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Ideal theory in Prüfer domains —An unconventional approach

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Abstract

In Prüfer domains of finite character, ideals are represented as finite intersections of special ideals which are proper generalizations of the classical primary ideals. We show that representations of ideals as shortest intersections of primal or quasi-primary ideals exist and are unique. Moreover, every non-zero ideal is the product of uniquely determined pairwise comaximal quasi-primary ideals.

Semigroups of primal and quasi-primary ideals with fixed associated primes are also investigated in arbitrary Prüfer domains. Their structures can be described in terms of the value groups of localizations. © 2002 Published by Elsevier Science (USA).

Introduction

In what follows, all rings R are commutative domains with identity. By 'ideal' we mean an integral ideal, unless stated otherwise. We will use the notation $\operatorname{Spec} R$ and $\operatorname{Max} R$ to denote the sets of prime and maximal ideals, respectively, in R.

We wish to study properties of ideals, in particular, their decompositions into intersections of ideals of special types in certain Prüfer domains. So far, the literature on this subject is sparse and mostly restricted to the question of when

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or which ideals admit decompositions as intersections of finitely many primary ideals. But this happens rarely in non-noetherian situations, since in general irreducible ideals fail to be primary. Thus, if we wish to obtain a decomposition theorem embracing all ideals, then we have to abandon the attractive primary ideals and to adopt an unconventional approach.

In a commutative noetherian ring, Noether [1] established four distinct decompositions of an ideal as (shortest) intersections of a finite number of

(i) irreducible,
(ii) primary,
(iii) relatively-prime-indecomposable, and
(iv) comaximally-indecomposable ideals.

The components in (iii) and (iv) are unique, while in (i) and (ii) only the associated prime ideals, but not the components themselves, are unique in general. In decomposition (i), several (but always the same number of) components may share the same associated prime. Each associated prime ideal that occurs in (i) also occurs in (ii), but with a single multiplicity. The decomposition (ii) is the most relevant one (see, e.g., [2])—this has served as a model for the study of ideals in general.

About a quarter of a century later, in [3,4] two different decompositions were added to the four noetherian decompositions: intersections of

(v) quasi-primary, and
(vi) primal ideals.

In these decompositions, only the minimal, respectively the maximal primes of (ii) occur as associated primes, each but once.

Recall that an ideal A of R is *primary* (Lasker [5], Macaulay [6]) if

(1) the radical $P = \sqrt{A}$ is a prime ideal, and
(2) P is the set of elements $r \in R$ which are not (relatively) prime to A.

Now, an ideal A is said to be *quasi-primary* if it has property (1) (i.e. its radical is prime), and *primal* if it satisfies (2) (i.e. the set of elements $r \in R$ which are not prime to A is a—necessarily prime—ideal, called the *adjoint prime*). These concepts make sense even if R has divisors of zero. In a Dedekind domain, the three concepts: primary, quasi-primary, and primal coincide; they all mean powers of prime ideals. (Various other decompositions were discussed, not restricted to the commutative case, by Lesieur–Croisot [7].)

We investigate decompositions of ideals in Prüfer domains into intersections of primal and quasi-primary ideals. As we intend to restrict our considerations to finite intersections, we assume to start with that our domains are *of finite* *character*; i.e. every non-zero element is contained but in a finite number of maximal ideals. Under this hypothesis, it is easy to reduce the existence problem to the local case: to valuation domains.

Our main results assert that in a Prüfer domain of finite character every non-zero ideal is the intersection of a finite number of primal ideals, and moreover, in shortest intersections the components are uniquely determined; see Theorem 3.2 below. A similar result is established for decompositions into the intersections (even into the products) of quasi-primary ideals; see Theorems 5.6, 5.7. It is remarkable that in Prüfer domains, the primal ideals turn out to be identical with the irreducible ideals.

We also study the collections of primal ideals with a fixed adjoint prime as well as the quasi-primary ideals with fixed radicals in arbitrary Prüfer domains.

The L-primal ideals turn out to be L-primal ideals also in the endomorphism ring $\operatorname{End} L$ of the prime ideal L. For a fixed prime L, the L-primal ideals form a totally ordered semigroup S_L. (In the Dedekind case, this semigroup is isomorphic to the additive semigroup of positive integers.) We show that the semigroup S_L can be characterized in terms of the positive cone of the completion of the value group of the valuation domain R_L in its interval topology. Finally, we introduce an equivalence relation between the L-primal ideals, and show that in Prüfer domains their equivalence classes form an abelian group under multiplication. This group is isomorphic to the Archimedean group $\operatorname{Arch} R_L$ of the valuation domain R_L, introduced by Bazzoni et al. [8] and developed by Fuchs and Salce [9].

The quasi-primary ideals with fixed prime radicals L form a subsemigroup in the semigroup of all ideals (under the usual definition of multiplication of ideals). If we restrict ourselves to those that are primal with adjoint primes L contained in a fixed maximal ideal P, then we can show that this is also a semigroup $Q_L(P)$ which is either a semigroup with one element, or is order-isomorphic to an extension of the Dedekind–MacNeille completion of an ordered subsemigroup of the positive cone of the value group of the valuation domain R_P.

Our results indicate that the ideal theory in Prüfer domains of finite character resembles the ideal theory of Dedekind domains more closely than previously expected: only the favorite decomposition (ii) does not exist in general, while the other ones are moreover unique. Furthermore, (i) and (vi) are identical, and so are (iii), (iv) and (v).

1. Domains of finite character

In this short section, we only assume that R is a domain of finite character.

