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Cyclic Dehn surgery and the A-polynomial

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Abstract

We present a necessary condition for Dehn surgery on a knot in $\mathbb{S}^3$ to be cyclic which is based on the A-polynomial of the knot. The condition involves a width of the Newton polygon of the A-polynomial, and provides a simple method of computing a list of possible cyclic surgery slopes. The width produces a list of at most three slopes for a hyperbolic knot which contains no closed essential surface in its complement (in agreement with the Cyclic Surgery Theorem). We conclude with an application to cyclic surgeries along non-boundary slopes of hyperbolic mutant knots. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In [3], Cooper et al. introduced a new two-variable polynomial knot invariant called the A-polynomial. The A-polynomial is derived from the set of representations of the knot group in $SL_2\mathbb{C}$, and it has a number of remarkable features. Foremost among these is that a certain polygon in the plane, called the Newton polygon of the A-polynomial, displays detailed information concerning both the topology and the geometry of the knot complement.

We shall investigate the relationship between cyclic surgery on hyperbolic knots in $\mathbb{S}^3$ and the Newton polygon. Our motivation for doing this is the Cyclic Surgery Theorem of Culler et al. The Cyclic Surgery Theorem was proved, in part, using the algebraic structure of the set of representations of the knot group in $SL_2\mathbb{C}$. Since the A-polynomial carries information regarding this set, it is not surprising that it would encode information

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concerning cyclic surgeries. We shall show that information about which surgeries are cyclic is encoded by a certain width of the Newton polygon of the A-polynomial.

Before stating the results, we establish some terminology and notation. Let K be a knot in $\mathbb{S}^3$, and let X denote the complement of an open regular neighborhood of K. Since ∂X is a torus, any simple closed curve in ∂X may be parameterized (up to isotopy) by a rational *slope* $p/q \in \mathbb{Q} \cup \infty$. The numerator p represents the number of times the curve wraps around ∂X in the meridional direction, and the denominator q the number of times in the longitudinal direction. A p/q *Dehn surgery* on K is the process of attaching a solid torus V to X so that the boundary of a meridional disk of V maps to a curve of slope p/q on ∂X. We shall denote the closed orientable three-manifold obtained from p/q surgery on K by $X(p/q)$. We call p/q a *cyclic surgery slope* if $\pi_1(X(p/q))$ is a cyclic group.

A surface in X is *essential* if it is properly embedded, orientable, incompressible, boundary-incompressible, and non-boundary parallel. If an essential surface meets ∂X, then it does so in a finite number of parallel curves. The slope of these curves is called the *boundary slope* of the surface. A slope is a *strict boundary slope* if it is the boundary slope of some essential surface which is not the fiber of any fibration of X over the circle.

The A-polynomial of a knot K will be denoted by $A_K(L,M)$. By definition, the A-polynomial defines a complex algebraic curve in $\mathbb{C}^2$ which is associated to a projection of the set of representations of $\pi_1(X)$ in $\mathrm{SL}_2\mathbb{C}$. It was shown by Culler and Shalen, in [6], that this curve provides information about essential surfaces in X; subsequently, in [3], it was shown that this information may be taken from the Newton polygon of the A-polynomial.

Definition 1.1. The *Newton polygon* of a polynomial $P(L,M)$, denoted by *Newt*(P), is the convex hull in $\mathbb{R}^2$ of $\{(a,b) \mid z_{a,b}L^aM^b$ is a term of $P(L,M)$ with $z_{a,b} \neq 0\}$.

In [1], Cooper shows that if $\pi_1(X)$ satisfies a technical condition (called property NCIS$^-$) and if p/q is a cyclic surgery slope, then the curve defined by $A_K(L,M)$ will intersect a particular curve associated to p/q surgery in a minimal set of points. We shall extend Cooper's result to include intersection multiplicity and ideal intersections of the projective (non-smooth) completions of these curves. We then use a classical theorem of algebraic geometry (Bézout's Theorem) to associate the algebraic number of intersections of these curves to a certain width of *Newt*(A_K).

Definition 1.2. The p/q *width* of *Newt*(A_K) is one less than the number of lines of slope p/q which intersect *Newt*(A_K) and contain a point of the integer lattice.

Let $w : \mathbb{Q} \cup \infty \to \mathbb{Z}$ be the *width function* on *Newt*(A_K) defined by $w(p/q) =$ the p/q width of *Newt*(A_K). Our main result is that the width function can be used to compute a list of possible cyclic surgery slopes.

Corollary 3.15. *Let K be a knot in $\mathbb{S}^3$ with $A_K(L,M) \neq 1$, and suppose that X contains no closed essential surface. If p/q surgery on K is cyclic, then p/q is not the slope of* *a side of Newt*(A_K)*. Moreover,* $w(p/q)$ *is the minimal value of* w *restricted to the set of slopes which are not the slope of a side of Newt*(A_K)*.*

If K is a hyperbolic knot, then there is a discrete faithful representation of $\pi_1(X)$ in $\mathrm{SL}_2\mathbb{C}$. Associated to this representation is a special factor $H_K(L,M)$ of $A_K(L,M)$. We shall use this factor to prove a reformulation of the Cyclic Surgery Theorem in terms of the p/q width.

Theorem 4.5. *Let* K *be a hyperbolic knot in* $\mathbb{S}^3$ *with no closed essential surface in its complement. Let* w *denote the width function on Newt*(H_K)*. Then there are at most three slopes* p/q *such that* p/q *is not a slope of a side of Newt*(H_K) *and* $w(p/q)$ *is minimal. Hence, there are at most three cyclic surgery slopes.*

We can use these results to compute a list of candidate slopes for cyclic surgery from the Newton polygon. The hope, however, is that new results regarding cyclic surgery will be produced from known properties of Newton polygons. An example in this vein is the following. Let $\Delta(p/q, r/s)$ denote the minimal geometric intersection number of two curves of slope p/q and r/s on the torus. If K and K' are mutant knots, then there is a common factor of $A_K(L,M)$ and $A_{K'}(L,M)$. When the mutants are hyperbolic, this common factor divides the hyperbolic factors of the two knots. This fact leads to the following result.

Theorem 5.1. *Let* K *and* K' *be hyperbolic mutant knots in* $\mathbb{S}^3$*. Suppose that both knot groups have property NCIS*$^-$*. If* p/q *and* r/s *are slopes such that*:

(1) p/q *surgery on* K *is cyclic,*
(2) r/s *surgery on* K' *is cyclic, and*
(3) *neither* p/q *nor* r/s *is a strict boundary slope,*

then $\Delta(p/q, r/s) \leqslant 1$.

2. Preliminaries

Throughout the paper we shall work with a fixed choice of basis, $\{\mu, \lambda\}$, of $\pi_1(\partial X)$. The generator μ is represented by the boundary of a meridional disk of a closed regular neighborhood of K, and λ generates the kernel of the inclusion map,

$$i : H_1(\partial X) \to H_1(X).$$

Henceforth, we shall refer to these two generators as the *meridian* and *longitude*, respectively.

A *representation* ρ of $\pi_1(X)$ in $\mathrm{SL}_2\mathbb{C}$ is a homomorphism of groups

$$\rho : \pi_1(X) \to \mathrm{SL}_2\mathbb{C}.$$

We shall let R denote the set of all representations of $\pi_1(X)$ in $\mathrm{SL}_2\mathbb{C}$. A representation is called *reducible* if there is a non-trivial proper subspace fixed by the entire image of the representation; otherwise, it is called *irreducible*.

Recall that if $\mathcal{I}$ is an ideal in $\mathbb{C}[X_1, X_2, \ldots, X_m]$, then the complex affine *algebraic set* defined by $\mathcal{I}$ is the common zero set in $\mathbb{C}^m$ of all polynomials in $\mathcal{I}$; equivalently, it is the common zero set of any generating set of polynomials of $\mathcal{I}$. A *curve* (or *affine curve*) is an algebraic set in $\mathbb{C}^2$ associated to a principal ideal in $\mathbb{C}[X_1, X_2]$; if $P(X_1, X_2)$ is a generator of this ideal, then we shall denote the curve by $\mathcal{V}(P)$. If P has no multiple factors, then we define the *degree* of $\mathcal{V}(P)$ to be the degree of the polynomial P, i.e.,

$$\deg\bigl(\mathcal{V}(P)\bigr) = \max\bigl\{a + b \mid z_{a,b} \neq 0\bigr\},$$

where

$$P(X_1, X_2) = \sum z_{a,b} X_1^a X_2^b.$$

Since X is compact, there is a finite presentation of $\pi_1(X)$. Representations in R may be thought of as an assignment of matrices in $\mathrm{SL}_2\mathbb{C}$ to the generators of $\pi_1(X)$. Therefore, given a presentation of $\pi_1(X)$ with n generators, representations in R correspond to points in $\mathbb{C}^{4n}$. The relations in $\pi_1(X)$ impose conditions on which points of $\mathbb{C}^{4n}$ correspond to representations. If the entries of the matrices are viewed as indeterminates, then each relation produces four polynomial equations. The set of simultaneous zeroes of these polynomial equations is precisely the subset of $\mathbb{C}^{4n}$ corresponding to R. Therefore, R is an algebraic set.

