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The Hadamard core of the totally nonnegative matrices☆

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Abstract

An *m*-by-*n* matrix *A* is called totally nonnegative if every minor of *A* is nonnegative. The Hadamard product of two matrices is simply their entry-wise product. This paper introduces the subclass of totally nonnegative matrices whose Hadamard product with any totally nonnegative matrix is again totally nonnegative. Many properties concerning this class are discussed including: a complete characterization for $\min\{m, n\} < 4$; a characterization of the zero–nonzero patterns for which all totally nonnegative matrices lie in this class; and connections to Oppenheim's inequality. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

The *Hadamard product* of two m-by-n matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is defined and denoted by

$$A \circ B = [a_{ij} b_{ij}].$$

The Hadamard product plays a substantial role within matrix analysis and in its applications (see, for example, [12, Chapter 5]). A matrix is called *totally positive*, TP (*totally nonnegative*, TN) if each of its minors is positive (nonnegative), see also [1,7,14]. This class arises in a long history of applications [10], and it has enjoyed increasing recent attention.

Some classes of matrices, such as the positive definite matrices, are closed under Hadamard multiplication (see [11, p. 458]), and given such closure, inequalities involving the Hadamard product, usual product, determinants and eigenvalues, etc. may be considered. For example, Oppenheim's inequality states that

$$\det(A \circ B) \geqslant \prod_{i=1}^{n} a_{ii} \det B$$

for any two n-by-n positive definite matrices $A = [a_{ij}]$ and B (see [11, p. 480]). Since Hadamard's inequality

$$\det A \leqslant \prod_{i=1}^{n} a_{ii}$$

also holds for positive definite matrices $A = [a_{ij}]$, it follows from Oppenheim that

$$\det(A \circ B) \geqslant \det(AB),$$

i.e., the Hadamard product dominates the usual product in determinant.

Unfortunately, it has long been known (see also [13,16]) that TN matrices are not closed under Hadamard multiplication; e.g., for

$$W = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad W^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \tag{1}$$

W is TN, but

$$W \circ W^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

is not. Similarly, TP is not Hadamard closed. Not surprisingly then inequalities such as Oppenheim's do not generally hold for TP or TN matrices. However, there has been interest in significant subclasses of the TP or TN matrices that are Hadamard closed, i.e., are such that arbitrary Hadamard products from them are TP or TN. Some of these subclasses include tridiagonal TN matrices, inverses of tridiagonal M-matrices, nonsingular totally nonnegative Routh–Hurwitz matrices, certain Vandermonde matrices, etc.; discussion of such classes may be found in [8,9,15–17,19].

Our interest here is similar but in a different direction: what may be said about those special TN matrices whose Hadamard product with *any* TN matrix is TN? Thus, we define the *Hadamard core* of the m-by-n TN matrices, $\mathrm{CTN}_{m,n}$, as follows:

$$\mathrm{CTN}_{m,n} = \{A \in \mathrm{TN} : B \in \mathrm{TN} \Rightarrow A \circ B \in \mathrm{TN}\}.$$

When the dimensions are clear from the context we may delete the dependence on m and n. It is a simple exercise that for $\min\{m, n\} \leqslant 2$, CTN $=$ TN, but as indicated by the nonclosure, CTN is properly contained in TN otherwise ($\min\{m, n\} > 2$). The Hadamard core of TP may be similarly defined, but, as its theory is not substantially different (because TN is the closure of TP), we do not discuss it here.

We first begin to describe CTN and are able to give a complete description when $\min\{m, n\} < 4$. Interestingly, perhaps the simplest description is via two test matrices, and we raise the question as to whether there is a finite set of test matrices in general. Surprisingly the core seems rather large. We also characterize the zero–nonzero patterns for which every TN matrix lies in the core. This gives insight into the core in general, as, for example, any tridiagonal TN matrix lies in the core. One motivation for considering the core is that we are able to show that Oppenheim's inequality does hold when, in addition to B being TN, A lies in the core. The proof requires noting facts about certain "retractibility" properties of TN matrices (see [5]), that are of independent interests. This work naturally raises further questions, some of which we mention at the conclusion.

2. Preliminaries and background

The set of all m-by-n matrices with real entries will be denoted by $M_{m,n}$, and if $m = n$, $M_{n,n}$ will be abbreviated to M_n. For $A \in M_{m,n}$ the notation $A = [a_{ij}]$ will indicate that the entries of A are $a_{ij} \in \mathbb{R}$, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The transpose of a given m-by-n matrix A will be denoted by A^{T}. For $A \in M_{m,n}$, $\alpha \subseteq \{1, 2, \ldots, m\}$, and $\beta \subseteq \{1, 2, \ldots, n\}$, the submatrix of A lying in rows indexed by α and the columns indexed by β will be denoted by $A[\alpha|\beta]$. Similarly, $A(\alpha|\beta)$ is the submatrix obtained from A by deleting the rows indexed by α and columns indexed by β. If $A \in M_n$ and $\alpha = \beta$, then the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$, and the complementary principal submatrix is $A(\alpha)$. If $x = [x_i] \in \mathbb{R}^n$, then we let $\mathrm{diag}(x_i)$ denote the n-by-n diagonal matrix with main diagonal entries x_i. We begin with some simple yet useful properties concerning matrices in CTN.

Proposition 2.1. *Suppose A and B are two m-by-n matrices in the Hadamard core. Then $A \circ B$, the Hadamard product of A and B, is in the Hadamard core.*

Proof. Let C be any m-by-n TN matrix. Then $B \circ C$ is TN since B is in CTN. Hence $A \circ (B \circ C)$ is TN. But $A \circ (B \circ C) = (A \circ B) \circ C$. Thus $A \circ B$ is in CTN, since C was arbitrary. □

Note that if $D = [d_{ij}]$ is a diagonal matrix, then $\det DA[\alpha|\beta] = \det D[\alpha] \det A[\alpha|\beta]$. Hence if A is TN, then DA is TN, for every entry-wise nonnegative (and hence totally nonnegative) diagonal matrix D. Moreover, observe that $D(A \circ B) = DA \circ B = A \circ DB$, from which it follows that DA is in CTN whenever D is a TN diagonal matrix and A is in CTN. The above facts aid in the proof of the following proposition.