If $A \neq 0$ is an ideal of R, then by definition there are but a finite number of maximal ideals, say $P_1, \ldots, P_k$, containing A. If P is any maximal ideal not containing A, then the localization $A_P = A \otimes_R R_P$ is equal to R_P. Consequently, in the (in general infinite) intersection $A = \bigcap_P A_P$, the R_P-ideals A_P are equal to R_P for almost all maximal ideals P. This shows that we have a finite intersection

$$A = A_{(P_1)} \cap \cdots \cap A_{(P_k)}, \tag{1}$$

where we have used the notation $A_{(P)} = A_P \cap R$. We have thus proved:

Theorem 1.1. *If R is a domain of finite character, then every ideal A is a finite intersection of ideals of the form $A_{(P)}$ for maximal ideals P of R.*

The main problem is of course to find out what kind of ideals these $A_{(P)}$ are. It is clear that they need not be primary, not even in the noetherian case. From now on, we concentrate on the Prüfer domain case, and investigate the ideals $A_{(P)}$ and their intersections (1).

2. Primal ideals in Prüfer domains

We repeat the definition: an ideal A is called *primal* if the elements of R that are not prime to A form an ideal; this ideal is always a prime ideal, called the *adjoint ideal* P of A (Fuchs [4]). In this case we also say that A is a *P-primal ideal*. Here an element $r \in R$ is called (*relatively*) prime to A if $rs \in A$ $(s \in R)$ implies $s \in A$; i.e. if the residual

$$A : r = \{s \in R \mid rs \in A\}$$

is equal to A. It is routine to verify that for the primal property of an ideal A it suffices to check that the sum of any two ring elements not prime to A is again not prime to A.

We will frequently need the following lemma.

Lemma 2.1. *Let J be an ideal of the localization R_S of the domain R at a subsemigroup S of $R\backslash 0$. If J is primal in R_S with adjoint prime P, then $J \cap R$ is primal in R with adjoint prime ideal $P \cap R$.*

Proof. Since P is prime in R_S, $P \cap R$ is prime in R. It only remains to show that $P \cap R$ is exactly the set of elements non-prime to $J \cap R$.

First, let $x \in R\backslash(P \cap R)$. Then $x \notin P$, and so $\{y \in R_S \mid xy \in J\} = J$. Consequently, $\{y \in R \mid xy \in J\} = J \cap R$. Hence such an x is prime to $J \cap R$.

Next, let $x \in P \cap R$; so $x \in P$. Thus there exists a $y \in R_S$ such that $xy \in J$, but $y \notin J$. Write $y = rs^{-1}$ with $r \in R$, $s \in S$; then clearly, $r \notin J$. Since $xy \in J$ implies $xr \in J \cap R$, we conclude that x is not prime to $J \cap R$. □

A sort of converse to Lemma 2.1 is contained in the next lemma.

Lemma 2.2. *Let A be a primal ideal of the domain R, with adjoint prime L.*

(i) *For every prime ideal P containing L, $AR_P \cap R = A$ holds.*
(ii) *If P is a prime ideal that does not contain L, then $AR_P \cap R > A$.*

Proof. (i) The inclusion $A \leqslant AR_P \cap R$ being obvious, pick an $x \in AR_P \cap R$. We can write $x = s^{-1}a$ with $s \in R \backslash P$, $a \in A$. Therefore, $sx = a \in A$. Since $s \notin L$, s is prime to A; so $x \in A$ follows.

(ii) If P does not contain L, then there is an element $x \in L \backslash P$. Let $y \in (A : x) \backslash A$. Then $xy \in A$ implies $y \in AR_P$, proving that $AR_P \cap R > A$. □

Next, we record a property of primal ideals, though we are not going to make any use of it. We remind the reader: a prime P is *associated to* the ideal A if $P = A : x$ holds for some $x \in R \backslash A$.

Lemma 2.3. *Let A be a non-zero ideal of the domain R. If P is a prime associated to A, then the ideal $B = AR_P \cap B$ is a P-primal ideal of R.*

Proof. By the definition of associated primes, there is a $c \in R \backslash A$ such that $P = A : c$. We claim that also $P = B : c$. On one hand, we have $Pc \subseteq A \subseteq B$, so $P \subseteq B : c$. On the other hand, $s \notin P$ cannot be contained in $B : c$, since $sc \in AR_P$ means $tsc \in A$ for some $t \notin P$, whence $ts \in A : c = P$ follows, a contradiction. As $c \notin B$, no element contained in P is prime to B.

It remains to show that every $x \notin P$ is relatively prime to B. Suppose $y \in B : x$. Then $xy \in AR_P$ together with $x \notin P$ implies $y \in AR_P$, i.e. $y \in B$. □

We now specialize our domains. Ideals whose endomorphism rings are isomorphic to the domain (and not to a proper overring) are called *Archimedean* (Matlis [10]). Principal ideals are always Archimedean. In a valuation domain with principal maximal ideal only the principal ideals are Archimedean. But if the maximal ideal P of the valuation domain V is not principal, then there are numerous Archimedean ideals. Their isomorphism classes form a group, called the *Archimedean group* of V, denoted by $\operatorname{Arch} V$. (See [9, p. 72].) We will refer to this group later on.

Lemma 2.4. (i) *In a valuation domain V, every ideal is primal.*

(ii) *The primal ideals of V whose adjoint prime is the maximal ideal of V are exactly the Archimedean ideals.*

Proof. (i) Let J be an ideal of the valuation domain V. The set

$$J^{\#} = \{r \in V \mid rJ < J\}$$

is a prime ideal (see [9, p. 69]). As $rJ < J$ is equivalent to $J < r^{-1}J$, it is clear that $J^\#$ coincides with the set of elements $r \in V$ which are not prime to J. This proves the claim.

(ii) Since $\operatorname{End} J \cong V_{J^\#}$ (the localization of V at the prime $J^\#$, see, e.g., [9, p. 69]), the claim should be evident in view of the argument in (i). □

By the way, the first part of this lemma is an immediate consequence of the fact that irreducible ideals in any commutative ring are primal (see [4]; cf. Lemma 2.6 below). As is well known, this property is not shared by primary ideals, in general, not even in valuation domains. In fact, it is easy to find examples of (irreducible) ideals in a valuation domain of Krull dimension > 1 which fail to be primary (cp. Theorem 7.2 *infra*).