Given $\rho \in R$ and $A \in \mathrm{SL}_2\mathbb{C}$, define ρ_A by $\rho_A(g) = A\rho(g)A^{-1}$. Then ρ_A is also a representation in R. The representations ρ and ρ_A are called *conjugate representations*. Conjugate representations encode the same information about $\pi_1(X)$. Much of the redundancy associated to conjugate representations in R can be avoided by restricting to the subset

$$R_U := \bigl\{\rho \in R \mid \rho(\mu) \text{ and } \rho(\lambda) \text{ are upper triangular}\bigr\}.$$

Notice that R_U is an algebraic subset of R. Moreover, no conjugacy class of representations is lost in this restriction because every representation is conjugate to one which is simultaneously upper triangular on μ and λ. We could avoid all redundancy associated to conjugate representations if we focused on the character variety. We shall not do this, however, since the definition of the A-polynomial is less cumbersome with R_U.

There is a natural projection of R_U into $\mathbb{C}^2$. Suppose that $\rho \in R_U$ has values

$$\rho(\lambda) = \begin{pmatrix} l & * \\ 0 & l^{-1} \end{pmatrix} \quad \text{and} \quad \rho(\mu) = \begin{pmatrix} m & * \\ 0 & m^{-1} \end{pmatrix}.$$

Define $\xi : R_U \to \mathbb{C}^2$ by $\xi(\rho) = (l, m)$. It is shown in [3] that the Zariski closure of the image of ξ is an algebraic set in $\mathbb{C}^2$. The definition of the A-polynomial is based on the fact that a complex dimension one algebraic set in $\mathbb{C}^2$ is a curve.

Definition 2.1. Let $\bigcup_{i=1}^n C_i$ be the union of the irreducible complex dimension one components of $\overline{\xi(R_U)}$ with $C_i \neq C_j$ when $i \neq j$. For each i, let $F_{C_i}(L, M)$ be an irreducible polynomial defining C_i. The *A-polynomial* of K is

$$A_K(L, M) := \frac{\prod_{i=1}^n F_{C_i}(L, M)}{L - 1}.$$

Notice that the A-polynomial is only well defined up to multiplication by a non-zero complex number. In [3], it is shown that one may scale so that the coefficients of the A-polynomial are integral. If we insist that the greatest common factor of the coefficients is 1, then the A-polynomial is well defined up to sign. In this paper, all A-polynomial's will be normalized in this manner.

The factor of $L-1$ in the denominator of Definition 2.1 arises as follows. Since X is a knot complement in $\mathbb{S}^3$, the abelianization of $\pi_1(X)$ is isomorphic to $\mathbb{Z}$, and the coset containing μ is a generator. Therefore, one gets an $\mathrm{SL}_2\mathbb{C}$'s worth of abelian representations by sending μ to an arbitrary matrix and all commutators to the identity. Since λ is in the commutator subgroup, every abelian representation sends λ to the identity. It follows that the abelian representations project to the curve $\mathcal{V}(L-1) \subset \overline{\xi(R_U)}$. Removing the factor of $L-1$ from $A_K(L,M)$ implies that there are only finitely many zeroes of $A_K(L,M)$ which correspond to abelian representations. Moreover, it is well known that if ρ in R_U is reducible, then $\xi(\rho(\lambda)) = (m,1)$. Therefore, removing the factor of $L-1$ also implies that there are only finitely many zeroes of $A_K(L,M)$ which correspond to reducible representations.

The following are some basic properties of the A-polynomial which we shall use throughout this paper. Proofs can be found in [3].

Proposition 2.2. *Suppose that K is a knot in $\mathbb{S}^3$.*
(1) *If K is the unknot, then $A_K(L,M)=1$.*
(2) *$A_K(L,M) = \pm L^a M^b A_K(L^{-1},M^{-1})$ for some $a,b\in\mathbb{Z}$.*
(3) *$A_K(L,M)$ involves only even powers of M.*
(4) *Neither L nor M is a factor of $A_K(L,M)$.*

Remark. Proposition 2.2 parts (2) and (4) imply that $Newt(A_K)$ is symmetric about its center of mass, lies in the first quadrant, and intersects both axes.

Example 2.3. The knot group of the figure-eight knot has a presentation with two meridional generators x and y. Since x and y are conjugate, an irreducible representation $\rho\in R_U$ may be conjugated so that:

$$\rho(x)=\begin{pmatrix} M & 1\\ 0 & M^{-1}\end{pmatrix},\quad \rho(y)=\begin{pmatrix} M & 0\\ q & M^{-1}\end{pmatrix}\quad\text{and}\quad \rho(\lambda)=\begin{pmatrix} L & *\\ 0 & L^{-1}\end{pmatrix}.$$

From the relation in the knot group, we obtain four polynomials of which there is an irreducible common factor $f(M,q)$. Every irreducible representation in R_U corresponds to a zero of $f(M,q)$. From the word in the knot group representing the longitude, we obtain a second polynomial $g(L,M,q)$. We compute the A-polynomial by taking the q-resultant of $f(M,q)$ and $g(L,M,q)$. This polynomial is:

$$-M^4 + L - LM^2 - 2LM^4 - LM^6 + LM^8 - L^2M^4.$$

(More information on calculations can be found in [4].)

The curve defined by an A-polynomial is not compact. We can compactify this curve by adjoining its points at infinity (or ideal points). In order to do this, we take the closure of an embedding of our curve in the complex projective plane. Recall that the *complex projective plane*, $\mathbb{CP}^2$, is the set all equivalence classes of points $(x, y, z) \in \mathbb{C}^3 \setminus (0, 0, 0)$ with $(x, y, z) \sim (kx, ky, kz)$ for all non-zero complex k. A point in $\mathbb{CP}^2$ shall be denoted by an ordered triple in square brackets, $[x, y, z]$. There is an embedding of $\mathbb{C}^2$ in $\mathbb{CP}^2$ defined by $(x, y) \mapsto [x, y, 1]$. We shall call $[x, y, z] \in \mathbb{CP}^2$ an *ideal point* if $z = 0$.

Curves in $\mathbb{CP}^2$ are defined by special types of polynomials. If $P(X, Y, Z)$ is in $\mathbb{C}[X, Y, Z]$, then P is called a *form* of degree d if each non-zero term of $P(X, Y, Z)$ has degree d. Notice that if P is a form of degree d and if $(x, y, z) \in \mathbb{C}^3$ is such that $P(x, y, z) = 0$, then $P(kx, ky, kz) = k^d P(x, y, z) = 0$ for all $k \in \mathbb{C} \setminus 0$. We define a complex *projective curve* to be the zero set in $\mathbb{CP}^2$ of a form in $\mathbb{C}[X, Y, Z]$.

The operation of homogenization of a two-variable polynomial is used to identify a curve in $\mathbb{C}^2$ with a projective curve in $\mathbb{CP}^2$. Suppose that $P(X, Y) = \sum_i z_i X^{a_i} Y^{b_i} \in \mathbb{C}[X, Y]$ has degree d. The *homogenization* of P with respect to Z, denoted by $\widetilde{P}(X, Y, Z)$, is $\sum_i z_i X^{a_i} Y^{b_i} Z^{d-a_i-b_i}$. Since $\widetilde{P}$ is a form, it defines a projective curve in $\mathbb{CP}^2$. Moreover, for every point (x, y) which is a zero of P, the point $[x, y, 1]$ is a zero of $\widetilde{P}$. In this way, we identify the curve $\mathcal{V}(P)$ in $\mathbb{C}^2$ with a dense subset of its *projective completion* $\mathcal{V}(\widetilde{P})$ in $\mathbb{CP}^2$. The points in $\mathcal{V}(\widetilde{P})$ of the form $[x, y, 0]$ will be called the ideal points of the curve $\mathcal{V}(P)$. The *degree* of a projective curve $\mathcal{V}(\widetilde{P})$ is defined to be the degree of the form $\widetilde{P}$ (which is also $\deg(P)$).

