitzer College, 1050 N. Mills Avenue,
Claremont, 91711 CA, USA*

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ABSTRACT

We compute Cayley graphs and automorphism groups for all finite n -quandles of two-bridge and torus knots and links, as well as torus links with an axis.

Keywords: Quandles; two-bridge knots; torus knots; Cayley graphs.

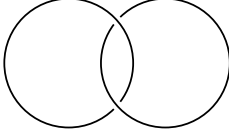
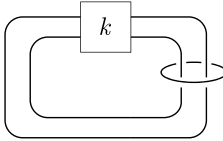
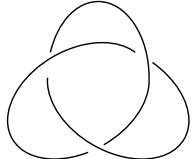
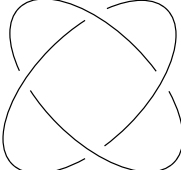
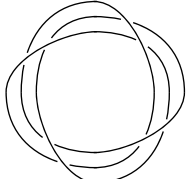
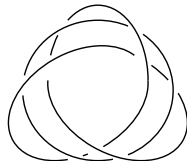
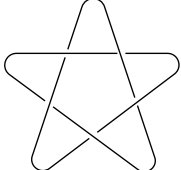
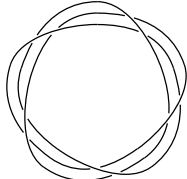
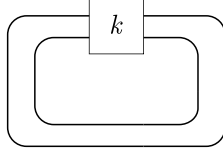
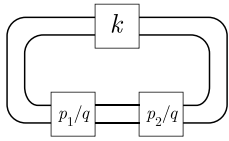
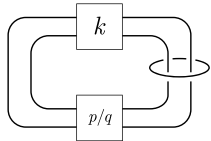
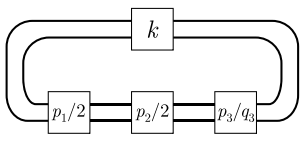
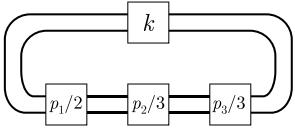
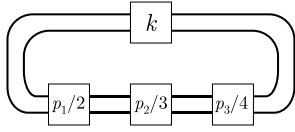
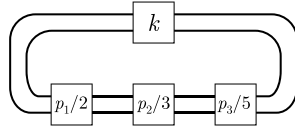
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1. Introduction

Associated to every oriented knot and link L is its fundamental quandle $Q(L)$. Except for the unknot and Hopf link, these quandles are infinite, but for each integer $n > 1$, a certain quotient of $Q(L)$, called the n -quandle of L , and denoted by $Q_n(L)$, may be finite. From results of Joyce [8, 9] and Winker [13], if the n -quandle $Q_n(L)$ is finite, then $\widetilde{M}_n(L)$, the n -fold cyclic branched cover of S^3 branched over L , has finite fundamental group. It was conjectured by Przytycki, and recently proven by Hoste and Shanahan [6], that the converse is also true. Using this result, together with Dunbar's [2] classification of all geometric, non-hyperbolic 3-orbifolds, a complete list of all knots and links in S^3 with finite n -quandle for some n was given in [6]. The links are listed in Table 1 (reproduced from [6]) as they are given by Dunbar and include all two-bridge links, some torus links, some torus links with "axis," and some Montesinos links. This paper is the second in a series of papers, beginning with [5], to give detailed descriptions of these finite n -quandles with the ultimate goal being a tabulation of all finite quandles that appear as n -quandles of links for some n . Here, we describe the Cayley graphs and automorphism groups of the

finite n -quandles associated to the two-bridge links, torus links, and torus links with axis. These links correspond to the links in the first three rows and the first entry in the fourth row of Table 1. (We warn the reader that it is not obvious that the first entry in the fourth row of this table represents all possible 2-bridge links.) The Montesinos links considered in [5] appear as the third entry in the fourth row. The four remaining links in Table 1 are infinite families which, like the Montesinos links, require substantially more analysis than the cases considered here. We intend to consider these families in future work.

Table 1. Links $L \in \mathbb{S}^3$ with finite $Q_n(L)$.

		
$n > 1$	$k \neq 0, n = 2$	$n = 3, 4, 5$
		
$n = 3$	$n = 2$	$n = 2$
		
$n = 3$	$n = 2$	$k \neq 0, n = 2$
		
$k + p_1/q + p_2/q \neq 0, n = 2$	$n = 2$	$k + p_1/2 + p_2/2 + p_3/q_3 \neq 0, n = 2$
		
$k + p_1/2 + p_2/3 + p_3/3 \neq 0, n = 2$	$k + p_1/2 + p_2/3 + p_3/4 \neq 0, n = 2$	$k + p_1/2 + p_2/3 + p_3/5 \neq 0, n = 2$

2. Link Quandles

We begin with a review of the definition of the fundamental quandle of a link and its associated n -quandles. We refer the reader to [4, 8, 9, 13] for more detailed information.

A *quandle* is a set Q equipped with two binary operations \triangleright and \triangleright^{-1} that satisfy the following three axioms:

- (A1.) $x \triangleright x = x$ for all $x \in Q$.
- (A2.) $(x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y$ for all $x, y \in Q$.
- (A3.) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ for all $x, y, z \in Q$.

Each element $x \in Q$ defines a map $S_x : Q \rightarrow Q$ by $S_x(y) = y \triangleright x$. The axiom A2 implies that each S_x is a bijection and the axiom A3 implies that each S_x is a quandle homomorphism, and therefore, an automorphism. We call S_x the *point symmetry at x* .

It is important to note that the operation \triangleright is, in general, not associative. In order to clarify the ambiguity caused by lack of associativity, we adopt the exponential notation introduced by Fenn and Rourke in [4] and denote $x \triangleright y$ as x^y and $x \triangleright^{-1} y$ as $x^{\bar{y}}$. With this notation, x^{y^z} will be taken to mean $(x^y)^z = (x \triangleright y) \triangleright z$ whereas $x^{y^{\bar{z}}}$ will mean $x \triangleright (y \triangleright z)$.

The following lemma from [4], which describes how to re-associate a product in an n -quandle given by a presentation, will be used repeatedly in this paper.

Lemma 2.1. *If a^u and b^v are elements of a quandle, then*

$$(a^u)^{(b^v)} = a^{u\bar{v}b^v} \quad \text{and} \quad (a^u)^{\overline{(b^v)}} = a^{u\bar{v}\bar{b}^v}.$$

Using Lemma 2.1, elements in a quandle given by a presentation $\langle S | R \rangle$ can be represented as equivalence classes of expressions of the form a^w , where a is a generator in S and w is a word in the free group on S (with \bar{x} representing the inverse of x).