There is another special property of primal ideals in valuation domains which is relevant for us. In general, it is not true that ideals module-isomorphic to primal ideals are again primal ideals: trivial counterexamples are PIDs, where all the ideals are module-isomorphic, but only the prime powers are primal. However, in the special case of valuation domains, we have the stronger statement.

Lemma 2.5. *Every ideal of a valuation domain V isomorphic to a primal ideal is itself primal with the same adjoint prime L.*

Proof. This is obvious from the observation that if I, J are ideals of V such that $J = qI$ (with $q \neq 0$ in the field of quotients of V), then $L = I^\# = (qI)^\# = J^\#$. □

Passing from the local case of valuation domains to the global case of Prüfer domains, we start with the following important observation. Recall that in a Dedekind domain, irreducibility means that the ideal is a power of a prime ideal (equivalently, it is primary).

Lemma 2.6. *In a Prüfer domain, an ideal is irreducible if and only if it is primal.*

Proof. The implication irreducible $\Rightarrow$ primal holds in every commutative ring [4]; the proof is immediate: if I is an irreducible ideal, and if both $I : a > I$ and $I : b > I$, then $I : (a - b) \geqslant (I : a) \cap (I : b) > I$ shows that along with a, b also $a - b$ is not prime to I.

For the reverse implication, we refer to Lemma 2.2. Suppose A is an L-primal ideal of the Prüfer domain R. If $A = B \cap C$ for ideals B, C, then $AR_P = BR_P \cap CR_P$ for every $P \in \operatorname{Max} R$. Since R_P is a valuation domain, either $BR_P = AR_P$ or $CR_P = AR_P$. If $P' \in \operatorname{Max} R$ contains L and if for this P' the first alternative holds, then $BR_{P'} \cap R = AR_{P'} \cap R = A$. Hence $B = \bigcap_{P \in \operatorname{Max} P} BR_P \leqslant BR_{P'} \cap R = A$, and A is irreducible □

For the proof of the following lemma we refer to the original source or to [9].

Lemma 2.7 (Bazzoni [11]). *Let $A \neq 0$ be an ideal and P a maximal ideal of the Prüfer domain R. Then*

(a) *there is a unique smallest prime ideal L (denoted by $Z_P(A)$) of R contained in P such that $AR_L = AR_P$;*
(b) *$Z_P(A) \leqslant P$ always; the equality $Z_P(A) = P$ holds whenever the maximal ideal P does not contain A;*
(c) *$AR_P = AR_{Z_P(A)}$ is an Archimedean ideal of the valuation domain $R_{Z_P(A)}$;*
(d) *$R_L = \mathrm{End}_R\, AR_P = \mathrm{End}_{R_P}\, AR_P$.*

We can now state the following lemma:

Lemma 2.8. *For every ideal $A \neq 0$ of a Prüfer domain R, and for every maximal ideal P of R containing A, the ideal $A_{(P)}$ is a primal ideal in R with adjoint prime contained in P.*

The adjoint prime of the primal ideal $A_{(P)}$ is exactly $L = Z_P(A)$. Thus, it is P if and only if A_P is an Archimedean ideal of R_P.

Proof. The first claim is obvious from Lemma 2.4, using Lemma 2.1 and the fact that R_P is a valuation domain.

In order to verify the second claim, observe that by Lemma 2.7(a), $A_{(P)} = AR_P \cap R = AR_L \cap R$. As AR_L is an Archimedean ideal in R_L (cf. Lemma 2.7(c)), by Lemma 2.4(ii) its adjoint prime is LR_L; so Lemma 2.1 shows that L is the adjoint prime of $A_{(P)}$. □

3. First decomposition: into primal ideals

Theorem 1.1 and Lemma 2.8 combined guarantee that every non-zero ideal in a Prüfer domain of finite character is a finite intersection of primal ideals. But we cannot stop here as we would like to have intersections with no superfluous components, and to identify the adjoint primes in such intersections. We are thus required to study the cases when intersections of primal ideals are again primal and when a primal ideal can contain an intersection of other primal ideals.

It is not easy to give a satisfactory answer to the question as to when the intersection of two primal ideals is again primal, because the answer depends not only on the inclusion relation between the adjoint primes: one primal ideal may contain another one, but the inclusion between the adjoint primes can be the opposite one. However, fortunately, for the primal ideals $A_{(P)}$ obtained from a fixed ideal A with various primes P, we can establish a strong claim, see Lemma 3.1 *infra*.

Recall that the prime ideals of a Prüfer domain form a tree under the inclusion relation; so there is a unique largest prime contained in two given prime ideals P

and P'; following Bazzoni [12], it will be denoted by the symbol $P \wedge P'$. Observe that $R_p\ R_{P'} = R_{P\wedge P'}$ holds for any two primes $P,\ P'$.

Lemma 3.1. *Let $A \neq 0$ be an ideal of the Prüfer domain R. If for the primal ideals $A_{(L_1)}, \dots, A_{(L_k)}, A_{(L)}$ (with adjoint primes $L_1, \dots, L_k, L$) we have*

$$A_{(L_1)} \cap \dots \cap A_{(L_k)} \leqslant A_{(L)},$$

then $A_{(L_i)} \leqslant A_{(L)}$ for some $i = 1, \dots, k$.

Proof. As primal means irreducible in Prüfer domains (see Lemma 2.6), and as the lattice of ideals in Prüfer domains is distributive, we can invoke a well-known theorem in distributive lattices D. This states that if $a, a_1, \dots, a_k \in D$ are meet-irreducible elements in D such that $a_1 \wedge \dots \wedge a_k \leqslant a$, then one of $a_1, \dots, a_k$ is $\leqslant a$ (see Birkhoff [13, p. 58]). Thus $A_{(L_i)} \leqslant A_{(L)}$ holds for some i. □

By a *shortest primal decomposition* $A = A_{L_1} \cap \dots \cap A_{(L_k)}$ of an ideal A we shall mean one where

(1) no component can be omitted, and
(2) where the adjoint prime ideals $L_1, \dots, L_k$ of the primal components $A_{(L_i)}$ are pairwise incomparable.