The Newton polygon contains information regarding the ideal points of a curve. Suppose that (x_n, y_n) is a sequence of points in $\mathcal{V}(P)$ which approach an ideal point of $\mathcal{V}(P)$. It follows that either $|x_n| \to \infty$ or $|y_n| \to \infty$ (or both). Without loss of generality, assume that $|y_n| \to \infty$. After passing to a subsequence, we may assume that there is a non-zero term $z_{a,b} x_n^a y_n^b$ of $P(x_n, y_n)$ whose modulus has the greatest order of magnitude for all n. If

$$\lim_{n\to\infty} \left| \frac{x_n^c y_n^d}{x_n^a y_n^b} \right| = 0$$

for all other non-zero terms $z_{c,d} x_n^c y_n^d$ of $P(x_n, y_n)$, then

$$\lim_{n\to\infty} \left| \frac{P(x_n, y_n)}{x_n^a y_n^b} \right| = |z_{a,b}|.$$

However, this would contradict the fact that $P(x_n, y_n) = 0$ for all n. Therefore, there must exist a second non-zero term $z_{c,d} x_n^c y_n^d$ of $P(x_n, y_n)$ so that

$$\lim_{n\to\infty} \left| \frac{x_n^c y_n^d}{x_n^a y_n^b} \right| = r > 0.$$

Taking logs of both sides and dividing by $\log|y_n|$ implies that

$$\lim_{n\to\infty} \left((c-a) \frac{\log|x_n|}{\log|y_n|} + (d-b) \right) = \lim_{n\to\infty} \frac{\log(r)}{\log|y_n|} = 0.$$

Therefore,

$$\lim_{n\to\infty} \frac{-\log|x_n|}{\log|y_n|} = \frac{d-b}{c-a}.$$

For each n, define the linear map $\phi_n : Newt(P) \to \mathbb{R}$ by

$$\phi(s,t) = s\log|x_n| + t\log|y_n|.$$

The level sets of ϕ are lines of slope $-\log|x_n|/\log|y_n|$. Since the terms $x_n^a y_n^b$ and $x_n^c y_n^d$ of $P(x_n, y_n)$ have maximum order of magnitude, $\phi_n(a,b) = \phi_n(c,d)$ is the maximum value of ϕ_n. Therefore, (a,b) and (c,d) lie in same level set on the boundary of $Newt(P)$. Moreover, since $-\log|x_n|/\log|y_n| \to (d-b)/(c-a)$, the slope of this side of $Newt(P)$ is $(d-b)/(c-a)$. Thus, sequences of points in $\mathcal{V}(P)$ approaching ideal points give rise to sides of the Newton polygon.

Suppose (l_n, m_n) is a sequence of points in $\mathcal{V}(A_K)$ which is approaching an ideal point. By above, the limit of $-\log|l_n|/\log|m_n|$ is the slope of a side of $Newt(A_K)$. On the other hand, in [6] it is shown that the limit of $-\log|l_n|/\log|m_n|$ is a boundary slope of the knot. We shall review this relationship below. For a more detailed account, consult [4].

A sequence of representations ρ_n is *blowing up* if there exists an element $g \in \pi_1(X)$ such that $\mathrm{trace}(\rho_n(g)) \to \infty$. There are two possibilities for a sequence of representations $\rho_n \in R_U$ which is blowing up:

Type 1: There is an element $g \in \pi_1(\partial X)$ such that $\mathrm{trace}(\rho_n(g)) \to \infty$ as $n \to \infty$. In this case, there is a unique (up to inverses) primitive element $\mu^p\lambda^q \in \pi_1(\partial X)$ such that $\mathrm{trace}(\rho_n(\mu^p\lambda^q))$ remains bounded as $n \to \infty$.

Type 2: For every $g \in \pi_1(\partial X)$, $\mathrm{trace}(\rho_n(g))$ remains bounded as $n \to \infty$.

In [6], it is shown that a sequence of representations which is blowing up gives rise to an essential surface in X. If ρ_n is a type 1 sequence of representations, then there is an essential surface in X with boundary slope p/q (in fact, it is shown that p/q is a strict boundary slope). Whereas, if ρ_n is a type 2 sequence, then there is a closed essential surface in X.

Let (l_n, m_n) be a sequence of points in $\mathcal{V}(A_K)$ which approach an ideal point of $\mathcal{V}(A_K)$. Since all but finitely many points of $\mathcal{V}(A_K)$ lift to representations in R_U, we may assume with no loss of generality that each point (l_n, m_n) lifts to a representation ρ_n. The sequence ρ_n is blowing up since (l_n, m_n) approaches an ideal point. If $-\log|l_n|/\log|m_n| \to p/q$, then we know that p/q is the slope of a side of $Newt(A_K)$. On the other hand, $-\log|l_n|/\log|m_n| \to p/q$ implies that $\mathrm{trace}(\rho_n(\mu^p\lambda^q))$ remains bounded as $n \to \infty$. So, ρ_n is a type 1 sequence, and p/q is a boundary slope. Therefore, boundary slopes that arise from type 1 sequences appear as the slope of a side of $Newt(A_K)$. The converse is also true, and is one of the main results of [3].

Theorem 2.4 (Cooper, Culler, Gillet, Long, Shalen). *The slopes of the sides of $Newt(A_K)$ are boundary slopes of incompressible surfaces in X which correspond to type* 1 *sequences of representations.*

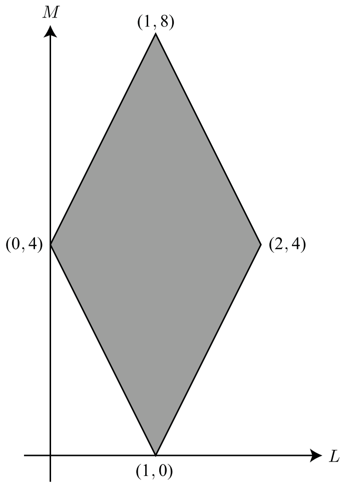

Fig. 1. The Newton polygon associated to the figure-eight knot.

Therefore, the slopes of the sides of $Newt(A_K)$ are (strict) boundary slopes of K. It is unknown if every strict boundary slope appears as the slope of a side of $Newt(A_K)$.

Example 2.5. From Example 2.3, the A-polynomial of the figure-eight knot is:

$$A_K(L, M) = -M^4 + L - LM^2 - 2LM^4 - LM^6 + LM^8 - L^2M^4.$$

The Newton polygon of $A_K(L, M)$ is shown in Fig. 1. It follows from Theorem 2.4 that the figure-eight knot has strict boundary slopes 4 and -4. The boundary slope 0 of the Seifert surface does not appear because it is not a strict boundary slope.

One technical problem that we wish to avoid is the existence of a zero (l, m) of $A_K(L, M)$ which does not correspond to a representation in R_U. If $(l, m) \in \overline{\xi(R_U)} - \xi(R_U)$ and if both l and m are non-zero, then call (l, m) a *hole* of $\mathcal{V}(A_K)$. Associated to each hole (l, m) of $\mathcal{V}(A_K)$, there is a type 2 sequence of representations ρ_n such that $\xi(\rho_n) \to (l, m)$. Therefore, in order to avoid holes, it suffices to require that there are no type 2 sequences of representations.

Definition 2.6. A knot group has *property NCIS*$^-$ if there is no sequence of representations $\rho_n \in R_U$ such that ρ_n is blowing up and trace$(\rho_n(g))$ remains bounded for all $g \in \pi_1(\partial X)$. In other words, a knot group has property NCIS$^-$ if there is no type 2 sequence of representations in R_U.

As mentioned, associated to each type 2 sequence is a closed essential surface in X. Therefore, if X contains no closed essential surface, then $\pi_1(X)$ has property NCIS$^-$. The converse is not true. In fact, it is unknown if holes exist.

3. Cyclic surgery and the Newton polygon

We are now ready to establish the relationship between the Newton polygon and cyclic surgery. An application of the Seifert–Van Kampen Theorem shows that $\pi_1(X(p/q))$ is $\pi_1(X)$ with the added relation $\mu^p\lambda^q = 1$. In order to show that $\pi_1(X(p/q))$ is non-cyclic, it suffices to find a representation of $\pi_1(X(p/q))$ in $\mathrm{PSL}_2\mathbb{C} := \mathrm{SL}_2\mathbb{C}/\{\pm I\}$ with non-cyclic image. Now, a representation $\rho \in R_U$ will induce a representation of $\pi_1(X(p/q))$ in $\mathrm{PSL}_2\mathbb{C}$ if and only if $\rho(\mu^p\lambda^q) = \pm I$. Moreover, a representation $\rho \in R_U$ such that $\rho(\mu^p\lambda^q) = \pm I$ will project to a point $(l, m) \in \mathcal{V}(A_K)$ with the property that $m^p l^q = \pm 1$. The following Theorem of Cooper [1] gives necessary conditions for such a point in $\mathcal{V}(A_K)$ to correspond to a representation with cyclic image.

Theorem 3.1 (Cooper). *If $\pi_1(X)$ has property NCIS$^-$, if $(l, m) \in (\mathbb{C} \setminus 0)^2$ is a root of $A_K(L, M)$ with the property that $m^p l^q = \pm 1$ for co-prime integers p and q, and if either l or m is not ± 1, then p/q surgery is not cyclic.*

We shall interpret this result in the context of curves as follows. For the remainder of this paper, we shall assume that p is non-negative and $\gcd(p, q) = 1$. Let

$$B_{p/q}(L, M) := \begin{cases} M^{2p}L^{2q} - 1 & \text{if } q \geqslant 0, \\ M^{2p} - L^{-2q} & \text{if } q < 0. \end{cases}$$

Notice that if $(l, m) \in \mathcal{V}(A_K) \cap \mathcal{V}(B_{p/q})$, then $A_K(l, m) = 0$ and $m^p l^q = \pm 1$. Hence, these points possibly correspond to representations of $\pi_1(X(p/q))$ in $\mathrm{PSL}_2\mathbb{C}$. Theorem 3.1 implies that if $\pi_1(X)$ has property NCIS$^-$ and p/q surgery is cyclic, then

$$\mathcal{V}(A_K) \cap \mathcal{V}(B_{p/q}) \subset \{-1, 0, 1\} \times \{-1, 0, 1\}.$$

One can say more about these points of intersection using the notion of intersection multiplicity from algebraic geometry. Before we describe the intersection multiplicities at the points in $\mathcal{V}(A_K) \cap \mathcal{V}(B_{p/q})$, we must discuss the slopes p/q for which our methods will not apply.