Proposition 2.2. *Any rank one totally nonnegative matrix lies in the Hadamard core.*

Proof. Let A be a rank one TN matrix, say $A = xy^{\mathrm{T}}$, in which $x = [x_i] \in \mathbb{R}^m$ and $y = [y_i] \in \mathbb{R}^n$ are entry-wise nonnegative vectors. Let $D = \mathrm{diag}(x_i)$ and $E = \mathrm{diag}(y_i)$. Then it is easy to show that $A = DJE$. (Observe that $J = ee^{\mathrm{T}}$, in which e is a vector of ones of appropriate size. Then $DJE = D(ee^{\mathrm{T}})E = (De)(e^{\mathrm{T}}E) = xy^{\mathrm{T}} = A$.) Since J is in CTN, we have that DJE is in CTN, in other words A is in CTN. □

Note that the example given in (1) implies that not all rank two TN matrices are in CTN, and in fact by direct summing the matrix A in (1) with an identity matrix follows that there exist TN matrices of all ranks greater than one that are not in CTN. We now note a very useful fact concerning an inheritance property for matrices in CTN.

Proposition 2.3. *If an m-by-n totally nonnegative matrix A lies in the Hadamard core, then every submatrix of A is in the corresponding Hadamard core.*

Proof. Suppose there exists a submatrix, say $A[\alpha|\beta]$, that is not in CTN. Then there exists a TN matrix B such that $A[\alpha|\beta] \circ B$ is not TN. Embed B into an m-by-n matrix $C = [c_{ij}]$ such that $C[\alpha|\beta] = B$, and $c_{ij} = 0$ otherwise. It is not difficult to show that C is TN, since any minor that does not lie in rows contained in α and columns contained in β is necessarily zero. Now consider $A \circ C$. Since $A[\alpha|\beta] \circ B$ is a submatrix of $A \circ C$ and $A[\alpha|\beta] \circ B$ is not TN, we have that $A \circ C$ is not TN. This completes the proof. □

The next result deals with the set of column vectors that can be inserted into a given matrix in CTN in such a way so that the resulting matrix remains in CTN. We say that a column m-vector v is *inserted in column* k ($k = 1, 2, \dots, n, n+1$) of an m-by-n matrix $A = [b_1, b_2, \dots, b_n]$, with columns $b_1, b_2, \dots, b_n$, if we obtain the new m-by-$(n+1)$ matrix of the form $[b_1, \dots, b_{k-1}, v, b_k, \dots b_n]$.

Proposition 2.4. *The set of columns (or rows) that can be inserted into an m-by-n TN matrix in the Hadamard core so that the resulting matrix remains in the Hadamard core is a nonempty convex set.*

Proof. Suppose A is an m-by-n TN matrix in CTN. Let S denote the set of columns that can be inserted into A so that the new matrix remains in CTN. It is easy to verify that $0 \in S$, hence $S \neq \emptyset$. We verify the second claim only in the case of inserting column vectors in position $n+1$, i.e., bordering A. The argument is similar for all other insertion positions. Let $x, y \in S$. Then the augmented matrices $[A|x]$ and $[A|y]$ are both in CTN. Suppose $t \in [0, 1]$ and consider the matrix $[A|tx + (1-t)y]$. Let $[B|z]$ be any m-by-$(n+1)$ TN matrix. Then

$$[A|tx + (1-t)y] \circ [B|z] = [A \circ B|t(x \circ z) + (1-t)(y \circ z)].$$

Since A is in CTN any submatrix of $A \circ B$ is TN. Therefore we only need to consider the submatrices of $[A|tx + (1-t)y] \circ [B|z]$ that involve column $n+1$. Let $[A'|tx' + (1-t)y'] \circ [B'|z']$ denote any such square submatrix of $[A|tx + (1-t)y] \circ [B|z]$. Then

$$\begin{aligned}
&\det([A'|tx' + (1-t)y'] \circ [B'|z']) \\
&\quad = \det([A' \circ B'|t(x' \circ z')]) + \det([A' \circ B'|(1-t)(y' \circ z')]) \\
&\quad = t\det([A' \circ B'|x' \circ z']) + (1-t)\det([A' \circ B'|y' \circ z']) \\
&\quad = t\det([A'|x'] \circ [B'|z']) + (1-t)\det([A'|y'] \circ [B'|z']) \geqslant 0,
\end{aligned}$$

since both $[A|x]$ and $[A|y]$ are in CTN. This completes the proof. □

An n-by-n matrix $A = [a_{ij}]$ is said to be a *tridiagonal matrix* if $a_{ij} = 0$ whenever $|i - j| > 1$. A nonobvious, but well-known fact is the next proposition which can be found in [7], where tridiagonal matrices are referred to as Jacobi matrices (see also [4] for a new proof of this fact).

Proposition 2.5 [7, p. 143]. *Let T be an n-by-n tridiagonal matrix. Then T is totally nonnegative if and only if T is an entry-wise nonnegative matrix with nonnegative principal minors.*

An n-by-n matrix A with nonpositive off-diagonal entries is called a (possibly singular) M-matrix if the principal minors of A are nonnegative (see [2, p. 149] or [6, p. 391]). An n-by-n matrix $C = [c_{ij}]$ is said to be *row diagonally dominant* if $|c_{ii}| \geqslant \sum_{j \neq i} |c_{ij}|$ for $i = 1, 2, \ldots, n$. Observe that if an M-matrix has nonnegative row sums, then it is row diagonally dominant. Keeping this observation in mind, Fiedler and Ptak essentially proved that A is an irreducible (possibly singular) M-matrix if and only if there exists a positive diagonal matrix D such that DAD^{-1} is row diagonally dominant (see [6, (5.8), (6.8)]). We are now in a position to extend a result of Markham [16] (see also [9]) concerning the Hadamard product of tridiagonal matrices.

Theorem 2.6. *Let T be an n-by-n totally nonnegative tridiagonal matrix. Then T is in the Hadamard core.*

Proof. It is enough to prove this result for the case in which T is irreducible, otherwise apply the following argument to each irreducible block and use the simple structure of a tridiagonal matrix. Let B be an arbitrary n-by-n TN matrix. Similarly we may assume B is irreducible, which implies $b_{ij} > 0$ for all i, j such that $|i - j| \leqslant 1$, i.e., B has positive "tri-diagonal part" (see [7, p. 139] and [4]). Since pre- and post-multiplication by positive diagonal matrices does not affect the property of being TN or whether or not a matrix is in CTN, we may assume that $b_{ii} = 1$ for $i = 1, 2, \ldots, n$ and that $b_{ij} = b_{ji}$ for all i, j with $|i - j| = 1$. Notice that if $S = \mathrm{diag}(1, -1, 1, -1, \ldots, \pm 1)$, then STS has nonpositive off-diagonal entries, and since T is TN, it follows that STS is a (possibly singular) M-matrix. Moreover, there exists a positive diagonal matrix D such that $DSTSD^{-1} = S(DTD^{-1})S$ is a row diagonally dominant matrix (see remarks preceding Theorem 2.6). Let $C = [c_{ij}] = S(DTD^{-1})S \circ B = S(DTD^{-1} \circ B)S$. Since B is TN with $b_{ii} = 1$ and $b_{ij} = b_{ji}$ whenever $|i - j| = 1$, it follows that $0 < b_{ij} \leqslant 1$ for all i, j with $|i - j| = 1$. Hence $DTD^{-1} \circ B$ is row diagonally dominant. Since $DTD^{-1} \circ B$ is tridiagonal, $S(DTD^{-1} \circ B)S$ has nonpositive off-diagonal entries, which implies $S(DTD^{-1} \circ B)S$ is a (possibly singular) M-matrix. Therefore $DTD^{-1} \circ B$ is an entry-wise nonnegative tridiagonal matrix with nonnegative principal minors. Hence, by Proposition 2.5, $DTD^{-1} \circ B$ is a TN matrix, and hence $T \circ B$ is a TN matrix. Thus T is in CTN. □

We obtain a result of Markham [16] (see also [9]) as a special case.