Requirement (2) is easy to satisfy, since $L_i \leqslant L_j$ implies $A_{(L_j)} \leqslant A_{(L_i)}$, so such an $A_{(L_i)}$ can simply be dropped from the intersection. Furthermore, Lemma 3.1 asserts that by omitting repetitions and canceling non-minimal components $A_{(L_i)}$ in an intersection like the one in the preceding paragraph, none of the remaining components is superfluous. Indeed, Lemma 3.1 assures that in this case none of the components can contain the intersection of the rest, so (1) will also be satisfied. Therefore, we obtain a shortest decomposition. This establishes the existence statement in the next

Theorem 3.2. *Let A be a non-zero ideal of the Prüfer domain R of finite character. A is representable as a shortest intersection*

$$A = A_{(L_1)} \cap \dots \cap A_{(L_k)} \tag{2}$$

with primal components $A_{(L_i)}$. The adjoint primes L_i are the maximal primes $Z_P(A)$ for which $R_{Z_{P(A)}} \neq R_P$ $(P \in \operatorname{Max} R)$.

(2) *is the only shortest primal representation of A: it is unique up to the order of the components.*

Proof. The first part of our theorem is evident in view of our preliminary remarks as well as by Lemma 2.8 which shows that $A_{(P)} = A_{(L)}$ is $L = Z_P(A)$-primal for any $P \in \operatorname{Max} R$ containing A.

In order to justify the uniqueness statement, it suffices to refer to the uniqueness of decompositions of elements into the intersection of meet-irreducible elements in distributive lattices. □

We turn our attention to additional information on the adjoint ideals of the primal decomposition (2).

Lemma 3.3. *Assume $A = A_{(L_1)} \cap \cdots \cap A_{(L_k)}$ is a shortest decomposition of an ideal into primal components*; *thus the adjoint primes $L_1, \ldots, L_k$ are pairwise incomparable. An element $s \in R$ is prime to A if and only if*

$$s \notin L_1 \cup \cdots \cup L_k.$$

Proof. Suppose first $s \notin L_1 \cup \cdots \cup L_k$. Then s is prime to every $A_{(L_i)}$; thus

$$A : s = A_{(L_1)} : s \cap \cdots \cap A_{(L_k)} : s = A_{(L_1)} \cap \cdots \cap A_{(L_k)} = A,$$

and s is prime to A.

Conversely, let $s \in R$ be prime to A; i.e. $A_{(L_1)} : s \cap \cdots \cap A_{(L_k)} : s = A_{(L_1)} \cap \cdots \cap A_{(L_k)}$. Localizing this equality at a maximal ideal P containing L_i, and then intersecting with R, we get $A_{(L_i)}$ on the right, since by Lemma 2.2 all other localizations are $> A_{(L_i)}$. Hence the localized left must also equal $A_{(L_i)}$. Since $A_{(L_j \wedge L_i)} : s \geqslant A_{(L_j \wedge L_i)} > A_{(L_i)}$ for $j \neq i$, only the term $A_{(L_i)} : s$ can be equal to $A_{(L_i)}$. But the equality $A_{(L_i)} : s = A_{(L_i)}$ means $s \notin L_i$. We conclude that $s \notin L_1 \cup \cdots \cup L_k$, indeed. □

It is not difficult to characterize the intersection $L_1 \cap \cdots \cap L_k$ of the adjoint primes of the decomposition (2) (borrowing an idea from [14]). Call an ideal I of R *non-prime* to A if every $x \in I$ is non-prime to A, and *strongly non-prime to A* if $I + J$ is non-prime to A for every ideal J non-prime to A. Then we have the following lemma.

Lemma 3.4. *The intersection $L_1 \cap \cdots \cap L_k$ of the adjoint primes of the decomposition* (2) *of an ideal A of a Prüfer domain R can be characterized as the unique largest ideal of R that is strongly non-prime to A.*

Proof. It is straightforward, and left to the reader. □

Let us point out that if we represent the L-primal component $A_{(L)}$ of an ideal A in a Prüfer domain R of finite character as $A_{(L)} = AR_L \cap R$, then—in the notation of Theorem 3.2—we get

$$A = AR_{L_1} \cap \cdots \cap AR_{L_k} \cap \bigcap_P R_P,$$

where P ranges over the set of maximal ideals not containing any of the primes $L_1, \ldots, L_k$.

4. The semigroup of L-primal ideals

Our next goal is to obtain more information about the set of L-primal ideals whenever the prime L is fixed, still assuming that R is a Prüfer domain. Ohm [15] proved that in such a domain the L-primary ideals form a semigroup under multiplication—we wish to prove the same for L-primal ideals and to get more information about the structure of this semigroup.

We will need detailed information on the endomorphism rings of ideals. To start with, observe that, for every ideal $A \neq 0$, $\operatorname{End} A$ is a fractional overring of R. This follows readily from the inclusion relation $\operatorname{End} A = \{q \in Q \mid qA \leqslant A\} \leqslant \{q \in Q \mid qA \leqslant R\}$. Also, it is well known that $\operatorname{End} A$ (as an overring of a Prüfer domain of finite character) is again a Prüfer domain of finite character.

Lemma 4.1 (Fontana et al. [16], Bazzoni [17]). *For a fractional ideal A of a Prüfer domain R of finite character, we have*

(i) $D = \operatorname{End} A = \bigcap_{P \in \operatorname{Max} R} R_{Z_P(A)}$;
(ii) $DR_P = R_{Z_P(A)}$ *for all* $P \in \operatorname{Max} R$;
(iii) *the maximal ideals of D are precisely the ideals XD where X ranges over the maximal members of the set* $\{Z_P(A) \mid P \in \operatorname{Max} R\}$.