Given $p/q \neq 1/0$, consider the family of lines with slope p/q which intersect *Newt*(A_K). Let α and β be the respective minimum and maximum M-intercepts of a line in this family. Since α and β are extrema, the lines of slope p/q through these points intersect *Newt*(A_K) in its boundary. Therefore, these lines must contain at least one vertex of *Newt*(A_K) (which is a point in the integer lattice corresponding to a non-zero term of $A_K(L, M)$). Let $A_K(L, M) = \sum_{i=1}^{n} z_i L^{a_i} M^{b_i}$. Define the *trailing edge* of $A_K(L, M)$ towards p/q to be the polynomial:

$$f^-_{p/q}(L, M) := \sum_{\{i \mid -pa_i + qb_i = q\alpha\}} z_i L^{a_i} M^{b_i},$$

and define the *leading edge* of $A_K(L, M)$ towards p/q to be the polynomial:

$$f^+_{p/q}(L, M) := \sum_{\{i \mid -pa_i + qb_i = q\beta\}} z_i L^{a_i} M^{b_i}.$$

Notice that the trailing edge is the sum of the terms of $A_K(L,M)$ corresponding to points of *Newt*(A_K) which lie along the line with slope p/q and M-intercept α. Similarly, the leading edge contains those terms of $A_K(L,M)$ corresponding to points on the line with slope p/q and M-intercept β.

If $p/q = 1/0$, then we let α and β be the respective minimum and maximum L-intercepts of vertical lines which intersect *Newt*(A_K). Notice that, since L is not a factor of $A_K(L,M)$, $\alpha = 0$. Define the trailing edge of $A_K(L,M)$ towards $1/0$ to be the polynomial:

$$f^-_{1/0}(L,M) := \sum_{\{i \mid a_i=0\}} z_i L^{a_i} M^{b_i},$$

and define the leading edge of $A_K(L,M)$ towards $1/0$ to be the polynomial:

$$f^+_{1/0}(L,M) := \sum_{\{i \mid a_i=\beta\}} z_i L^{a_i} M^{b_i}.$$

Example 3.2. For the figure-eight knot (see Example 2.5), $f^+_4(L,M) = -M^4 + LM^8$, $f^-_4(L,M) = L - L^2M^4$, $f^+_{1/0}(L,M) = -L^2M^4$, and $f^-_{1/0}(L,M) = -M^4$.

It follows from Proposition 2.2 that $f^+_{p/q}(L^{-1},M^{-1}) = \pm f^-_{p/q}(L,M)$ up to powers of L and M. Moreover, p/q is the slope of a side of *Newt*(A_K) if and only if the leading edge (hence, trailing edge by the previous comment) of $A_K(L,M)$ towards p/q has two or more terms. If p/q is the slope of a side of *Newt*(A_K), then the terms of $f^+_{p/q}(L,M)$ may be written in the form $L^aM^b(c_0 + c_1(L^qM^p) + \cdots + c_m(L^qM^p)^m)$. Define the *edge polynomial* of *Newt*(A_K) corresponding to the edge of slope p/q to be the polynomial

$$g_{p/q}(t) := c_0 + c_1t + \cdots + c_mt^m.$$

Since $f^+_{p/q}(L^{-1},M^{-1}) = \pm f^-_{p/q}(L,M)$ up to powers of L and M, defining $g_{p/q}$ with f^- gives the same polynomial up to sign.

The following type of slope will prove problematic in our study.

Definition 3.3. If p/q is the slope of a side of *Newt*(A_K), and if 1 or -1 is a root of the edge polynomial corresponding to p/q, then call p/q a *bad* slope; otherwise call p/q a *good* slope.

Example 3.4. For the figure-eight knot, $g_4(t) = -1 + t$, and $g_{-4}(t) = -1 + t$. Therefore, both 4 and -4 are bad slopes.

Adding the hypothesis that p/q is a good slope to Theorem 3.1 allows further restriction on the set $\mathcal{V}(A_K) \cap \mathcal{V}(B_{p/q})$.

Lemma 3.5. *Assume that $A_K(L,M) \neq 1$ and $\pi_1(X)$ has property NCIS$^-$. If p/q is a good cyclic surgery slope, then $(\mathcal{V}(A_K) \cap \mathcal{V}(B_{p/q})) \setminus (0,0) \subset \{-1,1\} \times \{-1,1\}$.*

Proof. Suppose that $(l,m) \in (\mathcal{V}(A_K) \cap \mathcal{V}(B_{p/q})) \setminus (0,0)$. It follows by Theorem 3.1 that $(l,m) \in (\{-1,0,1\} \times \{-1,0,1\}) \setminus (0,0)$. We shall show that neither l nor m can be 0.

By way of contradiction, assume that $(l,m) = (\pm 1, 0)$. Since $(\pm 1, 0) \in \mathcal{V}(B_{p/q})$, we have $B_{p/q}(\pm 1, 0) = 0$. By definition of $B_{p/q}(L, M)$, it follows that $p/q = 0/1$. On the other hand, $(\pm 1, 0) \in \mathcal{V}(A_K)$. So, $A_K(\pm 1, 0) = 0$. Consider the polynomial $A_K(L, 0)$. By Proposition 2.2, M is not a factor of $A_K(L, M)$. Hence, $A_K(L, 0)$ is not identically 0. On the other hand, $A_K(\pm 1, 0) = 0$. Thus, $A_K(L, 0)$ has at least two terms. However, $A_K(L, 0)$ is the trailing edge of $A_K(L, M)$ towards $0/1$. Therefore, $0/1$ is the slope of a side of *Newt*(A_K). Moreover, if $g_{0/1}(t)$ is the edge polynomial for slope $0/1$, then $A_K(t, 0) = t^a g_{0/1}(t)$ for some $a \in \mathbb{Z}$. Hence, $A_K(\pm 1, 0) = 0$ implies that $g_{0/1}(\pm 1) = 0$. Therefore, $0/1$ is a bad slope, and this contradicts our hypothesis.

By a similar argument, if $(l, m) = (0, \pm 1)$, then $1/0$ is a bad slope. Therefore, neither l nor m can be zero. □

The following propositions investigate the intersection multiplicities at affine and ideal points of $\mathcal{V}(A_K) \cap \mathcal{V}(B_{p/q})$ when p/q is a good cyclic surgery slope. The proofs of these propositions will incorporate ideas from both algebraic geometry and hyperbolic geometry. We shall briefly review these ideas below.

For two affine curves $\mathcal{U}$ and $\mathcal{V}$, let $I_p(\mathcal{U}, \mathcal{V})$ denote the *intersection multiplicity* of $\mathcal{U}$ and $\mathcal{V}$ at the point p. The intersection multiplicity is defined to be the generic algebraic number of intersections that occur between $\mathcal{U}$ and $\mathcal{V}$ near p after a small perturbation of these curves. For almost every linear subspace $\mathcal{L}$ containing p, $I_p(\mathcal{U}, \mathcal{L})$ has a fixed value. The value of $I_p(\mathcal{U}, \mathcal{L})$ is called the *multiplicity* of p as a point of $\mathcal{U}$, and will be denoted by $m_p(\mathcal{U})$. In order to simplify notation, if the polynomials F and G define the curves $\mathcal{U}$ and $\mathcal{V}$, respectively, then we shall let $I_p(F, G)$ denote $I_p(\mathcal{U}, \mathcal{V})$, and we shall let $m_p(F)$ denote $m_p(\mathcal{U})$. If $\mathcal{U}$ and $\mathcal{V}$ are projective curves defined by forms $F(X, Y, Z)$ and $G(X, Y, Z)$ and if $p = [x, y, 1]$, then we define:

$$I_p(\mathcal{U}, \mathcal{V}) := I_{(x,y)}\big(F(X, Y, 1), G(X, Y, 1)\big).$$

We make similar definitions if p is $[1, y, z]$ or $[x, 1, z]$.

The following are well known properties of the intersection multiplicity:

- If F, G, and H are polynomials, then $I_p(FG, H) = I_p(F, H) + I_p(G, H)$.
- $I_p(F, G) \geqslant m_p(F) \cdot m_p(G)$ with equality if and only if F and G have no common tangent line at p.

We shall also use the following classical theorem from algebraic geometry.