Corollary 2.7. *The Hadamard product of any two n-by-n tridiagonal totally nonnegative matrices is again totally nonnegative.*

3. Description of the core for min{m, n} < 4

The analysis of CTN in the 3-by-3 case differs significantly from the 2-by-2 case, and, unfortunately, unlike the 2-by-2 case, not all 3-by-3 totally nonnegative matrices are in the Hadamard core. Recall from (1) that the matrix

$$W = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

is not a member of CTN. We will see that W plays an important role in describing CTN. We begin our analysis of CTN with a preliminary lemma concerning a special class of 3-by-3 totally nonnegative matrices in CTN, that will aid the proof of the main result to follow.

Lemma 3.1. *Let*

$$A = \begin{bmatrix} 1 & 1 & a \\ 1 & 1 & a \\ a & a & 1 \end{bmatrix}.$$

Then A is in the Hadamard core if and only if A is totally nonnegative.

Proof. The necessity follows since CTN is always contained in TN. To verify sufficiency suppose A is TN. Let $B = [b_{ij}]$ be any 3-by-3 TN matrix. By virtue of the 2-by-2 case it is enough to show that $\det(A \circ B) \geqslant 0$. We make use of Sylvester's identity for determinants (see [11, p. 22]). Note that we may assume that $b_{22} > 0$, otherwise B is reducible in which case verification of $\det(A \circ B) \geqslant 0$ is trivial. Using Sylvester's identity we see that $\det B \geqslant 0$ is equivalent to

$$\frac{(b_{11}b_{22} - b_{12}b_{21})(b_{22}b_{33} - b_{23}b_{32})}{b_{22}} \geqslant \frac{(b_{12}b_{23} - b_{22}b_{13})(b_{21}b_{32} - b_{31}b_{22})}{b_{22}}.$$

Since A is TN, $0 \leqslant a \leqslant 1$. Observe that

$$\begin{aligned}
&\frac{(b_{11}b_{22} - b_{12}b_{21})(b_{22}b_{33} - b_{23}b_{32}a^2)}{b_{22}} \\
&\quad\geqslant \frac{(b_{11}b_{22} - b_{12}b_{21})(b_{22}b_{33} - b_{23}b_{32})}{b_{22}}, \text{ since } 0 \leqslant a \leqslant 1 \\
&\quad\geqslant \frac{(b_{12}b_{23} - b_{22}b_{13})(b_{21}b_{32} - b_{31}b_{22})}{b_{22}}, \text{ since } \det B \geqslant 0 \\
&\quad\geqslant a^2 \frac{(b_{12}b_{23} - b_{22}b_{13})(b_{21}b_{32} - b_{31}b_{22})}{b_{22}}, \text{ since } 0 \leqslant a \leqslant 1.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{(b_{11}b_{22} - b_{12}b_{21})(b_{22}b_{33} - b_{23}b_{32}a^2)}{b_{22}} \\
&\quad\geqslant a^2 \frac{(b_{12}b_{23} - b_{22}b_{13})(b_{21}b_{32} - b_{31}b_{22})}{b_{22}},
\end{aligned}$$

which implies $\det(A \circ B) \geqslant 0$, and hence A is in CTN. □

A similar conclusion holds (as in Lemma 3.1) for TN matrices of the form

$$\begin{bmatrix} 1 & a & a \\ a & 1 & 1 \\ a & 1 & 1 \end{bmatrix}.$$

The next two lemmas are verified separately from the main result to reduce the number of cases needed to prove the main result. The first is concerned with verifying a necessary condition for singular TN matrices to belong in the Core, while the second lemma reduces the analysis of describing elements in the Core to entry-wise positive TN matrices.

Lemma 3.2. *Let A be a* 3*-by-*3 *singular totally nonnegative matrix. If $A \circ W$ and $A \circ W^{\mathrm{T}}$ are both totally nonnegative, then A is in the Hadamard core.*

Proof. In light of the 2-by-2 case we may assume that A is irreducible. Moreover, up to positive diagonal equivalence we may also assume A is in the following form:

$$A = \begin{bmatrix} 1 & a & c \\ a & 1 & b \\ d & b & 1 \end{bmatrix}.$$

Since A is singular, $\det A = 1 + abc + abd - a^2 - b^2 - cd = 0$ or $1 + abc + abd = a^2 + b^2 + cd$. By hypothesis, $A \circ W$ and $A \circ W^{\mathrm{T}}$ are both totally nonnegative, hence $\det(A \circ W) = 1 + abd - a^2 - b^2 \geqslant 0$, and $\det(A \circ W^{\mathrm{T}}) = 1 + abc - a^2 - b^2 \geqslant 0$. Since $1 + abc + abd - a^2 - b^2 - cd = 0$ and $ab \geqslant c$, $d \geqslant 0$ (A is TN) it follows that equality must hold in $1 + abd - a^2 - b^2 \geqslant 0$. Similarly, equality holds for $1 + abc - a^2 - b^2 \geqslant 0$. This gives rise to one of the following four cases: (1) $c = 0$, and $ab = c$; (2) $c = 0$, and $d = 0$; (3) $d = 0$, and $ab = d$; (4) $ab = d$, and $ab = c$. Suppose B is an arbitrary 3-by-3 TN matrix, as with A, we may assume that B has the following form:

$$B = \begin{bmatrix} 1 & \alpha & \gamma \\ \alpha & 1 & \beta \\ \delta & \beta & 1 \end{bmatrix}.$$

Observe that cases (1) and (3) cannot occur since A was assumed to be irreducible. In case (2) A is tridiagonal, and hence is in CTN by Theorem 2.6. Finally, consider case (4). Then $\det A = 1 + (ab)^2 - a^2 - b^2 = (1 - a^2)(1 - b^2) = 0$. Therefore either $a = 1$ or $b = 1$. In either case A is of the form in Lemma 3.1 (or the remark after Lemma 3.1) and hence is in CTN. □