If n is a natural number, a quandle Q is an n -*quandle* if $x^{y^n} = x$ for all $x, y \in Q$, where by y^n we mean y repeated n times. In other words, each point symmetry S_x has order dividing n . A *trivial* quandle is one where $x^y = x^{\bar{y}} = x$ for all $x, y \in Q$ or, equivalently, a trivial quandle is a 1-quandle. A 2-quandle is also called an *involutory* quandle. Note that a quandle is involutory if and only if $\triangleright = \triangleright^{-1}$, that is, each point symmetry is an involution.

If L is an oriented knot or link in S^3 , then a presentation of its fundamental quandle, $Q(L)$, can be derived from a regular diagram D of L . See Joyce [9]. This process mimics the Wirtinger algorithm. Namely, assign a quandle generator x_1, x_2, \dots, x_n to each arc of D , then at each crossing introduce the relation $x_i = x_k^{x_j}$ as shown in Fig. 1. It is easy to check that the three Reidemeister moves do not change the quandle given by this presentation so that the quandle is indeed an invariant of the oriented link.

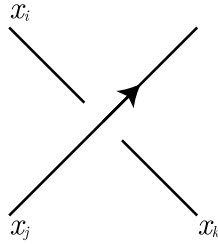


Fig. 1. The relation $x_i = x_k^{x_j}$ at a crossing.

The fundamental quandle of a link depends on the choice of orientation. If L is an oriented link, let $-L$ be its *reverse* obtained by reversing the orientation of all of its components and let L^* be its *obverse*, the mirror image of L . The *inverse* of L is defined to be $-L^*$. If D is a diagram of an oriented link and D' is obtained from D by both reflecting through a plane perpendicular to the plane of projection and reversing all orientations, then it is not hard to see that both D and D' produce exactly the same presentation for the fundamental quandle. Thus, the fundamental quandle is the same for an oriented link L and its inverse $-L^*$.

Given an oriented link L and a presentation $\langle S \mid R \rangle$ of $Q(L)$, a presentation of the quotient n -quandle $Q_n(L)$ is obtained by adding the relations $x^{y^n} = x$ for every pair of distinct generators x and y . As with the fundamental quandle, $Q_n(L)$ depends on the choice of orientation but $Q_n(L) = Q_n(-L^*)$. In the case $n = 2$, the relation $x_i = x_k^{x_j}$ is equivalent to the relation $x_k = x_i^{x_j}$. Hence, the 2-quandle does not depend on the orientation of the link, so $Q_2(L) = Q_2(-L) = Q_2(L^*) = Q_2(-L^*)$. In general, one should expect a significant loss of information in passing from $Q(L)$ to $Q_n(L)$.

The following result is a simple corollary of the main theorem of [6].

Proposition 2.2. *The fundamental quandle of an oriented link L is finite if and only if L is either the unknot or the Hopf link with any orientation. In these cases, the quandle is trivial of order 1 or 2, respectively.*

Proof. Suppose the fundamental quandle $Q(L)$ of the link L is finite. Hence, every quotient of $Q(L)$ is finite and so $Q_n(L)$ is finite for all $n > 1$. From the classification of links with finite n -quandles given in [6], the only non-trivial link for which this is true is the Hopf link. A simple computation completes the proof. \square

Given a presentation of an n -quandle, one can try to systematically enumerate its elements and simultaneously produce a Cayley graph of the quandle. Such a method was described in a graph-theoretic fashion by Winker in [13]. The method is similar to the well-known Todd-Coxeter process for enumerating cosets of a subgroup of a group [12] and has been extended to racks in [7]. (A rack is more general than a quandle, requiring only axioms A2 and A3.) We provide a brief description of

Winker’s method applied to the n -quandle of a link since it will be used extensively in this paper. Suppose $Q_n(L)$ is presented as

$$Q_n(L) = \langle x_1, x_2, \dots, x_g \mid x_{j_1}^{w_1} = x_{k_1}, \dots, x_{j_r}^{w_r} = x_{k_r} \rangle_n,$$

where each w_i is a word in the free group on $\{x_1, \dots, x_g\}$. Throughout this paper presentations of n -quandles will not explicitly list the n -quandle relations $x_i^{x_j^n} = x_i$ (nor the relations given by the quandle axioms) although we are implicitly assuming they hold. To avoid confusion we append the subscript n to presentations of n -quandles.

If y is any element of the quandle, then it follows from the relation $x_{j_i}^{w_i} = x_{k_i}$ and Lemma 2.1 that $y^{\bar{w}_i x_{j_i} w_i} = y^{x_{k_i}}$, and so

$$y^{\bar{w}_i x_{j_i} w_i \bar{x}_{k_i}} = y.$$

Winker calls this relation the *secondary relation* associated to the *primary relation* $x_{j_i}^{w_i} = x_{k_i}$. He also considers relations of the form $y^{x_j^n} = y$ for all y and $1 \leq j \leq g$. These relations are equivalent to the secondary relations of the n -quandle relations. In order to see this, notice that the secondary relation of the n -quandle relation $x_k^{x_j^n} = x_k$ is $y = y^{\bar{x}_j^n x_k x_j^n \bar{x}_k}$ for all elements y . Now given any z , if we let $y = z^{x_j^n}$ in this secondary relation, then $z^{x_j^n} = z^{x_j^n \bar{x}_j^n x_k x_j^n \bar{x}_k} = z^{x_k x_j^n \bar{x}_k}$. Hence, $z^{x_j^n x_k} = z^{x_k x_j^n}$ for all elements z . In a similar manner we find $z^{x_j^n \bar{x}_k} = z^{\bar{x}_k x_j^n}$ for all z . Now given any y we have $y = x_i^w$ for some $1 \leq i \leq g$ and, by what we just observed, we can commute x_j^n with w in the exponent of x_i . Therefore,

$$y^{x_j^n} = x_i^{w x_j^n} = x_i^{x_j^n w} = x_i^w = y.$$

Conversely, if $y^{x_j^n} = y$ for all y , then clearly $x_i^{x_j^n} = x_i$ as well.

Winker’s method now proceeds to build the Cayley graph associated to the presentation as follows:

- (1) Begin with g vertices labeled x_1, x_2, \dots, x_g and numbered $1, 2, \dots, g$.
- (2) Add an oriented loop at each vertex x_i and label it x_i . (This encodes the axiom A1.)
- (3) For each value of i from 1 to r , *trace* the primary relation $x_{j_i}^{w_i} = x_{k_i}$ by introducing new vertices and oriented edges as necessary to create an oriented path from x_{j_i} to x_{k_i} given by w_i . Consecutively number (starting from $g + 1$) new vertices in the order they are introduced. Edges are labeled with their corresponding generator and oriented to indicate whether x_i or \bar{x}_i was traversed.
- (4) Tracing a relation may introduce edges with the same label and same orientation into or out of a shared vertex. We identify all such edges, possibly leading to other identifications. This process is called *collapsing* and all collapsing is carried out before tracing the next relation.
- (5) Proceeding in order through the vertices, trace and collapse each n -quandle relation $y^{x_j^n} = y$ and each secondary relation (in order). All of these relations are traced and collapsed at a vertex before proceeding to the next vertex.