Theorem 3.6 (Bézout's Theorem). *If $\mathcal{U}$ and $\mathcal{V}$ are complex projective curves with no common component, and if $\mathcal{U}$ and $\mathcal{V}$ have degrees u and v, respectively, then*

$$\sum_{p \in \mathcal{U} \cap \mathcal{V}} I_p(\mathcal{U}, \mathcal{V}) = uv.$$

Let $\mathbb{H}^3$ denote hyperbolic three-space. We shall work with the upper-half space model of $\mathbb{H}^3$. In this model,

$$\mathbb{H}^3 = \big\{(x, y, z) \in \mathbb{R}^3 \mid z > 0\big\}.$$

The hyperbolic metric ds on the upper half-space is given by $ds = dx/z$ where dx is the Euclidean metric. The set of orientation-preserving isometries of $\mathbb{H}^3$ is isomorphic to $\mathrm{PSL}_2\mathbb{C}$. We see the action of a $\mathrm{PSL}_2\mathbb{C}$ matrix on $\mathbb{H}^3$ as follows. Identify the plane $z = 0$ with the complex plane. The Riemann sphere obtained by adjoining infinity to the plane $z = 0$ is called the *sphere at infinity* of $\mathbb{H}^3$. Given a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2\mathbb{C}$, there is an associated Mobius transformation $\omega \mapsto (a\omega + b)/(c\omega + d)$ acting on the sphere at infinity. The unique extension of this action on the sphere at infinity to $\mathbb{H}^3$ determines the isometry of $\mathbb{H}^3$ associated to the $\mathrm{PSL}_2\mathbb{C}$ matrix. An isometry of $\mathbb{H}^3$ is called *parabolic* if it fixes no point of $\mathbb{H}^3$ and a single point on the sphere at infinity. Parabolic isometries are represented by matrices in $\mathrm{PSL}_2\mathbb{C}$ which are not diagonalizable. Hence, parabolic isometries are represented by matrices which can be conjugated to have the form $\left(\begin{smallmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{smallmatrix}\right)$. A non-trivial isometry which is not parabolic fixes exactly two points on the sphere at infinity.

We are now ready for the first proposition.

Proposition 3.7. *Assume that $A_K(L, M) \neq 1$ and that $\pi_1(X)$ has property NCIS$^-$. If p/q is a good cyclic surgery slope, and if $(l, m) \in (\mathcal{V}(A_K) \cap \mathcal{V}(B_{p/q})) \setminus (0, 0)$, then $I_{(l,m)}(A_K, B_{p/q}) = m_{(l,m)}(A_K)$.*

Proof. Suppose that $(l, m) \in (\mathcal{V}(A_K) \cap \mathcal{V}(B_{p/q})) \setminus (0, 0)$. It follows by Lemma 3.5 that $(l, m) \in \{-1, 1\} \times \{-1, 1\}$. If $\mathcal{V}(A_K)$ and $\mathcal{V}(B_{p/q})$ have no common tangent line at (l, m), then $I_{(l,m)}(A_K, B_{p/q}) = m_{(l,m)}(A_K) \cdot m_{(l,m)}(B_{p/q})$. Moreover, for all $(l, m) \in \{-1, 1\} \times \{-1, 1\}$, $m_{(l,m)}(B_{p/q}) = 1$. Therefore, it suffices to show that $\mathcal{V}(A_K)$ and $\mathcal{V}(B_{p/q})$ have no common tangent line at (l, m).

We begin with the observation that the order of $\pi_1(X(p/q))$ is finite. By hypothesis, $\pi_1(X(p/q))$ is cyclic (hence, abelian), but not necessarily finite. However, since $\pi_1(X)$ is a knot group, it follows that $\pi_1(X(p/q))$ is generated by μ, and that $\mu^p = 1$. Hence, if $p \neq 0$, then the order of $\pi_1(X(p/q))$ is finite. If $p = 0$, then 0 surgery on K is infinite cyclic. Thus, by a theorem of Gabai [9], K is the unknot. This contradicts the hypothesis that $A_K(L, M) \neq 1$. Therefore, $\pi_1(X(p/q))$ is finite, and $p \neq 0$.

We now proceed to prove the proposition. By way of contradiction, assume that some component of $\mathcal{V}(A_K)$ does has common tangent line with $\mathcal{V}(B_{p/q})$ at (l, m). Choose a sequence of points (l_n, m_n) from this component so that $(l_n, m_n) \to (l, m)$. Since neither l nor m is 0, we may choose this sequence in $(\mathbb{C} \setminus 0)^2 - (l, m)$. Therefore, since $\pi_1(X)$ has property NCIS$^-$, there is a sequence of representations $\rho_n \in R_U$ such that: $\rho_n \to \rho$ as $n \to \infty$, $\xi(\rho_n) = (l_n, m_n)$ for each n, and $\xi(\rho) = (l, m)$. After possibly conjugating this sequence of representations by a family of matrices tending to the identity matrix, we may further assume that

$$\rho_n(\lambda) = \begin{pmatrix} l_n & c_n \\ 0 & 1/l_n \end{pmatrix} \to \rho(\lambda) = \begin{pmatrix} l & c \\ 0 & l \end{pmatrix}$$

and

$$\rho_n(\mu) = \begin{pmatrix} m_n & 1 \\ 0 & 1/m_n \end{pmatrix} \to \rho(\mu) = \begin{pmatrix} m & 1 \\ 0 & m \end{pmatrix}.$$

Remark. The reason we may assume that $\rho(\mu)$ is parabolic is the following. If $\rho(\mu)$ were not parabolic, then $\rho(\mu)$ would be diagonal. However, this implies that $\rho(\mu)=\pm I$ because the eigenvalues of $\rho(\mu)$ are either both 1 or both -1. In [2], it is shown that if $\rho\in R_U$ is such that $\rho(\mu)=\pm I$, then there is a neighborhood about ρ in R_U which contains only abelian representations. Hence, there would be infinitely many points in $\mathcal{V}(A_K)$ near (l,m) corresponding to abelian representations. This contradicts the fact that there are only finitely many points in $\mathcal{V}(A_K)$ corresponding to abelian representations.

Returning to the main line of the proof, since one of l_n or m_n is not ± 1, $\rho_n(\mu)$ and $\rho_n(\lambda)$ are sequences of non-parabolic isometries. Moreover, $\rho_n(\mu)$ and $\rho_n(\lambda)$ commute. Therefore, for each n, these isometries must fix the same two points on the sphere at infinity. Notice that the isometry $\rho_n(\mu)$ fixes the points ∞ and $(1/m_n-m_n)^{-1}$. Since $\rho_n(\lambda)$ must fix the same points, it follows that

$$c_n=\frac{1/l_n-l_n}{1/m_n-m_n}\to c. \tag{1}$$

There are four cases to consider.

Case 1: $(l,m)=(1,1)$. With this assumption, the unique tangent line to $\mathcal{V}(B_{p/q})$ at $(1,1)$ is $p(M-1)+q(L-1)=0$. The assumption that $\mathcal{V}(A_K)$ and $\mathcal{V}(B_{p/q})$ have a common tangent line at $(1,1)$ implies that

$$\frac{m_n-1}{l_n-1}\to -q/p \tag{2}$$

(recall that $p\neq 0$). Since $l_n\to 1$ and $m_n\to 1$, (1) and (2) imply that $1/c_n\to -q/p$. Hence, $q\neq 0$ because $c_n\to c$ and c is finite. Therefore, $c=-p/q$. Notice that:

$$\rho(\mu^p\lambda^q)=\begin{pmatrix}1&1\\0&1\end{pmatrix}^p\begin{pmatrix}1&-p/q\\0&1\end{pmatrix}^q=\begin{pmatrix}1&p\\0&1\end{pmatrix}\begin{pmatrix}1&-p\\0&1\end{pmatrix}=I.$$

So, ρ induces a representation of $\pi_1(X(p/q))$ in $\mathrm{SL}_2\mathbb{C}$. However, $\rho(\mu)$ has infinite order. This contradicts the fact that $\pi_1(X(p/q))$ is finite.

Case 2: $(l,m)=(-1,1)$. In this case, the tangent line to $\mathcal{V}(B_{p/q})$ at $(1,1)$ is $p(M-1)-q(L+1)=0$. So,

$$\frac{m_n-1}{l_n+1}\to\frac{q}{p}. \tag{3}$$

The limits in (1) and (3) imply that $c=p/q$. Hence,

$$\rho(\mu^p\lambda^q)=\begin{pmatrix}1&1\\0&1\end{pmatrix}^p\begin{pmatrix}-1&p/q\\0&-1\end{pmatrix}^q=\begin{pmatrix}1&p\\0&1\end{pmatrix}\begin{pmatrix}(-1)^q&(-1)^{q+1}p\\0&(-1)^q\end{pmatrix}=\pm I.$$

Therefore, ρ induces a representation of $\pi_1(X)$ in $\mathrm{PSL}_2\mathbb{C}$. As in case 1, the image of μ has infinite order, and this contradicts the fact that $\pi_1(X(p/q))$ is finite.

Case 3: $(l,m)=(1,-1)$. The tangent line to $\mathcal{V}(B_{p/q})$ at $(1,-1)$ is $-p(M+1)+q(L-1)=0$. It follows that $c=p/q$. Hence,

$$\rho(\mu^p\lambda^q)=\begin{pmatrix}-1&1\\0&-1\end{pmatrix}^p\begin{pmatrix}1&p/q\\0&1\end{pmatrix}^q=\begin{pmatrix}(-1)^p&(-1)^{p+1}p\\0&(-1)^p\end{pmatrix}\begin{pmatrix}1&p\\0&1\end{pmatrix}=\pm I.$$

Once again, ρ induces a representation of $\pi_1(X)$ in $\mathrm{PSL}_2\mathbb{C}$ with infinite order. This contradicts the assumption that $\pi_1(X(p/q))$ is finite.