Lemma 3.3. *Let A be a* 3*-by-*3 *totally nonnegative matrix with at least one zero entry. If $A \circ W$ and $A \circ W^{\mathrm{T}}$ are both totally nonnegative, then A is in the Hadamard core.*

Proof. It is enough to show that $\det(A \circ B) \geqslant 0$, for any TN matrix B. If $a_{ij} = 0$ for some i, j with $|i - j| \leqslant 1$, then A is reducible and the result follows. So assume either $a_{13} = 0$ or $a_{31} = 0$. If they are both zero, then A is a tridiagonal TN matrix and hence is in CTN, by Theorem 2.6. Thus assume, without loss of generality, that $a_{31} = 0$. In this case observe that $A \circ W^{\mathrm{T}} = A$, and $A \circ W = T$, in which T is a tridiagonal matrix. By hypothesis, T is TN, and therefore T is in CTN (Theorem 2.6). Moreover, $\det(A \circ B) \geqslant \det(T \circ B) \geqslant 0$ (the first inequality follows since A and B are TN, and the second inequality follows since T is in CTN). This completes the proof. □

We are now in a position to characterize all 3-by-3 TN matrices in the Hadamard core.

Theorem 3.4. *Let A be a* 3*-by-*3 *totally nonnegative matrix. Then A is in the Hadamard core if and only if $A \circ W$ and $A \circ W^{\mathrm{T}}$ are both totally nonnegative.*

Proof. The necessity is clear since W (and hence W^{T}) is TN. Assume that $A \circ W$ and $A \circ W^{\mathrm{T}}$ are both TN. By Lemmas 3.2 and 3.3 it suffices to assume that A is nonsingular and entry-wise positive. As was the case with the previous lemmas to show A is in CTN it is enough to verify that $\det(A \circ B) \geqslant 0$, for any TN matrix B. Before we proceed with the argument presented here we need the following simple and handy fact concerning TN matrices: increasing the $(1, 1)$ or (m, n) entry of an m-by-n TN matrix yields a TN matrix. Using this fact and (possibly) diagonal scaling it follows that any entry-wise positive nonsingular TN matrix can be written in the following form:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1+p & 1+p+q \\ 1 & 1+p+r & 1+s \end{bmatrix},$$

with $p, s > 0$ and $q, r \geqslant 0$ chosen accordingly, and up to transposition we may assume that $q \geqslant r$. Then, using this form for A, we have that

$$\det(A \circ W) \geqslant 0 \iff ps - p^2 - pr - pq - qr \geqslant r,$$

and

$$\det(A \circ W^{\mathrm{T}}) \geqslant 0 \iff ps - p^2 - pr - pq - qr \geqslant q.$$

The above two conditions are equivalent to

$$ps - (p^2 + pr + pq + qr) = ps - (p+q)(p+r) \geqslant q (\geqslant r).$$

Hence $s \geqslant ((p+q)(p+r) + q)/p$. Since s enters positively into $\det A$ and $\det(A \circ B)$, for any TN matrix B we can assume that equality holds, i.e., $s = ((p+q)(p+r) + q)/p$. Now assume that B is any 3-by-3 TP matrix that is of the form (similar to A)

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1+t & 1+t+u \\ 1 & 1+t+v & 1+w \end{bmatrix},$$

in which $0 < t, u, v, w$ are suitably chosen. Since w enters positively into $\det B$ and $\det(A \circ B)$ it is enough to prove $\det(A \circ B) \geqslant 0$ when w is chosen as small as possible, namely, $w = ((t+v)(t+u))/t$ (in which case $\det B = 0$). Now consider the matrix $A \circ B$ with the specified choices of s and w above. A routine computation reveals that

$$\begin{aligned}\det(A \circ B) = {} & u(q-r) \\ & + \frac{1}{pt}(qpuv + t^2 qpv + t^2 qpu + t^2 qru + tquv\end{aligned}$$

$$
\begin{aligned}
&+ p^2ruv + p^2quv + tqp + p^2uv + 2qpt^2 + qrt^2 \\
&+ t^3qp + t^3qr + t^2qv + t^2qv + t^2qu \\
&+ p^3uv + qt^2 + t^3q + tqpuv + tqruv + pqruv) \\
\geqslant\ & 0, \text{ since } q \geqslant r, \text{ by assumption.}
\end{aligned}
$$

Hence $A \circ B$ is TN for all TP matrices B (the 2-by-2 submatrices are necessarily TN). The fact that $A \circ B$ is TN for all 3-by-3 TN matrices B follows by a routine continuity argument since any TN matrix is the limit of TP matrices (see [1]). □

We now present some useful variations upon and consequences of Theorem 3.4.

Corollary 3.5. *Let $A = [a_{ij}]$ be a 3-by-3 totally nonnegative matrix. Then A is in the Hadamard core if and only if*

$$
\begin{aligned}
&a_{11}a_{22}a_{33} + a_{31}a_{12}a_{23} \geqslant a_{11}a_{23}a_{32} + a_{21}a_{12}a_{33}, \\
&a_{11}a_{22}a_{33} + a_{21}a_{13}a_{32} \geqslant a_{11}a_{23}a_{32} + a_{21}a_{12}a_{33}.
\end{aligned}
$$

Example 3.6 [*Polya matrix*]. Let $q \in (0, 1)$. Define the n-by-n Polya matrix Q whose (i, j)th entry is equal to q^{-2ij}. Then it is well known (see [20]) that Q is totally positive for all n (in fact Q is diagonally equivalent to a TP Vandermonde matrix). Suppose Q represents the 3-by-3 Polya matrix. We wish to determine when (if ever) Q is in CTN. By Corollary 3.5 and the fact that Q is symmetric, Q is in CTN if and only if $q^{-28} + q^{-22} \geqslant q^{-26} + q^{-26}$, which is equivalent to $q^{-28}(1 - q^2 - q^2(1 - q^4)) \geqslant 0$. This inequality holds if and only if $1 - q^2 \geqslant q^2(1 - q^4) = q^2(1 - q^2)(1 + q^2)$. Thus q must satisfy $q^4 + q^2 - 1 \leqslant 0$. It is easy to check that the inequality holds for $q^2 \in (0, 1/\mu)$, where $\mu = (1 + \sqrt{5})/2$ (the golden mean). Hence Q is in CTN for all $q \in (0, \sqrt{1/\mu})$.