The method will terminate in a finite graph if and only if the n -quandle is finite. The reader is referred to Winker [13] and [7] for more details.

Associated to every quandle Q is its automorphism group $\text{Aut}(Q)$. The *inner* automorphism group of Q , denoted by $\text{Inn}(Q)$, is the normal subgroup of $\text{Aut}(Q)$ generated by the point symmetries S_x . The *transvection* group of Q , denoted by $\text{Trans}(Q)$, is the subgroup of $\text{Inn}(Q)$ generated by all products $S_x S_y^{-1}$. The subgroup $\text{Trans}(Q)$ is normal in both $\text{Aut}(Q)$ and $\text{Inn}(Q)$. Moreover, $\text{Trans}(Q)$ is abelian if and only if

$$(x^y)^{(z^w)} = (x^z)^{(y^w)} \tag{2.1}$$

for all elements $x, y, z, w \in Q$. Quandles that satisfy the property (2.1) are called *medial* or *abelian*. See [9] for more details.

Some results in this paper, were obtained using the RIG package for GAP. The Cayley graph of a finite quandle Q can be used to produce the operation table for \triangleright which is encoded in a matrix M_Q . In RIG, a rack (or quandle) can then be defined using the command `RACK(M_Q)`. Once the quandle is entered into RIG, the built-in commands `AUTOMORPHISMGROUP`, `INNERGROUP`, and `TRANSVECTIONS-GROUP` will compute $\text{Aut}(Q)$, $\text{Inn}(Q)$, and $\text{Trans}(Q)$, respectively. Finally, the GAP command `STRUCTUREDESCRIPTION` will determine the structure of the group, such as $\mathbb{Z}_2 \times S_4$. No additional special code is required to reproduce our results.

3. Two-Bridge Links

In this section, we consider the involutory quandle of the non-trivial 2-bridge link $L_{p/q}$. Because the 2-fold cyclic cover of S^3 branched over a non-trivial two-bridge link is a lens space with finite fundamental group, these quandles are finite. We can easily find a presentation for the fundamental quandle of $L_{p/q}$, where $\text{gcd}(p, q) = 1$ and $0 < p < q$ using the Schubert normal form (see, for example, [1, 10]). As an example, consider the Schubert normal form of the figure-eight knot $L_{3/5}$ with orientation shown in Fig. 2. The fundamental quandle for this knot has the

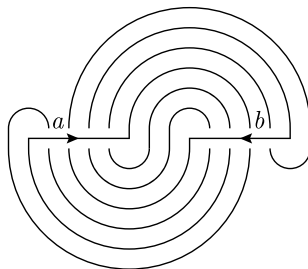


Fig. 2. The Schubert normal form of the two-bridge link $L_{3/5}$.

presentation

$$Q(L_{3/5}) = \langle a, b \mid a^{\bar{b}\bar{a}} = b, b^{\bar{a}\bar{b}} = a \rangle,$$

where the generators a and b are the arcs of the two-bridges.

For the involutory quandle, we may ignore the orientation of the link. Thus, the involutory quandle of $L_{3/5}$ (with any orientation) is presented by

$$Q_2(L_{3/5}) = \langle a, b \mid a^{baba} = b, b^{abab} = a \rangle_2 = \langle a, b \mid a^{baba} = b \rangle_2,$$

where the two presentations are equivalent because, in the involutory quandle, the second relation is equivalent to the first relation. In general, the presentation of $Q_2(L_{p/q})$ depends only on the number of undercrossings along one bridge in the Schubert normal form, which is equal to $q - 1$. We have two cases, depending on whether q is odd or even (i.e. whether $L_{p/q}$ is a knot or a link, respectively).

If $q = 2t + 1$ is odd, then a presentation of the involutory quandle is

$$Q_2(L_{p/q}) = \langle a, b \mid a^{(ba)^t} = b \rangle_2.$$

The secondary relation associated to the primary relation $a^{(ba)^t} = b$, is $x^{(ab)^{2t+1}} = x$. Tracing the primary relation by Winker’s method gives the diagram in Fig. 3 (where the vertex x_1 represents the element a and x_{2t+1} the element b).

Notice from Fig. 3 that

$$x_i^{(ab)^t} = \begin{cases} x_1, & i = 2t + 1, \\ x_{2t+2-i+(-1)^i}, & 1 \leq i < 2t + 1. \end{cases} \tag{3.1}$$

It is now straightforward to check that the secondary relation is satisfied at each vertex. For example, if $1 \leq i < 2t + 1$ is odd, then by (3.1) we have

$$x_i^{(ab)^{2t+1}} = x_i^{(ab)^t (ab)^t ab} = x_{2t+1-i}^{(ab)^t ab} = x_{i+2}^{ab} = x_i.$$

The remaining cases are similar. Therefore, Fig. 3 gives the Cayley graph of $Q_2(L_{p/q})$. The number of elements in the quandle is $2t + 1 = q$.

If $q = 2t$ is even, then a presentation of the involutory quandle is

$$Q_2(L_{p/q}) = \langle a, b \mid a^{(ba)^{t-1}b} = a, b^{(ab)^{t-1}a} = b \rangle_2.$$

In this case, there are two primary relations. However, the secondary relations associated to the two primary relations are the same: $x^{(ab)^{2t}} = x$. Tracing the

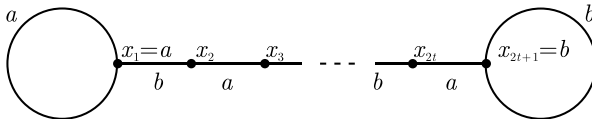


Fig. 3. The Cayley graph for $Q_2(L_{p/q})$ with $q = 2t + 1$.

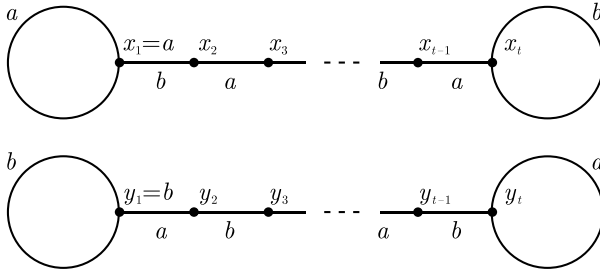


Fig. 4. The Cayley graph for $Q_2(L_{p/q})$ with $q = 2t$ and t odd.

primary relations gives the diagram in Fig. 4 when t is odd (where the vertex x_1 is the element a and y_1 is the element b). It is once again straightforward to check that the secondary relation is satisfied at each vertex, so this diagram is the Cayley graph of the involutory quandle. Notice that there are two *algebraic* components of the quandle, each given by a connected component of the Cayley graph. This reflects the fact that $L_{p/q}$ with q even is a 2-component link. In general, the number of link components is equal to the number of algebraic components of its quandle (see [9]). The number of elements in each algebraic component of $Q_2(L_{p/q})$ is t , and so the total number of elements is $2t = q$. A similar analysis applies to t even.