Case 4: $(l, m) = (-1, -1)$. In this final case, the tangent line to $\mathcal{V}(B_{p/q})$ at $(-1, -1)$ is

$$-p(M+1) - q(L+1) = 0.$$

Hence, $c = -p/q$, and

$$\rho(\mu^p\lambda^q) = \begin{pmatrix} (-1)^p & (-1)^{p+1}p \\ 0 & (-1)^p \end{pmatrix}\begin{pmatrix} (-1)^q & (-1)^q p \\ 0 & (-1)^q \end{pmatrix} = \pm I.$$

This gives the same contradiction as in the previous cases. □

We next develop a method to count intersection multiplicities at ideal points and $(0, 0)$. In doing so, we come across the following characterization of the p/q width.

Lemma 3.8. *Suppose $q \neq 0$. Let α and β be the respective minimum and maximum M-intercepts of a line of slope p/q which intersects Newt(A_K). Then*

$$(\beta - \alpha)|q| = w(p/q).$$

Proof. From Definition 1.2, $w(p/q)$ is one less than the number of lines of slope p/q which intersect *Newt*(A_K) and contain a point of the integer lattice. Since α and β are extrema, a line of slope p/q which contains α or β musts intersect *Newt*(A_K) in its boundary. Therefore, these lines must contain at least one vertex of *Newt*(A_K); hence, a point in the integer lattice. A line of slope p/q will contain a point of the integer lattice if and only if its M-intercept has the form k/q for some $k \in \mathbb{Z}$. It follows that there are $m, n \in \mathbb{Z}$ such that $\alpha = m/q$ and $\beta = n/q$. Furthermore, the number of lines of slope p/q which intersect *Newt*(A_K) and contain a point of the integer lattice is equal to the number of rational points of the form k/q in the interval $[m/q, n/q]$. Since there are $|n - m| + 1$ points of the form k/q in $[m/q, n/q]$, $w(p/q) = (|n - m| + 1) - 1 = |n - m|$. However, $|n - m| = (\beta - \alpha)|q|$. Therefore, $(\beta - \alpha)|q| = w(p/q)$. □

Remark. Recall that 0 is the minimal L-intercept of a vertical line which intersects *Newt*(A_K). Therefore, if β is the maximum L-intercept of a vertical line which intersects *Newt*(A_K), then $w(1/0) = \beta$.

Example 3.9. If K is the figure-eight knot, then $w(1/2) = 16$, $w(2) = 8$, and $w(1/0) = 2$ (see Fig. 1).

In the proof of Proposition 3.11, we shall appeal to the following technical lemma.

Lemma 3.10. *The following are equivalent*:

(1) *p/q is a bad slope,*
(2) *one of $f^-_{p/q}(t^{-p}, t^q)$, $f^-_{p/q}(-t^{-p}, t^q)$, or $f^-_{p/q}(t^{-p}, -t^q)$ is identically zero,*
(3) *one of $f^+_{p/q}(t^p, t^{-q})$, $f^+_{p/q}(-t^p, t^{-q})$, or $f^+_{p/q}(t^p, -t^{-q})$ is identically zero.*

Proof. (1) $\Leftrightarrow$ (2) Set

$$f^-_{p/q}(L,M)=L^aM^b\bigl(c_0+c_1(L^qM^p)+\cdots+c_m(L^qM^p)^m\bigr),\quad\text{and}$$
$$g_{p/q}(t)=c_0+c_1t+\cdots+c_mt^m.$$

The slope p/q is a bad slope if and only if either $g_{p/q}(1)=0$ or $g_{p/q}(-1)=0$. Notice that:

$$g_{p/q}(1)=c_0+c_1+\cdots+c_m=\frac{f^-_{p/q}(t^{-p},t^q)}{t^{-pa+qb}}.$$

Hence, $g_{p/q}(1)=0$ if and only if $f^-_{p/q}(t^{-p},t^q)\equiv 0$. On the other hand,

$$g_{p/q}(-1)=c_0-c_1+\cdots+(-1)^mc_m=\begin{cases}\frac{f^-_{p/q}(t^{-p},-t^q)}{t^{-pa}(-t)^{qb}} & \text{if } p \text{ is odd,}\\ \frac{f^-_{p/q}(-t^{-p},t^q)}{(-t)^{-pa}t^{qb}} & \text{if } p \text{ is even.}\end{cases}$$

So, $g_{p/q}(-1)=0$ if and only if either $f^-_{p/q}(t^{-p},-t^q)\equiv 0$ or $f^-_{p/q}(-t^{-p},t^q)\equiv 0$.

(2) $\Leftrightarrow$ (3) This follows directly from the fact that $f^+_{p/q}(L^{-1},M^{-1})=\pm f^-_{p/q}(L,M)$ up to powers of L and M. □

We are now prepared to count intersection multiplicities of $\mathcal{V}(A_K)$ and $\mathcal{V}(B_{p/q})$ at ideal points and $(0,0)$. Notice that the only possible ideal points of $\mathcal{V}(\widetilde{B}_{p/q})$ are $[1,0,0]$ or $[0,1,0]$.

Proposition 3.11. *Suppose that $\mathcal{V}(A_K)$ and $\mathcal{V}(B_{p/q})$ have no common component. Let $S=\mathcal{V}(\widetilde{A}_K)\cap\mathcal{V}(\widetilde{B}_{p/q})\cap\{[1,0,0],[0,1,0],[0,0,1]\}$. Then*

$$\sum_{x\in S} I_x\bigl(\widetilde{A}_K,\widetilde{B}_{p/q}\bigr)\geqslant \deg(A_K)\cdot\deg(B_{p/q})-2w(p/q). \tag{4}$$

Moreover, we have equality in (4) *when p/q is a good slope.*

Proof. Let

$$A_K(L,M)=\sum_{i=1}^n z_iL^{a_i}M^{b_i},\quad\text{and}\quad d=\deg(A_K).$$

If $q\neq 0$, then let α and β be the respective minimum and maximum M-intercepts of a line of slope p/q which intersects *Newt*(A_K). If $q=0$, then let β be the maximum L-intercept of a vertical line which intersects *Newt*(A_K). There are three cases to consider.

Case 1: Suppose that $q>0$. With this assumption, $\deg(B_{p/q})=2(p+q)$, and $S\subset\{[1,0,0],[0,1,0]\}$. Therefore,

$$\sum_{x\in S} I_x\bigl(\widetilde{A}_K,\widetilde{B}_{p/q}\bigr)=I_{[1,0,0]}\bigl(\widetilde{A}_K,\widetilde{B}_{p/q}\bigr)+I_{[0,1,0]}\bigl(\widetilde{A}_K,\widetilde{B}_{p/q}\bigr).$$

We first compute $I_{[1,0,0]}(\widetilde{A}_K,\widetilde{B}_{p/q})$. By definition,

$$I_{[1,0,0]}\bigl(\widetilde{A}_K,\widetilde{B}_{p/q}\bigr)=I_{(0,0)}\bigl(\widetilde{A}_K(1,M,Q),\widetilde{B}_{p/q}(1,M,Q)\bigr).$$

Moreover, since $\widetilde{B}_{p/q}(1,M,Q) = (M^p - Q^{p+q})(M^p + Q^{p+q})$,

$$I_{[1,0,0]}\big(\widetilde{A}_K, \widetilde{B}_{p/q}\big)$$
$$= I_{(0,0)}\big(\widetilde{A}_K(1,M,Q), M^p - Q^{p+q}\big) + I_{(0,0)}\big(\widetilde{A}_K(1,M,Q), M^p + Q^{p+q}\big).$$

In order to compute $I_{(0,0)}(\widetilde{A}_K(1,M,Q), M^p - Q^{p+q})$, we parameterize $\mathcal{V}(M^p - Q^{p+q})$ by setting $M = t^{p+q}$ and $Q = t^p$. Then $I_{(0,0)}(\widetilde{A}_K(1,M,Q), M^p - Q^{p+q})$ is the multiplicity of 0 as a root of $\widetilde{A}_K(1, t^{p+q}, t^p)$. Since $\widetilde{A}_K$ is a form of degree d,

$$\widetilde{A}_K(1, t^{p+q}, t^p) = t^{pd} A_K(t^{-p}, t^q).$$

So, the multiplicity of 0 as a root of $\widetilde{A}_K(1, t^{p+q}, t^p)$ is equal to the sum of pd and the multiplicity of 0 as a root of $A_K(t^{-p}, t^q)$. Notice that the hypothesis that $\mathcal{V}(A_K)$ and $\mathcal{V}(B_{p/q})$ have no common component ensures that $A_K(t^{-p}, t^q)$ is not identically 0. If $e_k t^k$ is a term in $A(t^{-p}, t^q)$, then

$$e_k = \sum_{\{i \mid -pa_i + qb_i = k\}} z_i.$$

Therefore, the multiplicity of 0 as a root of $A(t^{-p}, t^q)$ is the minimum value of $k = -pa_i + qb_i$ such that $e_k \neq 0$. If we let

$$k_0 = \min_{1 \leqslant i \leqslant n} \{-pa_i + qb_i\},$$

then

$$I_{(0,0)}\big(\widetilde{A}_K(1,M,Q), M^p - Q^{p+q}\big) \geqslant pd + k_0.$$

Moreover, we have equality provided $e_{k_0} \neq 0$. For each i, let y_i be the M-intercept of the line of slope p/q containing (a_i, b_i). Since $qy_i = -pa_i + qb_i$ and $q > 0$, it follows that

$$k_0 = \min_{1 \leqslant i \leqslant n} \{qy_i\} = q \min_{1 \leqslant i \leqslant n} \{y_i\} = q\alpha.$$