Corollary 3.7. *Let $A = [a_{ij}]$ be a 3-by-3 totally nonnegative matrix. Suppose $B = [b_{ij}]$ is the unsigned classical adjoint matrix. Then A is in the Hadamard core if and only if $a_{11}b_{11} - a_{12}b_{12} \geqslant 0$, and $a_{11}b_{11} - a_{21}b_{21} \geqslant 0$; or, equivalently,*

$$a_{11} \det A[\{2, 3\}] - a_{12} \det A[\{2, 3\}|\{1, 3\}] \geqslant 0,$$

and

$$a_{11} \det A[\{2, 3\}] - a_{21} \det A[\{1, 3\}|\{2, 3\}] \geqslant 0.$$

Even though Corollary 3.7 is simply a recapitulation of Corollary 3.5, the conditions rewritten in the above form aid in the proof of the next fact. Recall that if A is a nonsingular TN matrix, then $SA^{-1}S$ is a TN matrix, in which $S = \operatorname{diag}(1, -1, 1, -1, \dots, \pm 1)$ (see, e.g., [7, p. 109]).

Theorem 3.8. *Suppose A is a* 3*-by-*3 *nonsingular TN matrix in the Hadamard core. Then* $SA^{-1}S$ *is in the Hadamard core.*

Proof. Observe that $SA^{-1}S$ is TN and, furthermore $SA^{-1}S = (1/\det A)B$, where $B = [b_{ij}]$ is the unsigned classical adjoint of A. Hence $SA^{-1}S$ is in CTN if and only if B is a member of CTN. Observe that the inequalities in Corollary 3.7 are symmetric in the corresponding entries of A and B. Thus B is in CTN. This completes the proof. □

Corollary 3.9. *Let A be a* 3*-by-*3 *totally nonnegative matrix whose inverse is tridiagonal. Then A is in the Hadamard core.*

Proof. Proof follows from Theorems 2.6 and 3.8. □

Gantmacher and Krein [7] proved that the set of all inverse tridiagonal totally nonnegative matrices is closed under Hadamard multiplication. (In the symmetric case, which can be assumed without loss of generality, an inverse tridiagonal matrix is often called a Green's matrix as was the case in [7,8].) The above result strengthens this fact in the 3-by-3 case. However, it is not true in general that inverse tridiagonal totally nonnegative matrices are contained in CTN. For $n \geqslant 4$, CTN does not enjoy the "inverse closure" property as in Theorem 3.8. Consider the following example.

Example 3.10. Let

$$A = \begin{bmatrix} 1 & a & ab & abc \\ a & 1 & b & bc \\ ab & b & 1 & c \\ abc & bc & c & 1 \end{bmatrix},$$

where $a, b, c > 0$ are chosen so that A is positive definite. Then it is easy to check that A is TN, and the inverse of A is tridiagonal. Consider the upper right 3-by-3 submatrix of A, namely

$$M = \begin{bmatrix} a & ab & abc \\ 1 & b & bc \\ b & 1 & c \end{bmatrix},$$

which is TN. By Proposition 2.3, if A is in CTN, then M is in CTN. However, $\det(M \circ W) = abc(b^2 - 1) < 0$, since $b < 1$. Thus A is not in CTN.

For $3 \leqslant k \leqslant n$, let $W^{(k)} = (w_{ij}^{(k)})$ be the 3-by-n totally nonnegative matrix consisting of entries:

$$w_{ij}^{(k)} = \begin{cases} 0 & \text{if } i = 1,\ j \geqslant k, \\ 1 & \text{otherwise.} \end{cases}$$

For $1 \leqslant k \leqslant n-2$, let $U^{(k)} = (u_{ij}^{(k)})$ be the 3-by-n totally nonnegative matrix consisting of entries:

$$u_{ij}^{(k)} = \begin{cases} 0 & \text{if } i=3,\ 1 \leqslant j \leqslant k, \\ 1 & \text{otherwise.} \end{cases}$$

For example, if $n=5$ and $k=3$, then

$$W^{(3)} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

and

$$U^{(3)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Theorem 3.11. *Let A be a* 3*-by-n* ($n \geqslant 3$) *totally nonnegative matrix. Then A is in the Hadamard core if and only if* $A \circ W^{(k)}$ *is totally nonnegative for* $3 \leqslant k \leqslant n$ *and* $A \circ U^{(j)}$ *is totally nonnegative for* $1 \leqslant j \leqslant n-2$.

Proof. The necessity is obvious, since $W^{(k)}$and $U^{(j)}$ are both TN. Observe that it is enough to show that every 3-by-3 submatrix of A is in CTN, by Proposition 2.3. Let B be any 3-by-3 submatrix of A. Consider the matrices $A \circ W^{(k)}$ and $A \circ U^{(j)}$ for $3 \leqslant k \leqslant n$ and $1 \leqslant j \leqslant n-2$. By hypothesis $A \circ W^{(k)}$ and $A \circ U^{(j)}$ are TN. Hence by considering appropriate submatrices, it follows that $B \circ W$ and $B \circ W^{\mathrm{T}}$ are both TN. Therefore B is in CTN by Theorem 3.4. Thus A is in CTN. □

Of course by transposition, we may obtain a similar characterization of CTN in the n-by-3 case. At present no characterization of the Hadamard core for 4-by-4 totally nonnegative matrices is known, but we offer some ideas and conjectures on this issue in Section 6.

4. Patterns for which all TN matrices lie in the core

In this section we consider zero–nonzero patterns (which in our case will always be zero-positive (or $(0,+)$)-patterns) of totally nonnegative matrices in the Hadamard core. Recall that an m-by-n $(0,+)$-*sign pattern* is an m-by-n array of symbols chosen from $\{+,0\}$, and a *realization of a sign pattern*, S, is a real m-by-n matrix A such that:

$$a_{ij} > 0 \text{ when } s_{ij} = +, \quad \text{and} \quad a_{ij} = 0 \text{ when } s_{ij} = 0.$$

There are two natural mathematical notions associated with various sign-pattern problems. They are the notions of *require* and *allow*. We say an m-by-n sign pattern S

requires property P if every realization of S has property P. On the other hand we say a sign pattern S *allows property P* if there exists a realization of S with property P. We begin our analysis here by completely characterizing all the sign patterns S that require a TN matrix to be in the Hadamard core of the totally nonnegative matrices.

Definition 4.1. Given an m-by-n sign pattern S, that allows TN, we say that *S requires Hadamard coreness of a TN matrix* if any totally nonnegative realization of S is in the Hadamard core.

Observe that in order for a given sign pattern, S to require Hadamard coreness, it is necessary that S be in double echelon form described below. In the following definition and throughout this paper the symbol $*$ in a matrix means the corresponding entry is nonzero.