In summary, for any non-trivial two-bridge knot or link, the involutory quandle $Q_2(L_{p/q})$ has order q . In fact, $Q_2(L_{p/q})$ is isomorphic to the *dihedral quandle* which is the set $R_q = \mathbb{Z}/q\mathbb{Z}$ with quandle operations defined by $i^j = i^{\bar{j}} = 2j - i \pmod{q}$.

Proposition 3.1. *If $L_{p/q}$ is a non-trivial two-bridge knot or link, then $Q_2(L_{p/q})$ is isomorphic to the dihedral quandle R_q .*

Proof. There are two cases depending on whether q is even or odd. We present only the odd case; the remaining case is similar. Assume $q = 2t + 1$. From above we have the presentation

$$Q_2(L_{p/q}) = \langle a, b \mid a^{(ba)^t} = b \rangle_2.$$

Define a map $\phi : Q_2(L_{p/q}) \rightarrow R_q$ by defining ϕ on the generators by $\phi(a) = 0$ and $\phi(b) = 1$ and then extending the map over $Q_2(L_{p/q})$ using the quandle operation. (That is, if $\phi(x) = i$ and $\phi(y) = j$, then define $\phi(xy) = i^j$.) To verify that ϕ is well-defined, it suffices to check that the image of the relation $a^{(ba)^t} = b$ is satisfied in R_q . Notice that if $\phi(x) = i$, then,

$$\phi(x^{ba}) = (i^1)^0 = i - 2 \pmod{q}.$$

It then follows that

$$\phi(a^{(ba)^t}) = -2t = 1 = \phi(b) \pmod{q}.$$

The map ϕ is a homomorphism by definition. Moreover, it is surjective because $\phi(a) = 0$ and $\phi(b) = 1$ generate R_q . This follows inductively from the observation

Table 2. The automorphism groups of $Q_2(L_{p/q})$.

Q_n	$ Q_n $	Aut	Inn	Trans
$Q_2(L_{p/q})$	q	$\mathbb{Z}_q \rtimes \mathbb{Z}_q^*$	$D_{q/\gcd(2,q)}$	$\mathbb{Z}_{q/\gcd(2,q)}$

that $i^{i+1} = i + 2 \pmod{q}$ for all $i \in R_q$. Finally, since $|Q_2(L_{p/q})| = |R_q| = q$, we have that ϕ is an isomorphism. \square

In [3], Elhamdadi, MacQuarrie, and Restrepo prove that $\text{Aut}(R_n)$ is the semi-direct product $\mathbb{Z}_n \rtimes \mathbb{Z}_n^*$ and $\text{Inn}(R_n)$ is the dihedral group D_n of order $2n$ if n is odd and $D_{n/2}$ if n is even. Here, \mathbb{Z}_n^* is the multiplicative group of units in \mathbb{Z}_n , and the semidirect product is given by the homomorphism $\phi : \mathbb{Z}_n \rightarrow \text{Aut}(\mathbb{Z}_n)$ defined by $\phi(a)(x) = ax$. Thus, we obtain the results given in Table 2. We further note that $\text{Trans}(Q_2(L_{p/q}))$ is generated by a rotation in $\text{Inn}(Q_2(L_{p/q}))$.

4. Torus Links

As mentioned in the introduction, the n -quandle of a link L is finite if and only if the n -fold cyclic branched cover of S^3 branched over L has finite fundamental group. These covers, in the case where L is the torus link $T_{p,q}$, were classified by Milnor in [11] and as a result, $Q_n(T_{p,q})$ is finite if and only if $\frac{1}{p} + \frac{1}{q} + \frac{1}{n} > 1$. The inequality holds for the following values of p, q , and n given in the following table.

Since $T_{2,q}$ is a two-bridge link, the links $T_{2,q}$ with $n = 2$ in Table 3 were considered in the previous section. For each of the remaining cases, it is a simple matter to derive a presentation from a link diagram and then employ Winker’s method to create a Cayley graph of the associated quandle. Diagrams for $T_{2,q}$ and $T_{3,q}$ with choice of orientation and quandle generators are shown in Fig. 5. In the case of $T_{2,q}$, the box labeled q contains q right handed half twists. The remainder of this

Table 3. Values of p, q , and n for which $Q_n(T_{p,q})$ is finite.

(p, q)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	$(2, q), q > 5$	(3, 3)	(3, 4)	(3, 5)
n	$n > 1$	2, 3, 4, 5	2, 3	2, 3	2	2	2	2

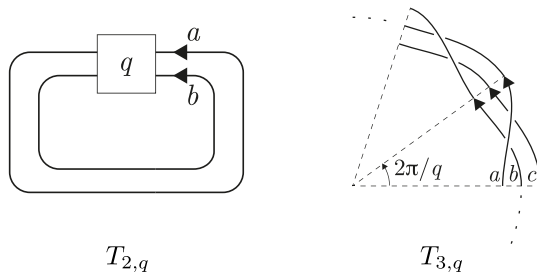


Fig. 5. The oriented links $T_{2,q}$ and $T_{3,q}$ with choice of generators.