Hence,

$$e_{k_0} = \sum_{\{i \mid -pa_i + qb_i = q\alpha\}} z_i = \frac{f^-_{p/q}(t^{-p}, t^q)}{t^{q\alpha}}.$$

By Lemma 3.10, if p/q is a good slope, then $f^-_{p/q}(t^{-p}, t^q)$ is not identically zero. So, $e_{k_0} \neq 0$ when p/q is a good slope. It follows that,

$$I_{(0,0)}\big(\widetilde{A}_K(1,M,Q), M^p - Q^{p+q}\big) \geqslant pd + q\alpha, \tag{5}$$

and we have equality when p/q is a good slope.

The computation for $I_{(0,0)}(\widetilde{A}_K(1,M,Q), M^p + Q^{p+q})$ is similar. We parametrize $\mathcal{V}(M^p + Q^{p+q})$ by setting $M = -t^{p+q}$ and $Q = t^p$ if p is odd, or by setting $M = -t^{p+q}$ and $Q = -t^p$ if p is even. Since $\widetilde{A}_K$ is a form of degree d, we have

$$\widetilde{A}_K(1, -t^{p+q}, t^p) = t^{pd} A_K(t^{-p}, -t^q) \qquad \text{if } p \text{ odd, and}$$
$$\widetilde{A}_K(1, -t^{p+q}, -t^p) = \pm t^{pd} A_K(-t^{-p}, t^q) \qquad \text{if } p \text{ even.}$$

Therefore, if

$$k_0 = \min_{1\leqslant i\leqslant n}\{-pa_i + qb_i\},$$

then

$$I_{(0,0)}\big(\widetilde{A}_K(1,M,Q), M^p + Q^{p+q}\big) \geqslant pd + k_0 = pd + q\alpha. \tag{6}$$

Furthermore, if e_{k_0} is the coefficient of the t^{k_0} term, then

$$e_{k_0} = \begin{cases} \sum_{\{i\mid -pa_i+qb_i=q\alpha\}} (-1)^{b_i} z_i = \frac{f^-_{p/q}(t^{-p},-t^q)}{t^{q\alpha}} & \text{if } p \text{ is odd,}\\ \sum_{\{i\mid -pa_i+qb_i=q\alpha\}} (-1)^{a_i} z_i = \frac{f^-_{p/q}(-t^{-p},t^q)}{t^{q\alpha}} & \text{if } p \text{ is even.}\end{cases}$$

Therefore, by Lemma 3.10, we have equality in (6) when p/q is a good slope.

Summing Eqs. (5) and (6) implies:

$$I_{[1,0,0]}\big(\widetilde{A}_K, \widetilde{B}_{p/q}\big) \geqslant 2pd + 2q\alpha, \tag{7}$$

and we have equality in (7) when p/q is a good slope.

We apply a similar argument to the intersection at [0, 1, 0]. We first note that

$$I_{[0,1,0]}\big(\widetilde{A}_K, \widetilde{B}_{p/q}\big)$$
$$= I_{(0,0)}\big(\widetilde{A}_K(L,1,Q), L^q - Q^{p+q}\big) + I_{(0,0)}\big(\widetilde{A}_K(L,1,Q), L^q + Q^{p+q}\big).$$

To compute $I_{(0,0)}(\widetilde{A}_K(L,1,Q), L^q - Q^{p+q})$, we parametrize $\mathcal{V}(L^q - Q^{p+q})$ by setting $L = t^{p+q}$ and $Q = t^q$. It follows that $I_{(0,0)}(\widetilde{A}_K(L,1,Q), L^q - Q^{p+q})$ is the multiplicity of 0 as a root of $\widetilde{A}_K(t^{p+q},1,t^q)$. However,

$$\widetilde{A}_K(t^{p+q},1,t^q) = t^{qd} A_K(t^p, t^{-q}).$$

Thus, $I_{(0,0)}(\widetilde{A}_K(L,1,Q), L^q - Q^{p+q})$ is equal to the sum of qd and the multiplicity of 0 as a root of $A_K(t^p, t^{-q})$. If $e_k t^k$ is a term of $A_K(t^p, t^{-q})$, then

$$e_k = \sum_{\{i\mid pa_i - qb_i = k\}} z_i.$$

Therefore, the multiplicity of 0 as a root of $A_K(t^p, t^{-q})$ is the minimal value of $k = pa_i - qb_i$ such that $e_k \neq 0$. Let

$$k_1 := \min_{1\leqslant i\leqslant n}\{pa_i - qb_i\}.$$

Then

$$I_{(0,0)}\big(\widetilde{A}_K(L,1,Q), L^q - Q^{p+q}\big) \geqslant qd + k_1,$$

and we have equality when $e_{k_1} \neq 0$. Once again, let y_i be the M-intercept of the line of slope p/q containing (a_i, b_i). Since $-qy_i = pa_i - qb_i$ and $q > 0$, it follows that

$$k_1 = \min_{1\leqslant i\leqslant n}\{-qy_i\} = -q \max_{1\leqslant i\leqslant n}\{y_i\} = -q\beta.$$

Hence,

$$e_{k_1} = \sum_{\{i\mid -pa_i+qb_i=q\beta\}} z_i = t^{q\beta} f^+_{p/q}(t^p, t^{-q}).$$

If p/q is a good slope, then $e_{k_1} \neq 0$ by Lemma 3.10. Therefore,

$$I_{(0,0)}\big(\widetilde{A}_K(L,1,Q), L^q - Q^{p+q}\big) \geqslant qd - q\beta, \tag{8}$$

and we have equality when p/q is a good slope.

A similar computation gives:

$$I_{(0,0)}\big(\widetilde{A}_K(L,1,Q), L^q + Q^{p+q}\big) \geqslant qd - q\beta. \tag{9}$$

Therefore, summing (8) and (9) yields:

$$I_{[0,1,0]}\big(\widetilde{A}_K, \widetilde{B}_{p/q}\big) \geqslant 2qd - 2q\beta, \tag{10}$$

with equality in (10) when p/q is a good slope.

The proof for case 1 is completed by summing (7) and (10), then rewriting the right-hand side of the inequality using Lemma 3.8:

$$\sum_{x\in S} I_x\big(\widetilde{A}_K, \widetilde{B}_{p/q}\big) \geqslant d\cdot 2(p+q) + 2q(\alpha-\beta) = \deg(A_K)\cdot \deg(B_{p/q}) - 2w(p/q).$$

Case 2: Suppose that $q<0$. The proof here is similar to case 1. However, there are two subcases.

Subcase 1: $p \geqslant -q$. With this assumption, $\deg(B_{p/q}) = 2p$, and $S \subset \{[0,0,1],[1,0,0]\}$. Hence,

$$\sum_{x\in S} I_x\big(\widetilde{A}_K, \widetilde{B}_{p/q}\big) = I_{[0,0,1]}\big(\widetilde{A}_K, \widetilde{B}_{p/q}\big) + I_{[1,0,0]}\big(\widetilde{A}_K, \widetilde{B}_{p/q}\big).$$

We first compute $I_{[0,0,1]}(\widetilde{A}_K, \widetilde{B}_{p/q})$. Notice that

$$\begin{aligned} &I_{[0,0,1]}\big(\widetilde{A}_K, \widetilde{B}_{p/q}\big)\\ &\quad = I_{(0,0)}\big(A_K(L,M), M^p - L^{-q}\big) + I_{(0,0)}\big(A_K(L,M), M^p + L^{-q}\big). \end{aligned}$$

If we parametrize $\mathcal{V}(M^p - L^{-q})$ by $M = t^{-q}$ and $L = t^p$, then $I_{(0,0)}(A_K(L,M), M^p - L^{-q})$ is the multiplicity of 0 as a root of $A_K(t^p, t^{-q})$. Thus, if

$$k_1 := \min_{1\leqslant i\leqslant n}\{pa_i - qb_i\},$$

then

$$I_{(0,0)}\big(A_K(L,M), M^p - L^{-q}\big) \geqslant k_1.$$

As before, let y_i denote the M-intercept of the line of slope p/q containing (a_i, b_i). Since $-qy_i = pa_i - qb_i$ and $q<0$,

$$k_1 = \min_{1\leqslant i\leqslant n}\{-qy_i\} = -q\min_{1\leqslant i\leqslant n}\{y_i\} = -q\alpha.$$

Therefore,

$$I_{(0,0)}\big(A_K(L,M), M^p - L^{-q}\big) \geqslant -q\alpha. \tag{11}$$

Moreover, as in Case 1, Lemma 3.10 implies that we have equality in (11) when p/q is a good slope.