We continue with a simple lemma.

Lemma 4.2. *For a prime ideal L of a Prüfer domain R, an L-primal ideal A of R is also an L-primal ideal in the Prüfer domain $D = \operatorname{End} L$.*

Proof. Observe that if A is a primal ideal with adjoint prime L, then

$$D = \operatorname{End} A = R_L \cap \bigcap_{P \notin \Omega(L)} R_{Z_P(A)}.$$

For the proof, it suffices to note that for $P \in \Omega(L)$, we clearly have $Z_P(A) = L$.

Since $A = AR_L \cap \bigcap_{P \notin \Omega(L)} AR_{Z_P(A)}$ and $D = R_L \cap \bigcap_{P \notin \Omega(L)} R_{Z_P(A)}$ show that $DA = A$, it follows that A is a D-ideal. From $AR_L = AD_L$ and $D_P = R_P$ $(P \in \operatorname{Max} R \backslash \Omega(L))$ we infer that

$$A = AR_L \cap R = AR_L \cap \bigcap_{P \notin \Omega(L)} R_P = AD_L \cap \bigcap_{P \notin \Omega(L)} D_P = AD_L \cap D,$$

so A is L'-primal in D for some prime $L' \leqslant L$. Since AD_L is Archimedean, $L' = L$ follows. □

In view of the preceding lemma, the L-primal ideals of R can be treated, if convenient, as L-primal ideals of the Prüfer domain $D = \operatorname{End} L$. (Note that L becomes a maximal ideal in D.)

In our study of the set of L-primal ideals for a fixed prime L, the starting point is the following observation.

Lemma 4.3. *Let R be a Prüfer domain, and L a prime ideal of R. The L-primal ideals of R form a semigroup S_L under the multiplication of ideals.*

Proof. Let A, B be two L-primal ideals of R, so $A = AR_L \cap R$ and $B = BR_L \cap R$ with Archimedean ideals AR_L, BR_L of R_L. Evidently,

$$AB = (AR_L \cap R)(BR_L \cap R) = ABR_L \cap AR_L \cap BR_L \cap R = ABR_L \cap R.$$

Since ABR_L as the product of two Archimedean ideals is an Archimedean ideal of R_L, the proof is completed. □

We will view $\mathcal{S}_L$ furnished with the natural order relation: $A \leqslant B$ in $\mathcal{S}_L$ means inclusion; i.e. A is contained in B. Needless to say, this order relation is compatible with the semigroup operation, so $\mathcal{S}_L$ may be viewed as an ordered semigroup.

If R is a Prüfer domain, then its localization R_L at any prime L is a valuation domain. We denote the value group of R_L by Γ_L; this is a totally ordered (additive) abelian group. We consider this group in its interval topology. If the maximal ideal LR_L is principal, then this topology is discrete; so Γ_L is complete in the interval topology. If LR_L is not principal, then we form the completion $\widetilde{\Gamma}_L$ of Γ_L in this topology. It is well known (and easy to see) that $\widetilde{\Gamma}_L$ carries a total order, extending the order of its subgroup Γ_L. Observe that the set of inequivalent Cauchy sequences in the positivity domain $\Gamma_L^+ = \{\delta \in \Gamma_L \mid \delta \geqslant 0\}$ is in a bijective correspondence with the set of non-principal Archimedean ideals of R_L (see [18]), and hence the multiplicative group of non-principal Archimedean fractional ideals of R_L is order-isomorphic to the additive group $\widetilde{\Gamma}_L$.

In order to describe $\mathcal{S}_L$, we recall a standard construction in the theory of semigroups, which we now extend to ordered semigroups. Let $\mathcal{S}$ be a semigroup and T a subsemigroup. By a *retract extension* $\mathcal{S} \propto T$ of $\mathcal{S}$ by T we mean the semigroup defined on the set $\mathcal{S}^* = \mathcal{S} \cup T^*$ where T^* is an isomorphic copy of T (write $t^* \leftrightarrow t$ for corresponding elements) with the following operation:

(1) $\mathcal{S}$ and T^* are subsemigroups of $\mathcal{S}^*$; and
(2) $st^* = st$, $t^*s = ts$ for all $s \in \mathcal{S}$, $t^* \in T^*$.

If, moreover, $\mathcal{S}$ is totally ordered, then we make $\mathcal{S}^*$ into a totally ordered semigroup by preserving the order relations in $\mathcal{S}$ and T^*, and setting

(3) for $s \in \mathcal{S}$, $t^* \in T^*$, define $s < t^*$ or $t^* < s$, according as $s \leqslant t$ or $t < s$.

We apply this construction to the semigroup $\widetilde{\Gamma}_L^+$ and to its subsemigroup $\Gamma_L^{++} = \{\delta \in \Gamma_L \mid \delta > 0\}$ of strictly positive elements of Γ_L. Let $\Sigma_L = \widetilde{\Gamma}_L^+ \propto \Gamma_L^{++}$ denote this totally ordered semigroup.

Theorem 4.4. *Let R be a Prüfer domain and L a prime ideal of R.*

(i) *If LR_L is a principal R_L-ideal, then the multiplicative semigroup S_L of L-primal ideals of R is order-isomorphic to the additive semigroup of strictly positive elements of the value group Γ_L.*

(ii) *If LR_L is not a principal ideal, then $\mathcal{S}_L$ is order-isomorphic to the totally ordered semigroup $\Sigma_L = \widetilde{\Gamma}_L^+ \propto \Gamma_L^{++}$.*

Proof. For $A \in \mathcal{S}_L$, we have $A = AR_L \cap R$ with an Archimedean ideal AR_L of R_L. The correspondences

$$A \mapsto AR_L \quad (A \in \mathcal{S}_L) \quad \text{and}$$
$$J \mapsto J \cap R \quad (J \text{ an archimedean ideal of } R_L)$$

are inverse to each other. They obviously preserve inclusion relations, and a simple calculation, like in the proof of Lemma 4.3, can convince us that they preserve operations as well. Thus we have an order-isomorphism between $\mathcal{S}_L$ and the multiplicative semigroup of all proper Archimedean ideals R_L.