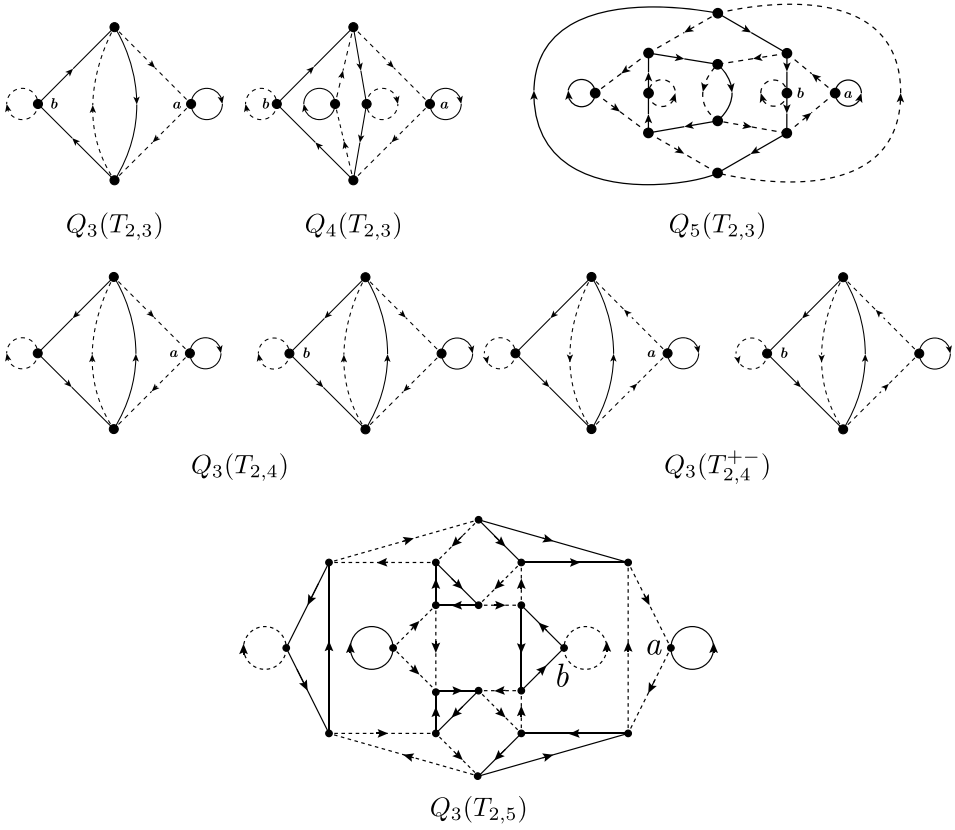


Fig. 6. Cayley graphs of $Q_n(T_{2,q})$ with $n > 2$.

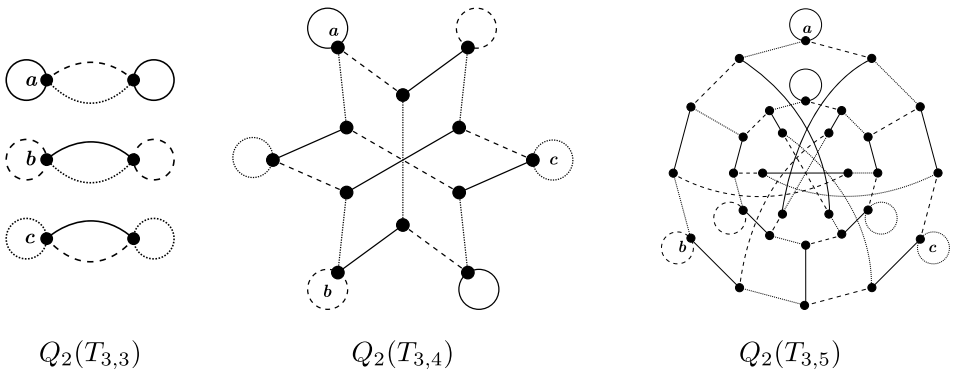


Fig. 7. Cayley graphs of $Q_2(T_{3,q})$.

Table 4. Order and automorphism groups of finite torus link n -quandles.

Q_n	$ Q_n $	Aut	Inn	Trans
$Q_2(T_{2,q})$	q	$\mathbb{Z}_q \rtimes \mathbb{Z}_q^*$	$D_{q/\gcd(2,q)}$	$\mathbb{Z}_q/\gcd(2,q)$
$Q_3(T_{2,3})$	4	A_4	A_4	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$Q_4(T_{2,3})$	6	S_4	S_4	A_4
$Q_5(T_{2,3})$	12	A_5	A_5	A_5
$Q_3(T_{2,4})$	8	S_4	A_4	A_4
$Q_3(T_{2,4}^{+-})$	8	$\mathbb{Z}_2 \times A_4$	A_4	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$Q_3(T_{2,5})$	20	S_5	A_5	A_5
$Q_2(T_{3,3})$	6	$\mathbb{Z}_2 \times S_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$Q_2(T_{3,4})$	12	$\mathbb{Z}_2 \times S_4$	A_4	A_4
$Q_2(T_{3,5})$	30	$\mathbb{Z}_2 \times S_5$	A_5	A_5

section lists the results. Note that only in the case of the two-component link $T_{2,4}$ with $n = 3$ does orientation matter. We denote by $T_{2,4}^{+-}$ the oriented link obtained from $T_{2,4}$ by reversing the orientation of the second component (the one labeled b).

Cayley graphs of these quandles are given in Figs. 6 and 7. In these graphs, solid, dashed, and dotted edges correspond to the generators a , b and c , respectively. The automorphism groups were computed using GAP and are given in Table 4.

5. Torus Links with Axis

Given the torus link $T_{p,q}$ lying on the torus F that separates S^3 into two solid tori, an *axis* of $T_{p,q}$ is the core of either solid torus. In this section, we consider the oriented torus links with axis shown in Fig. 8. As in the last section, q represents q right handed half twists. For each torus link $T_{2,q}$, we adjoin the axis A with linking number $+2$ (as opposed to $+q$). In the case of the trefoil, we also include the oriented axis B with linking number $+3$. Only the involutory quandles of these links are finite, hence the orientations are immaterial. Since $T_{2,-q} \cup A = (T_{2,q} \cup A)^*$ and $T_{2,1} \cup A$ is the 2-bridge link $L_{1/4}$, we may further assume $q > 1$. A choice of generators for each link is given in the figure.

First, we consider the link $T_{2,3} \cup B$. After deriving a presentation for $Q_2(T_{2,3} \cup B)$ from the diagram and employing Winker’s method, we obtain the Cayley graph

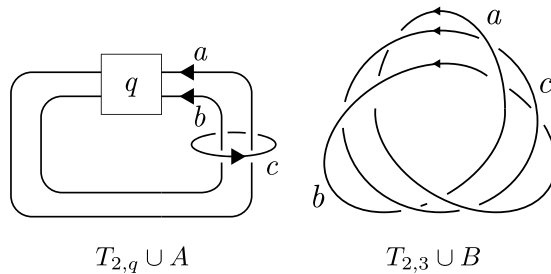


Fig. 8. The oriented links $T_{2,q} \cup A$ and $T_{2,3} \cup B$ with choice of generators.

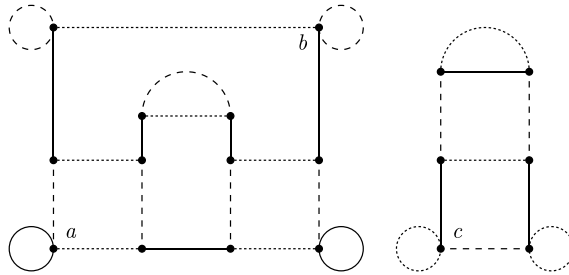


Fig. 9. The Cayley graph of $Q_2(T_{2,3} \cup B)$.

shown in Fig. 9. Solid, dashed, and dotted edges correspond to the generators a , b , and c , respectively, in the figure. Using GAP to compute the automorphism groups of this quandle we find that $\text{Aut}(Q_2(T_{2,3} \cup B)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times S_4$, $\text{Inn}(Q_2(T_{2,3} \cup B)) \cong S_4$ and $\text{Trans}(Q_2(T_{2,3} \cup B)) \cong S_4$.