Definition 4.2. An m-by-n matrix A with no zero rows or columns is said to be *in double echelon form* if:

(i) Each row of A has one of the following forms:
   1. $(*, *, \ldots, *)$,
   2. $(*, \ldots, *, 0, \ldots, 0)$,
   3. $(0, \ldots, 0, *, \ldots, *)$ or
   4. $(0, \ldots, 0, *, \ldots, *, 0, \ldots, 0)$.

(ii) The first and last nonzero entries in row $i+1$ are not to the left of the first and last nonzero entries in row i, respectively ($i = 1, 2, \ldots, m-1$).

Thus, a matrix in double echelon form appears as follows:

$$\begin{bmatrix} * & * & 0 & \cdots & 0 \\ * & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * & * \end{bmatrix}.$$

It is not difficult to see that any TN matrix with no zero rows or columns must be in double echelon form (see also [7]). We say that a $(0, +)$-pattern S is in double echelon form if every realization of S is in double echelon form (i.e., S requires matrices to be in double echelon form).

Example 4.3. It is an easy exercise to show that any 1-by-1 or 2-by-2 sign pattern in double echelon form requires Hadamard coreness of a TN matrix. We denote the following 3-by-3 sign patterns as:

$$F = \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix}, \quad W = \begin{bmatrix} + & + & 0 \\ + & + & + \\ + & + & + \end{bmatrix} \quad \text{or} \quad W^{\mathrm{T}} = \begin{bmatrix} + & + & + \\ + & + & + \\ 0 & + & + \end{bmatrix}.$$

Then any 3-by-3 double echelon sign pattern other than F, W or W^{T} requires Hadamard coreness of a TN matrix. To verify this, first observe that by the example in (1) and Example 3.6 there exist matrices with the above sign patterns that are not in CTN. Thus, suppose S is a 3-by-3 sign pattern different from the three patterns above. Then, S is either reducible or a tridiagonal pattern (with possibly more zeros), and hence S requires Hadamard coreness of a TN matrix (the latter following from Theorem 2.6).

Lemma 4.4. *Suppose A is an m-by-n totally nonnegative matrix with no zero rows or columns, and let $\mathscr{X}$ be any 1-by-n sign pattern. Then $[\frac{A}{\mathscr{X}}]$ allows TN if and only if $[\frac{A}{\mathscr{X}}]$ is in double echelon form.*

Proof. The above condition is obviously necessary. Suppose $[\frac{A}{\mathscr{X}}]$ is in double echelon form. Assume that $\mathscr{X}$ is in following form: $\mathscr{X} = [0\ \cdots\ 0,\ \ +\ \cdots\ +,\ \ 0\ \cdots\ 0]$, in which the plus signs span columns j to $j+k \leqslant n$. Observe that if $j+k<n$, then columns $j+k+1, \ldots, n$ of A must be all zero columns since $[\frac{A}{\mathscr{X}}]$ is in double echelon form. Thus, since removal of zero columns does not change total nonnegativity, it is enough to prove this lemma for the case $j+k=n$. Hence, $\mathscr{X} = [0\ \cdots\ 0,\ \ +\ \cdots\ +]$, in which the first plus sign occurs in the jth column. Let $x = [x_i]$ be a realization of $\mathscr{X}$ to be determined, and let $C = [\frac{A}{x}]$. We will choose values for $x_i,\ i \geqslant j$ sequentially. It is not difficult to see that we may choose x_j positive so that

$$C\left[\{1,2,\ldots,m+1\}|\{1,\ldots,j\}\right] = \begin{bmatrix} A\left[\{1,2,\ldots,m\}|\{1,\ldots,j\}\right] \\ 0\ \cdots\ 0,\quad x_j \end{bmatrix}$$

is TN. Applying similar arguments (since x_{j+1} is in the bottom right entry of the corresponding matrix), we may choose x_{j+1} large enough so that $C[\{1,2,\ldots,m+1\}|\{1,\ldots,j+1\}]$ is TN. Continuing in this manner until we choose x_n such that $C = [\frac{A}{x}]$ is TN. Observe that at each stage, $x_i\ (i \geqslant j)$ enters positively into each minor that includes x_i, and there is no upper bound for the choice of x_i. This completes the proof. $\square$

It is well known (see [1]) that if A is TN, then A^{T} and the matrix obtained from A by reversing $(i \to n-i+1)$ the rows and the columns are both TN. This simple observation along with Lemma 4.4 implies the next result.

Corollary 4.5. *Suppose A is an m-by-n totally nonnegative matrix with no zero rows or columns, and let $\mathscr{X}$ be any 1-by-n sign pattern. Then $[A|\mathscr{X}^{\mathrm{T}}]$, $[\mathscr{X}^{\mathrm{T}}|A]$ or $[\frac{\mathscr{X}}{A}]$ allows TN if and only if each is in double echelon form.*

Theorem 4.6. *Let S be an n-by-n* $(0, +)$*-pattern with no zero rows or columns. Then*, *S requires Hadamard coreness of a TN matrix if and only if S is in double echelon form and does not contain any one of the sign patterns* F, W *or* W^{T} *as a subpattern.*

Proof. Suppose S is in double echelon form with W as a subpattern. The analysis is similar for the other two patterns. First, observe that we may assume this subpattern occurs as a contiguous pattern (i.e., based on consecutive rows and columns), since S is in double echelon form. Suppose this 3-by-3 subpattern is indexed by rows $j, j+1, j+2$ of S. Let B be a 3-by-3 totally nonnegative matrix with sign pattern W that is not in CTN (recall the example in (1)). By Lemma 4.4 and Corollary 4.5, extend B to a 3-by-n TN matrix $\bar{B}$ such that the sign pattern of $\bar{B}$ equals the sign pattern in rows $j, j+1, j+2$ of S. Now, apply Lemma 4.4 and Corollary 4.5 to construct an n-by-n TN matrix $\tilde{B}$, from $\bar{B}$, with sign pattern S. However, $\tilde{B}$ is not in CTN since $\tilde{B}$ contains a submatrix that is not in CTN (see Proposition 2.3).

On the other hand suppose S is in double echelon form and does not contain F, W or W^{T} as a subpattern. We proceed by using induction on n. This claim has already been verified for $n \leqslant 3$ (see Example 4.3), so assume the result is true for all such patterns of size less than or equal to $n-1$. Let S be as assumed above. Observe that, by induction, any TN realization of S has all of its proper submatrices in CTN. Thus, we only need to verify that $\det(A \circ B) \geqslant 0$, where A is any realization of S and B is TN. We consider three cases:

Case 1. Suppose the ith diagonal entry of S is zero for some $i = 1, 2, \ldots, n$. Then, S contains a zero block of size $n - i + 1 + i = n + 1$. Hence, $A \circ B$ has a zero block of size $n + 1$ for any realization A of S. But, in this case, $\det(A \circ B) = 0$ (see [18]). Thus, A is in CTN.