(i) In case LR_L is a principal ideal, there are no non-principal Archimedean ideals in R_L. Hence the L-primal ideals are of the form $rR_L \cap R$, thus $\mathcal{S}_L$ is in a bijective correspondence with the set of proper principal ideals of R_L.

(ii) If LR_L is not principal, then the totally ordered semigroup of all proper Archimedean ideals in R_L is easily seen to be order-isomorphic to the semigroup Σ_L. □

To conclude, let us point out a different point of view about the semigroup $\mathcal{S}_L$. We start with a preliminary lemma.

Lemma 4.5. *Let $A < B$ be L-primal ideals of the Prüfer domain R. If $AR_L \cong BR_L$, then there exists an L-primal D-ideal J such that $A = BJ$.*

Proof. Write $AR_L = rBR_L$ where $r \in R$ may be assumed. Define $J = rR_L \cap R$. This must be an L-primal ideal, since rR_L is an Archimedean ideal of R_L. We clearly have $BJ = (BR_L \cap R)(rR_L \cap R) = rBR_L \cap R = AR_L \cap R = A$. □

The ideal $J = rR_L \cap R$ in the preceding lemma may be called an *L-component of a principal ideal*. It is evident that L-components of principal ideals form a subsemigroup in $\mathcal{S}_L$, corresponding to Γ_L^{++} in Σ_L.

We introduce an equivalence relation in $\mathcal{S}_L$ as follows. We shall call $A, B \in \mathcal{S}_L$ equivalent if they are equal or differ in a factor that is the L-component of a principal ideal. In other words, if, e.g., $A < B$, then $A = BJ$ where $J = aR_L \cap R$ ($a \in R$).

Theorem 4.6. *Excluding the L-components of principal ideals, for any prime ideal L, the equivalence classes of the other L-primal ideals (if such exist) in a Prüfer domain R form an abelian group under multiplication. This group is isomorphic to the Archimedean group* $\operatorname{Arch} R_L$ *of the localization R_L.*

Proof. The correspondence $\phi : A \mapsto [AR_L]$ carries an L-primal ideal A of R to the isomorphism class $[AR_L]$ of the Archimedean ideal AR_L of R_L. The image of this map ϕ contains evidently every isomorphism class of Archimedean ideals in R_L, and—as is readily seen—ϕ also preserves multiplication. It is moreover clear that two equivalent L-primal ideals have the same image under ϕ, and in view of Lemma 4.5 the converse is also true. Hence the claim is evident. □

It is worth while mentioning that the kernel of the map $\phi : \mathcal{S}_L \to \operatorname{Arch} R_L$ consists of the subsemigroup of the L-components of principal ideals, thus it is order-isomorphic to the positive cone of the value group Γ_L (with 0 omitted).

5. Second decomposition: into quasi-primary ideals

Recall that an ideal A is called *quasi-primary* if its radical

$$\sqrt{A} = \{x \in R \mid x^n \in A \text{ for some } n > 0\}$$

is a prime ideal (Fuchs [3]). The following elementary result will be needed in the proofs to follow.

Lemma 5.1 (Fuchs [3]). *If A, B are two quasi-primary ideals in a ring R, then the ideal $A \cap B$ is again quasi-primary if and only if the prime radicals $\sqrt{A}$ and $\sqrt{B}$ are comparable. In that case, the radical of $A \cap B$ is the smaller of the primes $\sqrt{A}$ and $\sqrt{B}$.*

The next lemma works also for arbitrary commutative rings R (using the concept 'isolated S-component' $A_{(S)} = \{x \in R \mid sx \in A \text{ for some } s \in S\}$ rather than localization), but we stay with domains R.

Lemma 5.2. *If J is a quasi-primary ideal of the localization R_S (for a multiplicative subsemigroup S of $R \backslash 0$) with prime radical P, then $J \cap R$ is a quasi-primary ideal of R with prime radical $P \cap R$.*

Proof. Since P is prime in R_S, all that remains to be verified is that $P \cap R$ is the radical of $J \cap R$. If $r^n \in J \cap R$ for some $r \in R$ and integer $n > 0$, then $r^n \in J$

implies $r \in P$; so $r \in P \cap R$. Conversely, if $r \in P \cap R$, then $r \in P$; so $r^n \in J \cap R$ for some $n > 0$, whence the claim is evident. □

The following simple fact should be kept in mind.

Lemma 5.3. *For every ideal A and prime ideal P, we have*

$$\sqrt{AR_P} = \sqrt{A}\, R_P.$$

Proof. If $x \in \sqrt{AR_P}$, then there is an integer $n \geqslant 1$ such that $x^n = as^{-1}$ for some $a \in A$, $s \in R \setminus P$. Hence $(sx)^n \in A$; so $sx \in \sqrt{A}$ and $x \in \sqrt{A} R_P$. The same argument backwards proves the converse inclusion. □

In the special case of valuation domains, we have the crucial fact.

Lemma 5.4. (i) *Every ideal in a valuation domain V is quasi-primary.*
(ii) *An ideal whose radical is a maximal ideal is primary.*

Proof. (i) Let P denote the radical of the ideal J of V. If $r, s \in V$ such that $rs \in P$, then $(rs)^n \in J$ for some integer $n > 0$. If $r \mid s$ in V, then $s^{2n} \in J$, so $s \in P$.
(ii) is well known. □

We point out right away that Lemmas 5.2 and 5.4 combined show that the ideals $A_{(P)}$ in (1) are quasi-primary provided R is a Prüfer domain. This already guarantees the existence of quasi-primary decompositions in these domains, but we want to establish the existence of *shortest quasi-primary decompositions*; i.e. those where no component can be omitted and the prime radicals of the quasi-primary components are pairwise incomparable.