Now we analyze the infinite family $T_{2,q} \cup A$. There are two cases depending on whether q is even or odd. Assume $q = 2t + 1$ with $t > 0$. A presentation of the involutory quandle in this case is

$$Q_2(T_{2,q} \cup A) = \langle a, b, c \mid c^{ab} = c, a^{(ba)^t bc} = b, b^{(ab)^t c} = a \rangle_2.$$

In order to make Winker's method simpler, we change to the equivalent presentation

$$Q_2(T_{2,q} \cup A) = \langle a, b, c \mid \{c^{(ab)^i} = c\}_{i=1}^{2t}, a^{c(ba)^t} = b, \\ a^{(ba)^t c} = b, a^{cac} = a, b^{cbc} = b \rangle_2.$$

To see that the presentations are equivalent, we need to show that each set of relations can be derived from the other. Let P_1 , P_2 , and P_3 be the relations in the first presentation and S , R_2 , R_3 , R_4 , and R_5 be the relations in the second presentation in the order given. The relations $c^{(ab)^i} = c$ in the set S follow from induction and relation P_1 . The relation R_2 follows immediately from P_3 . The secondary relation associated to P_1 implies $x^{cba} = x^{bac}$ for all x and thus $x^{c(ba)^j} = x^{(ba)^j c}$ for $j \geq 1$ follows by induction. Similarly, we have $x^{c(ab)^j} = x^{(ab)^j c}$ for $j \geq 1$. The relation R_2 now follows from this observation and P_3 . The relation R_4 is then derived as follows:

$$a^{cac} \stackrel{P_3}{=} b^{(ab)^t ac} = b^{b(ab)^t ac} = b^{(ba)^{t+1} c} = b^{c(ba)^{t+1}} \stackrel{P_2}{=} a^a = a.$$

The derivation of R_5 is similar,

$$b^{cbc} \stackrel{R_3}{=} a^{(ba)^t bc} \stackrel{P_2}{=} b.$$

In a similar manner, P_1 , P_2 , and P_3 can be derived from S , R_2 , R_3 , R_4 , and R_5 .

We now proceed with the diagramming method. We begin with three vertices $x_1 = a$, $y_{2t+1} = b$ and $z_1 = c$ and loops labeled a , b and c , respectively, at each vertex. We next trace the $2t + 4$ primary relations which introduces the vertices

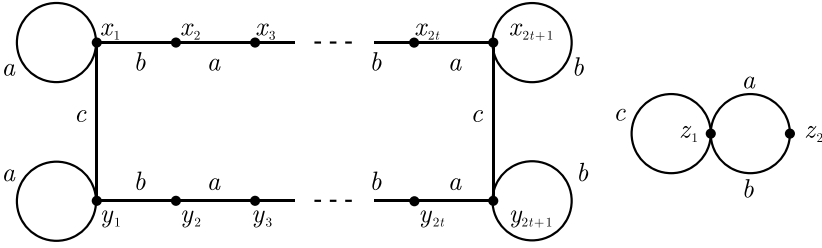


Fig. 10. The primary relation graph for $Q_2(T_{2,q} \cup A)$ with $q = 2t + 1$.

$z_2, y_1, \dots, y_{2t}, x_2, \dots, x_{2t+1}$ in that order. Tracing the primary relations gives the graph shown in Fig. 10.

Finally, we consider the secondary relations $x^{w_j} = x$ for $1 \leq j \leq 5$ where,

$$\begin{aligned} w_1 &= (ba)^i c(ab)^i c, \quad 1 \leq i \leq 2t, \\ w_2 &= (ab)^t c a c b (ab)^t, \\ w_3 &= c(ab)^{2t} a c b, \\ w_4 &= (ca)^4, \\ w_5 &= (cb)^4. \end{aligned}$$

We trace the secondary relations at each vertex in the order the vertices were introduced. Notice that the formulas in (3.1) from Sec. 3 apply to both x_i and y_i in Fig. 10. Tracing the secondary relation w_1 for $1 \leq i \leq 2t$ at vertex x_1 introduces $2t - 1$ edges labeled c that connect the vertices x_i to y_i . At this point we claim that all secondary relations are satisfied at vertices x_i and y_i for $1 \leq i \leq 2t + 1$. There are several cases to consider. We will verify the relations $x_i^{w_2} = x_i$ and $x_i^{w_3} = x_i$ for $i \neq 1$ and odd and leave the remaining cases to the reader. Using (3.1) and Fig. 10, we have

$$x_i^{(ab)^t c a c b (ab)^t} = x_{2t+1-i}^{c a c b (ab)^t} = x_{2t+3-i}^{(ab)^t} = x_i.$$

Similarly,

$$x_i^{c(ab)^{2t} a c b} = y_i^{(ab)^{2t} a c b} = y_{2t+1-i}^{(ab)^t a c b} = y_{i+2}^{a c b} = x_i.$$

Finally, consider tracing the secondary relations at vertex z_1 . The relations $z_1^{w_1} = z_1$ for $1 \leq i \leq 2t$ are already satisfied. Tracing $z_1^{w_2} = z_1$ introduces a loop at z_2 labeled c . This gives the graph in Fig. 11. It is not hard to then verify that all remaining secondary relations are satisfied at z_1 and z_2 . Therefore, Fig. 11 is the Cayley graph of $Q_2(T_{2,q} \cup A)$ with q odd. From this we see that $Q_2(T_{2,q} \cup A)$ with q odd has order $2q + 2$. Notice that the graph has two connected components which was expected since $T_{2,q} \cup A$ is a link of 2 components.

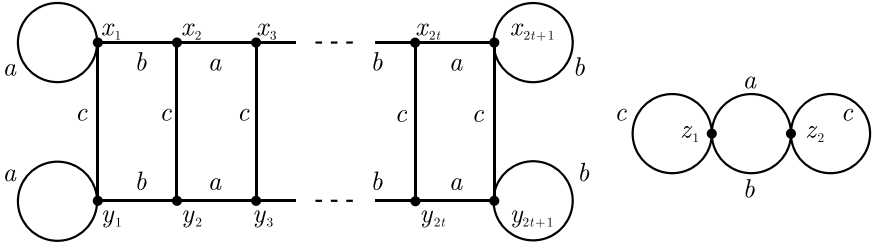


Fig. 11. The Cayley graph for $Q_2(T_{2,q} \cup A)$ with $q = 2t + 1$.

If $q = 2t$ is even and $t > 0$, then a presentation of the involutory quandle from Fig. 8 is

$$Q_2(T_{2,q} \cup A) = \langle a, b, c \mid c^{ab} = c, a^{(ba)^{t-1}bc} = a, b^{(ab)^t c} = b \rangle_2.$$

As in the odd case, the Cayley graph is easier to produce using the following equivalent presentation:

$$Q_2(T_{2,q} \cup A) = \langle a, b, c \mid \{c^{(ab)^i} = c\}_{i=1}^{2t-1}, a^{(ba)^{t-1}c} = a, \\ b^{(ab)^{t-1}c} = b, a^{cac} = a, b^{cbc} = b \rangle_2.$$

Applying the diagramming method to this presentation we find the Cayley graph shown in Fig. 12. The edges labeled $a(b)$ are a when t is even and b when t is odd (and the reverse for the edges labeled $b(a)$). As in the odd case, the order of the involutory quandle is $2k + 2$.