Case 2. Suppose S has positive main diagonal entries, but that some entry on the superdiagonal is zero (similar arguments hold if an entry on the subdiagonal is zero). Assume the $(i, i+1)$st entry of S is zero for some $i = 1, 2, \ldots, n-1$. Since S has positive main diagonal entries in addition to being in double echelon form, it follows that S contains a block of zeros of size $n - i + i = n$. Hence, S is block triangular, and by induction, we have $\det(A \circ B) \geqslant 0,$ for any realization A of S and B is TN.

Case 3. Finally, suppose S has positive main, super, and subdiagonal entries. Since S does not contain any of the three subpatterns (by assumption), it follows that the $(i, i+2)$ and $(i+2, i)$ entries of S must be zero for $i = 1, \ldots, n-2$. Since S is in double echelon form, it follows that S is a tridiagonal pattern. Thus, any realization A of S is in CTN by Theorem 2.6. □

Note that if A is m-by-n with $n \geqslant m$ (without loss of generality), then A is in CTN if and only if every m-by-m submatrix of A is in CTN. This follows from

Proposition 2.3. The above remark combined with Theorem 4.6 gives rise to the following corollary.

Corollary 4.7. *Let S be any rectangular m-by-n* $(0, +)$*-pattern with no zero rows or columns. Then, S requires Hadamard coreness of a TN matrix if and only if S is in double echelon form and does not contain* F, W *or* W^{T} *as a subpattern.*

5. Oppenheim's inequality

Suppose A and B are two n-by-n positive semidefinite matrices. Then by a classical result of Schur (see [11, p. 458]), $A \circ B$ is again positive semidefinite. Therefore, in particular, $\det(A \circ B) \geqslant 0$ in this case. However, even more is true. Oppenheim proved that if A and B are positive semidefinite, then $\det(A \circ B) \geqslant \det B \prod_{i=1}^{n} a_{ii}$ (see [11, p. 480]).

For the case in which A and B are n-by-n totally nonnegative matrices it is certainly not true that $\det(A \circ B) \geqslant 0$ (see the example in (1)). Markham [16], however, showed that Oppenheim's inequality holds for the special class of tridiagonal TN matrices. We generalize this result by making use of matrices in CTN. If A is in CTN, then $A \circ B$ is totally nonnegative (whenever B is TN) and $\det(A \circ B) \geqslant 0$. Furthermore, Oppenheim's inequality holds in this case, which is much more general than that of [16].

Theorem 5.1. *Let A be an n-by-n totally nonnegative matrix in the Hadamard core, and suppose B is any n-by-n totally nonnegative matrix. Then*

$$\det(A \circ B) \geqslant \det B \prod_{i=1}^{n} a_{ii}.$$

Proof. If B is singular, then there is nothing to show, since $\det(A \circ B) \geqslant 0$, as A is in CTN. Assume B is nonsingular. If $n = 1$, then the inequality is trivial. Suppose, by induction, that Oppenheim's inequality holds for all $(n-1)$-by-$(n-1)$ TN matrices A and B with A in CTN. Suppose A and B are n-by-n TN matrices and assume that A is in CTN. Let A_{11} (B_{11}) denote the principal submatrix obtained from A (B) by deleting row and column 1. Then by induction $\det(A_{11} \circ B_{11}) \geqslant \det B_{11} \prod_{i=2}^{n} a_{ii}$. Since B is nonsingular, by Fischer's inequality (see [7, p. 129]) B_{11} is nonsingular. Consider the matrix $\tilde{B} = B - xE_{11}$, where $x = \det B / \det B_{11}$ and E_{11} is the standard basis matrix with a 1 in the $(1, 1)$ position and zeros otherwise. Then $\det \tilde{B} = 0$, and $\tilde{B}$ is TN (see [5]). Therefore $A \circ \tilde{B}$ is TN and $\det(A \circ \tilde{B}) \geqslant 0$. Observe that $\det(A \circ \tilde{B}) = \det(A \circ B) - xa_{11}\det(A_{11} \circ B_{11}) \geqslant 0$. Thus

$$\begin{aligned}\det(A \circ B) &\geqslant x a_{11} \det(A_{11} \circ B_{11}) \\ &\geqslant x a_{11} \det B_{11} \prod_{i=2}^{n} a_{ii} \\ &= \det B \prod_{i=1}^{n} a_{ii},\end{aligned}$$

as desired. □

Since any TN matrix $A = [a_{ij}]$ satisfies Hadamard's inequality ($\det A \leqslant \prod_{i=1}^{n} a_{ii}$, see [7, p. 129]) the next result follows from Hadamard's inequality and Theorem 5.1.

Corollary 5.2. *Let A be an n-by-n totally nonnegative matrix in the Hadamard core, and suppose B is any n-by-n totally nonnegative matrix. Then*

$$\det(A \circ B) \geqslant \det(AB).$$

We close this section with some further remarks concerning Oppenheim's inequality. In the case in which $A = [a_{ij}]$ and $B = [b_{ij}]$ are n-by-n positive semidefinite matrices it is clear from Oppenheim's inequality that

$$\det(A \circ B) \geqslant \max\left\{\det B \prod_{i=1}^{n} a_{ii},\ \det A \prod_{i=1}^{n} b_{ii}\right\}.$$

However, in the case in which A is in the Hadamard core and B is an n-by-n TN matrix it is not true in general that $\det(A \circ B) \geqslant \det A \prod_{i=1}^{n} b_{ii}$. Consider the following example.

Example 5.3. Let A be any 3-by-3 totally positive matrix in CTN, and let $B = W$, the 3-by-3 totally nonnegative matrix equal to

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then since the (1,3) entry of A enters positively into $\det A$ it follows that $\det(A \circ B) < \det A = \det A \prod_{i=1}^{3} b_{ii}$.

If, however, both A and B are in CTN, then we have the next result, which is a direct consequence of Theorem 5.1.

Corollary 5.4. *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two n-by-n matrices in CTN. Then*

$$\det(A \circ B) \geqslant \max\left\{\det B \prod_{i=1}^{n} a_{ii},\ \det A \prod_{i=1}^{n} b_{ii}\right\}.$$

The following example sheds some light on the necessity that A be in CTN in order for Oppenheim's inequality to hold. In particular, we show that if A and B are TN and $A \circ B$ is TN, then Oppenheim's inequality need not hold.