Before stating our existence theorem, we prove a lemma needed in the uniqueness statements of subsequent theorems.

Lemma 5.5. *Let R be a Prüfer domain. If A and B are quasi-primary ideals with incomparable radicals, then $A + B = R$.*

Proof. If $A + B < R$, then $A + B$ is contained in a maximal ideal P of R. Since $\sqrt{A} + \sqrt{B} \leqslant \sqrt{A+B} \leqslant P$, by the tree property of $\operatorname{Spec} R$ in Prüfer domains, the prime ideals $\sqrt{A}$ and $\sqrt{B}$ must be comparable. This contradicts the hypothesis. □

Recall that the most one can establish in general concerning uniqueness of intersections in noetherian rings is that the primes associated with the components in a shortest quasi-primary decomposition are uniquely determined by the ideal A.

But, surprisingly, in the Prüfer domain case we can have the strongest possible uniqueness statement.

Theorem 5.6. *Let R be a Prüfer domain of finite character and A a non-zero ideal of R.*

(i) *A is a finite intersection of quasi-primary ideals with incomparable prime radicals.*
(ii) *The components in such a decomposition are uniquely determined by A.*
(iii) *The intersection of the prime radicals of the components is exactly the radical $\sqrt{A}$ of A.*

Proof. (i) As noted above, from Theorem 1.1, Lemmas 5.2 and 5.4 we obtain a decomposition of any ideal $A \neq 0$ of R as a finite intersection of quasi-primary ideals. In view of Lemma 5.1, the intersection of two quasi-primary ideals with the same or comparable radicals is again quasi-primary, so by replacing all such intersections in (1) with single components, we are led to a decomposition of A where the radicals of the components are all incomparable. It is also clear that the radicals of the components are precisely the minimal primes of the ideal A. That this is a shortest intersection is a consequence of (iii).

(ii) Uniqueness will be an immediate consequence of the next theorem.

(iii) The claim concerning the radical is obvious in view of the familiar behavior of radicals toward intersections. □

We can even improve on Theorem 5.6 and conclude that the ideals admit product representations with quasi-primary factors.

Theorem 5.7. *Every non-zero ideal A in a Prüfer domain of finite character is the product of a finite number of pairwise comaximal quasi-primary ideals, uniquely determined by A.*

Proof. By Lemma 5.5, quasi-primary ideals whose radicals are incomparable are comaximal. For comaximal ideals, intersection equals product, hence from Theorem 5.6 we infer that A is the product of pairwise comaximal quasi-primary ideals. As products of indecomposable comaximal ideals are always unique (this fact goes back to Noether [1]), the proof of uniqueness is complete. □

Consequently, while in Dedekind domains every non-zero ideal is the product of primary ideals (which are powers of prime ideals), in Prüfer domains of finite character an analogous result holds: just primary ideals have to be replaced by quasi-primary ideals.

6. The semigroup of L-quasi-primary ideals

We now proceed to investigate the structure of this semigroup. We shall see that satisfactory results can be obtained if we restrict ourselves to subsemigroups that are linearly ordered.

In order to identify easily the L-quasi-primary ideals, we will require the following lemma.

Lemma 6.1. *Let P be a prime ideal in the Prüfer domain R and L a prime contained in P. A P-primal ideal A of R is L-quasi-primary if and only if LR_P is the radical of the R_P-ideal AR_P.*

Proof. Sufficiency has been verified in Lemma 5.2, while necessity follows at once from Lemma 5.3. □

Our study of L-quasi-primary ideals continues with the important observation.

Lemma 6.2. *In a Prüfer domain, for any fixed prime L, the L-quasi-primary ideals form a semigroup Q_L under multiplication.*

Proof. Let $A, B \in Q_L$. Then $\sqrt{AB} = \sqrt{A} \cap \sqrt{B}$ implies that $AB \in Q_L$; so Q_L is a semigroup. □

For a prime ideal L of a Prüfer domain R, let L' denote the union of all primes in R properly contained in L.

Lemma 6.3. *In a Prüfer domain R, the L-quasi-primary ideals A are in a bijective correspondence with the L/L'-quasi-primary ideals of the Prüfer domain R/L' under the natural correspondence $A \mapsto A/L'$.*

Proof. It is readily seen that $\sqrt{A/L'} = \sqrt{A}/L'$. □

It is evident that if $L' = L$, then L is the only L-quasi-primary ideal in R. If $L' < L$, then an ideal A of R is L-quasi-primary exactly if it satisfies

$$L' < A \leqslant L.$$

This interval is, in general, not totally ordered. In fact, the shortest primal representation of an L-quasi-primary ideal A contains more than one component unless A is primal. Therefore, we will concentrate on the subsemigroup $Q_L(P)$ of Q_L which consists of the L-quasi-primary ideals of the form $A_{(P)} = AR_P \cap R$ for a fixed $P \in \operatorname{Max} R$, where A can be any ideal of R. It is readily checked that $Q_L(P)$ is a totally ordered semigroup.

Let Δ_L denote the convex subgroup of Γ_P which is the kernel of the natural map $\Gamma_P \longrightarrow \Gamma_L$ for primes $L \leqslant P \in \operatorname{Max} R$. Here Γ_P and Γ_L stand for the value groups of the valuation domains R_P and R_L, respectively.

Theorem 6.4. *Let R be a Prüfer domain and P a maximal ideal of R.*

(i) *If a prime $L \leqslant P$ is the union of primes properly contained in it, then Q_L is a singleton.*
(ii) *For a prime $L \leqslant P$ that is not the union of primes properly contained in L, the L-quasi-primary ideals A of the form $A = AR_P \cap R$ form a totally ordered semigroup $Q_L(P)$ under multiplication.*

$Q_L(P)$ is order-isomorphic to the totally ordered semigroup $\Delta^{++}_{LL'}$ if $\Delta_{LL'} = \Delta_{L'}/\Delta_L$ is discrete in the induced order, otherwise to $\widehat{\Delta}^{+}_{LL'} \propto \Delta^{++}_{LL'}$, where the hat denotes the Dedekind–MacNeille completion.