We conclude by determining the automorphism groups for $Q_2(T_{2,q} \cup A)$.

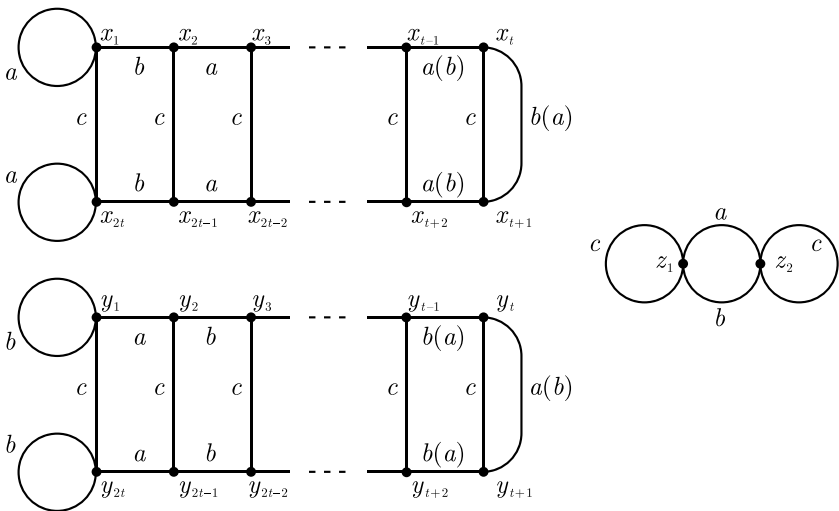


Fig. 12. The Cayley graph for $Q_2(T_{2,q} \cup A)$ with $q = 2t$.

Proposition 5.1. For $q \geq 2$, $\text{Aut}(Q_2(T_{2,q} \cup A)) \cong \mathbb{Z}_2 \times \text{Aut}(Q_2(L_{1/2q})) \cong \mathbb{Z}_2 \times (\mathbb{Z}_{2q} \rtimes \mathbb{Z}_{2q}^*)$.

Proof. We will consider the case when $q = 2t + 1$ is odd; the case when q is even is similar. From the presentation in Sec. 3, notice that there is a natural inclusion $\eta : Q_2(L_{1/2q}) \rightarrow Q_2(T_{2,q} \cup A)$ determined by $\eta(a) = a$ and $\eta(b) = b$. Using the relation $x^{c(ab)^j} = x^{(ab)^j c}$ discussed in the derivation of second presentation of $Q_2(T_{2,q} \cup A)$ we see that

$$\eta(a^{(ba)^{2t}b}) = a^{(ba)^{2t}b} \stackrel{R_3}{=} b^{c(ba)^t b} \stackrel{R_3}{=} b^{cb(ba)^t b} = b^{c(ab)^t} = b^{(ab)^t c} \stackrel{R_2}{=} a^{cc} = a = \eta(a).$$

In a similar manner, η preserves the second relation $b^{(ab)^{2t}a} = b$ of $Q_2(L_{1/2q})$. So, η is a quandle homomorphism. Clearly, η is onto the set $H = \{x_1, \dots, x_{2t+1}, y_1, \dots, y_{2t+1}\}$ which has the same order, $2q$, as $Q_2(L_{1/2q})$. Thus, $\eta : Q_2(L_{1/2q}) \rightarrow H$ is a quandle isomorphism onto the subquandle H of $Q_2(T_{2,q} \cup A)$. Notice that H is the subquandle generated by $\{a, b\}$ and that the set $S = \{z_1, z_2\} = Q_2(T_{2,q} \cup A) \setminus H$ is also a subquandle.

For any $f \in \text{Aut}(Q_2(L_{1/2q}))$, we define $\varphi_f \in \text{Aut}(Q_2(T_{2,q} \cup A))$ by $\varphi_f(a) = \eta \circ f(a)$, $\varphi_f(b) = \eta \circ f(b)$ and $\varphi_f(c) = c$ (from now on, we will abuse notation and use f in place of $\eta \circ f$, as long as there is no confusion). We also define $\psi \in \text{Aut}(Q_2(T_{2,q} \cup A))$ by $\psi(a) = a$, $\psi(b) = b$ and $\psi(c) = c^a$. We will show that $\text{Aut}(Q_2(T_{2,q} \cup A)) = \langle \psi, \varphi_f \rangle \cong \mathbb{Z}_2 \times \text{Aut}(Q_2(L_{1/2q}))$.

It is clear that $\langle \varphi_f \rangle \cong \text{Aut}(Q_2(L_{1/2q}))$. Also $\psi^2 = id$, since $\psi^2(c) = \psi(c^a) = \psi(c)^{\psi(a)} = c^{aa} = c$, so $\langle \psi \rangle \cong \mathbb{Z}_2$. Finally, we compare $\phi_f \circ \psi$ and $\psi \circ \phi_f$. Note that since $f(a)$ (respectively, $f(b)$) involves only the generators a and b , $\psi(f(a)) = f(a)$ (respectively, $\psi(f(b)) = f(b)$).

$$\begin{aligned} \phi_f \circ \psi(a) &= f(a), & \psi \circ \phi_f(a) &= \psi(f(a)) = f(a), \\ \phi_f \circ \psi(b) &= f(b), & \psi \circ \phi_f(b) &= \psi(f(b)) = f(b), \\ \phi_f \circ \psi(c) &= \phi_f(c^a) = c^{f(a)}, & \psi \circ \phi_f(c) &= \psi(c) = c^a. \end{aligned}$$

$f(a) = a^w$ or b^w , where w is a word in a and b . Without loss of generality, suppose $f(a) = a^w$. Then $c^{f(a)} = c^{a^w} = c^{\bar{w}aw}$. Since the word $\bar{w}aw$ has odd length, $c^{\bar{w}aw} = c^a$. Hence, $\phi_f \circ \psi = \psi \circ \phi_f$ for every ϕ_f . This implies that $\langle \psi, \phi_f \rangle \cong \langle \psi \rangle \times \langle \phi_f \rangle \cong \mathbb{Z}_2 \times \text{Aut}(Q_2(L_{1/2q}))$.