Example 5.5. Let

$$A = \begin{bmatrix} 1 & 0.84 & 0.7 \\ 0.84 & 1 & 0.84 \\ 0 & 0.84 & 1 \end{bmatrix},$$

and $B = A^{\mathrm{T}}$. Then A (and hence B) is TN, and $\det A = \det B = 0.08272$. Now

$$A \circ B = \begin{bmatrix} 1 & 0.7056 & 0 \\ 0.7056 & 1 & 0.7056 \\ 0 & 0.7056 & 1 \end{bmatrix},$$

and it is not difficult to verify that $A \circ B$ is TN with $\det(A \circ B) \approx 0.00426$. However, in this case

$$\det(A \circ B) \approx 0.00426 < 0.08272 = \begin{cases} \det\ A \prod_{i=1}^{3} b_{ii} \\ \det\ B \prod_{i=1}^{3} a_{ii}. \end{cases}$$

The next remark settles the issue of the possibility that Oppenheim's inequality offers a characterization of all TN matrices in CTN, namely, if a given TN matrix A satisfies Oppenheim's inequality (i.e., $\det(A \circ B) \geqslant \det B \prod_{i=1}^{n} a_{ii}$ for every TN matrix B), then A is in CTN. If $n \leqslant 3$ and A satisfies Oppenheim's inequality for every TN matrix B, then $\det(A \circ B) \geqslant 0$, and all the 2-by-2 submatrices of $A \circ B$ will be TN, for any TN matrix B. In particular, A is in CTN. For $n = 4$ consider the following matrix. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Suppose B is any 4-by-4 TN matrix. Then since the $(1, 4)$ entry enters negatively into $\det B$ it follows that $\det(A \circ B) \geqslant \det B = \det B \prod_{i=1}^{4} a_{ii}$. Hence A satisfies Oppenheim's inequality, but A is not in CTN since A contains a submatrix ($A[\{1, 2, 3\}|\{2, 3, 4\}]$) that is not in CTN. We also note here that, we can get by with less than A in CTN in the proof of Theorem 5.1. We simply need that the principal submatrices of A, of the form $A[\{k, k+1, \ldots, n\}]$ ($k = 1, 2, \ldots, n$), satisfy $\det(A[\{k, k+1, \ldots, n\}] \circ B') \geqslant 0$, for all appropriately sized TN matrices B'.

6. Further discussion

At present no characterization of CTN for 4-by-4 totally nonnegative matrices is known. One reason for the complications regarding a characterization of CTN in the 4-by-4 case is that we do not have a solid conceptual understanding for the description of CTN in the 3-by-3 case. The proof offered here for Theorem 3.4 (and in fact all known proofs of which there are few) are computational in nature. We believe there is more to learn about CTN in the 3-by-3 case, and that these difficulties have impeded our progress in the 4-by-4 case.

In any event the question here is: Is there a finite collection of (test) matrices that are needed to determine membership in CTN? If so, must they have some special structure? For example, in the 3-by-3 case (and the proposed test matrices in the 4-by-4 case below) all of the entries of the test matrices are either zero or one. After examination of the 3-by-3 and 3-by-n test matrices, a list of potential 4-by-4 test matrices was proposed. This list includes the following six matrices as well as their transposes:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

We refer to these matrices as V_1–V_6, respectively. In the 4-by-4 case we propose the following conjecture.

Conjecture 6.1. Let A be a 4-by-4 totally nonnegative matrix. Then A is in the Hadamard core if and only if $A \circ V_i$, $A \circ V_i^{\mathrm{T}}$, are totally nonnegative matrices, for $i = 1, 2, \ldots, 6$.

Unfortunately, we have been unable to determine relevant determinantal inequalities relating these matrices to each other or to A.

Finally, it would be an interesting and worthy exercise to determine exact conditions on a totally nonnegative matrix (or a subclass of TN) which ensure that Oppenheim's inequality holds among the class TN for that matrix (or that subclass of TN). The final remark in Section 5 demonstrates that it is not necessary to belong to CTN in order to guarantee that Oppenheim's inequality holds.

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References

[1] T. Ando, Totally positive matrices, Linear Algebra Appl. 90 (1987) 165–219.
[2] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Academic Press, New York, 1979.
[3] A.S. Crans, S.M. Fallat, The Hadamard core for totally nonnegative matrices, An unpublished paper from a National Science Foundation Research Experiences for Undergraduates program held at the College of William and Mary in the summer of 1998.
[4] S.M. Fallat, Totally nonnegative matrices, Ph.D. Dissertation, Departments of Applied Science and Mathematics, College of William and Mary, Williamsburg, VA, 1999.
[5] S.M. Fallat, C.R. Johnson, R.L. Smith, The general totally positive matrix completion problem with few unspecified entries, Electron. J. Linear Algebra 7 (2000) 1–20.
[6] M. Fiedler, V. Ptak, On matrices with nonpositive off-diagonal elements and positive principal minors, Czechoslovak Math. J. 17 (1967) 420–433.
[7] F.R. Gantmacher, M.G. Krein, Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen Mechanischer Systeme, Akademie, Berlin, 1960.
[8] J. Garloff, D.G. Wagner, Hadamard products of stable polynomials are stable, J. Math. Anal. Appl. 202 (1996) 797–809.
[9] J. Garloff, D.G. Wagner, Preservation of total nonnegativity under Hadamard products and related topics, In: M. Gasca, C.A. Micchelli (Eds.), Total Positivity and its Applications, Mathematics and its Applications, vol. 359, Kluwer Academic Publishers, Dordrecht, 1996, pp. 97–102.
[10] M. Gasca, C.A. Micchelli, Total Positivity and its Applications, Mathematics and its Applications, vol. 359, Kluwer Academic Publishers, Dordrecht, 1996.
[11] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, New York, 1985.
[12] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, New York, 1991.
[13] C.R. Johnson, Closure properties of certain positivity classes of matrices under various algebraic operations, Linear Algebra Appl. 97 (1987) 243–247.
[14] S. Karlin, Total Positivity, vol. I, Stanford University Press, Stanford, 1968.
[15] T.L. Markham, On oscillatory matrices, Linear Algebra Appl. 3 (2) (1970) 143–156.
[16] T.L. Markham, A semigroup of totally nonnegative matrices, Linear Algebra Appl. 3 (2) (1970) 157–164.
[17] M. Neumann, A conjecture concerning the Hadamard product of inverses of M-matrices, Linear Algebra Appl. 285 (1998) 277–290.
[18] H.J. Ryser, Combinatorial Mathematics, The Mathematical Association of America, Rahway, 1963.
[19] D.G. Wagner, Total positivity of Hadamard products, J. Math. Anal. Appl. 163 (1992) 459–483.
[20] A. Whitney, A reduction theorem for totally positive matrices, J. Anal. Math. 2 (1952) 88–92.