Proof. (i) and the first part of (ii) are obvious in view of the preceding arguments. The last claim follows from the fact that the L-quasi-primary ideals in $Q_L(P)$ are in a bijective correspondence with the ideals of R_P which are between $L'R_P$ and LR_P, and these are in a bijective correspondence with the Dedekind cuts in $\Delta^{+}_{L'}$, that fail to belong to Δ^{+}_{L}. □

7. Possible decomposition: into primary ideals

We return to the ubiquitous question as to when the ideals can be represented as finite intersections of primary ideals (such rings are called *Laskerian*) or products of primary ideals. This question has been addressed by several authors (e.g., Gilmer [19], or Anderson and Mahaney [20]), and our next theorem is not new. We state it for the sake of completeness, and derive it from our earlier results.

We start with the observation that an ideal that is both quasi-primary and primal need not be primary: it is primary exactly if its radical and its adjoint prime are identical. The following result is well known.

Lemma 7.1. *Let R be a valuation domain. Every ideal is primary if and only if the value group of R is Archimedean* (*i.e. R is of Krull dimension* 1).

Proof. If R is of Krull dimension 1, then it has but one non-zero prime: its maximal ideal P. This P is then the radical and the adjoint prime to every non-zero ideal of R; thus all the ideals are primary. Conversely, if $P' < P$ are non-zero prime ideals in R, then pick $0 \neq a \in P'$, and consider the ideal $I = aP$. It satisfies $I^{\#} = P^{\#} = P$, while $\sqrt{aP} \leqslant P'$. This shows that I cannot be primary. □

We can now conclude the following theorem.

Theorem 7.2. *Let R be a Prüfer domain. Every non-zero ideal of R is the intersection of a finite number of primary ideals if and only if*

(a) *R is of finite character, and*
(b) *R is of Krull dimension* 1.

Then every ideal is uniquely the product of primary ideals.

Proof. First assume R has the indicated property, and let A be an ideal of its localization R_P at some $P \in \operatorname{Max} R$. Thus $A \cap R$ is a primary ideal of R, so its radical and adjoint ideal coincide. This can happen only if A was a primary ideal. Thus in every localization of R, all the ideals are primary, so the preceding lemma implies that R has Krull dimension 1. Now every ideal $I \neq 0$ of R is a finite intersection of primary ideals, and I is contained in no primes other than the associated primes, whence the finite character of R is evident.

Conversely, let R satisfy (a) and (b). Then a quasi-primary ideal whose radical is a maximal ideal in R has to be primary. A simple reference to Theorem 5.7 completes the proof. □

Ohm [15] classified the primary ideals in Prüfer domains and proved that for a fixed prime L, the L-primary ideals form a semigroup $\mathcal{P}_L$. It is not difficult to characterize this semigroup.

As above, let L' denote the union of all primes in R properly contained in the prime ideal L. If $L = L'$, then L is the only L-primary ideal in R. If $L' < L$, then the L-primary ideals are between L' and L, but in this interval only those ideals are L-primary which are L-primal as well. The correspondence $A \mapsto AR_L$ is a bijection between the sets of L-primary ideals of R and the set of LR_L-primary ideals of R_L; the latter is just the set of all ideals between $L'R_L$ and LR_L, which is in a bijective correspondence with the non-zero proper ideals of the valuation domain $R_L/L'R_L$ (Ohm [15]). As these correspondences respect multiplication and order relations, this leads us to the following theorem.

Theorem 7.3. *Let R be a Prüfer domain, and L a prime ideal of R that contains properly the union L' of all primes properly contained in L. The semigroup $\mathcal{P}_L$ of L-primary ideals of R is order-isomorphic to the totally ordered semigroup $\Gamma_{LL'}^{++}$ or to $\widetilde{\Gamma}_{LL'}^{++} \propto \Gamma_{LL'}^{++}$, (where $\Gamma_{LL'}$ denotes the value group of the valuation domain $R_L/L'R_L$) according as $\Gamma_{LL'}$ is discrete or not.*

Proof. The remarks above along with the arguments in the proof of Theorem 4.4 furnish a proof. □

Note that the order of $\Gamma_{LL'}$ is Archimedean, so its completion in the interval topology is the same as its Dedekind completion.

It is worth while pointing out an interesting relation between L-primal and L-primary ideals (besides the fact that all L-primary ideals are L-primal): in the equivalence relation between L-primal ideals (defined above before Theorem 4.6), every equivalence class contains L-primary ideals. This is a consequence of the following lemma (which also shows that two L-primary ideals belonging to the same equivalence class differ only by a factor that is the isolated L-primary component of a principal ideal).

Lemma 7.4. *Every L-primal ideal of Prüfer domain R is the product of an L-primary ideal and the L-component of a principal ideal.*

Proof. It suffices to verify the claim in case $L' < L$ where L' has the same meaning as above. Let A be an L-primal ideal of R. Then $J = AR_L$ is an LR_L-primal ideal of R_L; so it is an Archimedean ideal in R_L. Therefore, $J^\# = LR_L$. By [9, p. 70. (g)], there is an ideal $J_0 \geqslant J$ of R_L between $L'R_L$ and LR_L that is isomorphic to J; i.e. $J = rJ_0$ for some $r \in R$. In view of Lemma 2.5, J_0 is LR_L-primal, and since its radical must be LR_L, it is even LR_L-primary. Hence $A = J \cap R = (J_0 \cap R)(rR_L \cap R)$ establishes the claim. □

Note added in proof

Added in the proof (May 27, 2002). Theorem 5.7 is a special case of a more general theorem by J.W. Brewer and W.J. Heinzer, On decomposing ideals into products of comaximal ideals, to appear.

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