Finally, suppose $\alpha \in \text{Aut}(Q_2(T_{2,q} \cup A))$. Then α fixes H and S setwise, since H is the only subquandle of order $2q$. The restriction of α to H gives an automorphism of $\text{Aut}(Q_2(L_{1/2q}))$; let $f = \alpha|_H$. Since α also fixes S , $\alpha(c) = c$ or c^a . If $\alpha(c) = c$, then $\alpha = \phi_f$; on the other hand, if $\alpha(c) = c^a$, then $\alpha = \psi \circ \phi_f$. So every automorphism is in $\langle \psi, \phi_f \rangle$. Hence, $\text{Aut}(Q_2(T_{2,q} \cup A)) = \langle \psi, \varphi_f \rangle \cong \mathbb{Z}_2 \times \text{Aut}(Q_2(L_{1/2q}))$. \square

Corollary 5.2. For $q > 2$, $\text{Inn}(Q_2(T_{2,q} \cup A)) \cong D_{2q/\text{gcd}(2,q)}$ and $\text{Trans}(Q_2(T_{2,q} \cup A)) \cong D_q$.

Proof. We will first consider the case when q is odd. The inner automorphism group is generated by the symmetries S_a, S_b, S_c . We consider the Cayley graph from Fig. 11, and use the vertex labelings from that diagram. Then we can describe the action of each symmetry as a permutation on the set of vertices of the Cayley graph.

$$\begin{aligned}
 S_a &= (x_1)(x_2x_3)(x_4x_5) \cdots (x_{q-1}x_q) \cdot (y_1)(y_2y_3)(y_4y_5) \cdots (y_{q-1}y_q) \cdot (z_1z_2), \\
 S_b &= (x_1x_2)(x_3x_4) \cdots (x_{q-2}x_{q-1})(x_q) \cdot (y_1y_2)(y_3y_4) \cdots (y_{q-2}y_{q-1})(y_q) \cdot (z_1z_2), \\
 S_c &= (x_1y_1)(x_2y_2) \cdots (x_qy_q) \cdot (z_1)(z_2).
 \end{aligned}$$

These permutations correspond to symmetries of a regular $2q$ -gon. We embed the vertices of the $2q$ -gon in the xy -plane, centered at the origin, and label them with x_i and y_i as shown on the left in Fig. 13 (if $q \cong 1 \pmod{4}$, then x_{q-1} and y_q are on the left side; if $q \cong 3 \pmod{4}$, then x_{q-1} and y_q are on the right). The vertices z_1 and z_2 are embedded on the z -axis above and below the polygon. Then the actions of S_a and S_b are 180° rotations around the lines through $\{x_1, y_1\}$ and $\{x_q, y_q\}$, respectively, and the action of S_c is the 180° rotation about the z -axis. S_a and S_b generate the symmetries of a q -gon (the dotted polygon on the left in Fig. 13). Combined with the half-turn rotation of S_c (since q is odd), this generates the group of symmetries of the $2q$ -gon. So the inner automorphism group is isomorphic to D_{2q} .

Similarly, when q is even, we consider the Cayley graph from Fig. 12. Once again, we describe S_a, S_b and S_c as permutations of the vertices in the Cayley graph.

$$\begin{aligned}
 S_a &= (x_1)(x_2x_3)(x_4x_5) \cdots (x_{q-2}x_{q-1})(x_q) \cdot (y_1y_2)(y_3y_4) \cdots (y_{q-1}y_q) \cdot (z_1z_2), \\
 S_b &= (x_1x_2)(x_3x_4) \cdots (x_{q-1}x_q) \cdot (y_1)(y_2y_3)(y_4y_5) \cdots (y_{q-2}y_{q-1})(y_q) \cdot (z_1z_2), \\
 S_c &= (x_1x_q)(x_2x_{q-1}) \cdots (x_{q/2}x_{q/2+1}) \cdot (y_1y_q)(y_2y_{q-1}) \cdots (y_{q/2}y_{q/2+1}) \cdot (z_1)(z_2).
 \end{aligned}$$

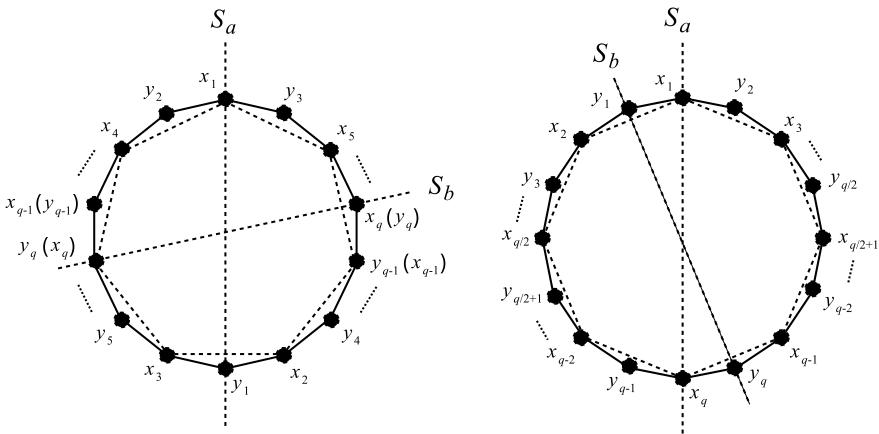


Fig. 13. Realizing S_a and S_b as symmetries for q odd (on left) and even (on right).

Table 5. The automorphism groups of $Q_2(T_{2,3} \cup A)$ and $Q_2(T_{2,3} \cup B)$.

Q_n	$ Q_n $	Aut	Inn	Trans
$Q_2(T_{2,q} \cup A)$	$2 + 2 q $	$\mathbb{Z}_2 \times (\mathbb{Z}_{2q} \rtimes \mathbb{Z}_{2q}^*)$	$D_{2q/\gcd(2,q)}$	D_q
$Q_2(T_{2,3} \cup B)$	18	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_4$	S_4	S_4

As before, we embed the vertices as the vertices of a $2q$ -gon embedded in the xy -plane and centered at the origin, as shown on the right in Fig. 13 (once again, z_1 and z_2 are embedded on the z -axis; $x_{q/2}$ is on the left side if $q/2$ is even, and on the right if $q/2$ is odd). As in the odd case, S_a and S_b generate the symmetries of the q -gon (the dotted polygon on the right in Fig. 13); however, in this case, the half-turn rotation S_c is already in this group of symmetries. So when q is even, the inner automorphism group is just D_q .

Since we are in an involutory quandle, each of S_a , S_b and S_c has order 2, and the transvection group is generated by their products $S_a S_b$, $S_a S_c$, and $S_b S_c$ (and their inverses). Regardless of whether q is odd or even, $S_a S_b$ is a rotation which generates the rotation subgroup of D_q , and $S_a S_c$ and $S_b S_c$ are reflections that have the same angle between their axes as S_a and S_b . So in both cases, these motions generate the symmetries of a q -gon, and the transvection group is isomorphic to D_q . \square

We summarize our results on automorphism groups of torus links with axes in Table 5.

Remark. Aside from the 2-bridge links $L_{p/q}$, for which the involutory quandle depends only on q , all of the links in this paper are distinguished by the finite n -quandles described. This can be easily seen by comparing the data in Tables 2, 3, 4 and 5.

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