A similar computation gives:

$$I_{(0,0)}\big(A_K(L,M), M^p + L^{-q}\big) \geqslant -q\alpha. \tag{12}$$

Therefore, from (11) and (12), we have

$$I_{[0,0,1]}\big(\widetilde{A}_K, \widetilde{B}_{p/q}\big) \geqslant -2q\alpha, \tag{13}$$

with equality when p/q is a good slope.

On the other hand,

$$\begin{aligned} &I_{[1,0,0]}\big(\widetilde{A}_K, \widetilde{B}_{p/q}\big) \\ &\quad = I_{(0,0)}\big(\widetilde{A}_K(1,M,Q), M^p - Q^{p+q}\big) + I_{(0,0)}\big(\widetilde{A}_K(1,M,Q), M^p + Q^{p+q}\big). \end{aligned}$$

If we parametrize $\mathcal{V}(M^p - Q^{p+q})$ by $M = t^{p+q}$ and $Q = t^p$, then $I_{(0,0)}(\widetilde{A}_K(1,M,Q), M^p - Q^{p+q})$ is the multiplicity of 0 as a root of $\widetilde{A}_K(1, t^{p+q}, t^p)$. However, since $\widetilde{A}_K$ is a form of degree d, this multiplicity is equal to the sum of pd and the multiplicity of 0 as a root of $A_K(t^{-p}, t^q)$. If

$$k_0 := \min_{1\leqslant i\leqslant n}\{-pa_i + qb_i\},$$

then

$$I_{(0,0)}\big(\widetilde{A}_K(1,M,Q), M^p - Q^{p+q}\big) \geqslant pd + k_0.$$

However, since $qy_i = -pa_i + qb_i$ and $q < 0$,

$$k_0 = \min_{1\leqslant i\leqslant n}\{qy_i\} = q\max_{1\leqslant i\leqslant n}\{y_i\} = q\beta.$$

Therefore,

$$I_{(0,0)}\big(\widetilde{A}_K(1,M,Q), M^p - Q^{p+q}\big) \geqslant pd + q\beta, \tag{14}$$

and Lemma 3.10 implies that we have equality when p/q is a good slope.

In a similar manner, we compute

$$I_{(0,0)}\big(\widetilde{A}_K(1,M,Q), M^p + Q^{p+q}\big) \geqslant pd + q\beta. \tag{15}$$

Therefore, from (14) and (15),

$$I_{[1,0,0]}\big(\widetilde{A}_K, \widetilde{B}_{p/q}\big) \geqslant 2pd + 2q\beta, \tag{16}$$

with equality when p/q is a good slope.

Summing (13) and (16) gives

$$\sum_{x\in S} I_x\big(\widetilde{A}_K, \widetilde{B}_{p/q}\big) \geqslant d\cdot 2p + 2q(\beta - \alpha).$$

However, since $q < 0$, $w(p/q) = -q(\beta - \alpha)$ by Lemma 3.8. Therefore,

$$\sum_{x\in S} I_x\big(\widetilde{A}_K, \widetilde{B}_{p/q}\big) \geqslant \deg(A_K)\cdot\deg(B_{p/q}) - 2w(p/q),$$

and we have equality when p/q is a good slope.

Subcase 2: $p < -q$. The argument here is essentially the same. In this case, $S \subset \{[0,0,1],[0,1,0]\}$, and $\deg(B_{p/q}) = -2q$. Computing intersection multiplicities as in the other cases we see:

$$I_{[0,0,1]}(\widetilde{A}_K, \widetilde{B}_{p/q}) \geqslant -2q\alpha, \quad \text{and}$$
$$I_{[0,1,0]}(\widetilde{A}_K, \widetilde{B}_{p/q}) \geqslant -2qd + 2q\beta.$$

Therefore,

$$\sum_{x \in S} I_x(\widetilde{A}_K, \widetilde{B}_{p/q}) \geqslant d \cdot (-2q) + 2q(\beta - \alpha) = \deg(A_K) \cdot \deg(B_{p/q}) - 2w(p/q).$$

Moreover, as before, we have equality if p/q is a good slope.

Case 3: Suppose that $q = 0$. Then $\deg(B_{1/0}) = 2$, and $S \subset \{[1,0,0]\}$. Therefore,

$$\sum_{x \in S} I_x(\widetilde{A}_K, \widetilde{B}_{1/0}) = I_{[1,0,0]}(\widetilde{A}_K, \widetilde{B}_{1/0}).$$

A computation similar to the previous cases shows:

$$I_{[1,0,0]}(\widetilde{A}_K, \widetilde{B}_{1/0}) \geqslant 2d - 2\beta.$$

Furthermore, Lemma 3.10 will again imply that we have equality when $1/0$ is a good slope. Therefore, since $w(1/0) = \beta$,

$$\sum_{x \in S} I_x(\widetilde{A}_K, \widetilde{B}_{1/0}) \geqslant d \cdot 2 - 2\beta = \deg(A_K) \cdot \deg(B_{1/0}) - 2w(1/0). \qquad \square$$

Our main theorem combines Proposition 3.7 and Proposition 3.11 using Bézout's Theorem. In order for Bézout's Theorem to apply to the curves $\mathcal{V}(\widetilde{A}_K)$ and $\mathcal{V}(\widetilde{B}_{p/q})$, they must have no common component. This will be true when p/q is a good slope.

Lemma 3.12. *If p/q is a good slope, then $\mathcal{V}(\widetilde{A}_K)$ and $\mathcal{V}(\widetilde{B}_{p/q})$ have no common component.*

Proof. Assume that q is non-negative. The proof for q negative is similar. By way of contradiction, assume that $\mathcal{V}(\widetilde{A}_K)$ and $\mathcal{V}(\widetilde{B}_{p/q})$ have a common component. Since neither of these curves has a component at infinity, it follows that $\gcd(A_K, B_{p/q}) \neq 1$. However, $B_{p/q}(L, M)$ has precisely two irreducible factors: $M^p L^q - 1$ and $M^p L^q + 1$. If $M^p L^q - 1$ is a factor of $A_K(L, M)$, then $A_K(t^{-p}, t^q) \equiv 0$. It follows that $f^-_{p/q}(t^{-p}, t^q) \equiv 0$. So, by Lemma 3.10, p/q is a bad slope. This contradicts our hypothesis. Similarly, if $M^p L^q + 1$ is a factor of $A_K(L, M)$, then either $f^-_{p/q}(-t^{-p}, t^q) \equiv 0$ or $f^-_{p/q}(t^{-p}, -t^q) \equiv 0$. Once again, by Lemma 3.10, either outcome would contradict the hypothesis that p/q is a good slope. $\square$

Theorem 3.13. *Suppose that $A_K(L, M) \neq 1$ and $\pi_1(X)$ has property NCIS$^-$. If p/q is a good cyclic surgery slope, and if r/s is any slope, then either*:

- *$w(p/q) \leqslant w(r/s)$, or*
- $\gcd(A_K, B_{r/s}) \neq 1$.

Proof. Assume that r/s is any slope such that $\gcd(A_K(L,M), B_{r/s}(L,M)) = 1$. Therefore, $\mathcal{V}(\widetilde{A}_K)$ and $\mathcal{V}(\widetilde{B}_{r/s})$ have no common component. We shall apply Bézout's Theorem twice; first to the curves $\mathcal{V}(\widetilde{A}_K)$ and $\mathcal{V}(\widetilde{B}_{p/q})$, and then to $\mathcal{V}(\widetilde{A}_K)$ and $\mathcal{V}(\widetilde{B}_{r/s})$.

Since p/q is a good slope, it follows from Lemma 3.12 that $\mathcal{V}(\widetilde{A}_K)$ and $\mathcal{V}(\widetilde{B}_{p/q})$ have no common component. By Bézout's Theorem,

$$\sum_{x \in \mathcal{V}(\widetilde{A}_K) \cap \mathcal{V}(\widetilde{B}_{p/q})} I_x\big(\widetilde{A}_K, \widetilde{B}_{p/q}\big) = \deg(A_K) \cdot \deg(B_{p/q}). \tag{17}$$

The set $\mathcal{V}(\widetilde{A}_K) \cap \mathcal{V}(\widetilde{B}_{p/q})$ can be divided into two disjoint subsets:

$$T := \big(\mathcal{V}(\widetilde{A}_K) \cap \mathcal{V}(\widetilde{B}_{p/q})\big) \setminus \big\{[1,0,0], [0,1,0], [0,0,1]\big\}, \quad \text{and}$$
$$S := \mathcal{V}(\widetilde{A}_K) \cap \mathcal{V}(\widetilde{B}_{p/q}) \cap \big\{[1,0,0], [0,1,0], [0,0,1]\big\}.$$

By Lemma 3.5, $T \subset \{-1,1\} \times \{-1,1\} \times \{1\}$, and by Proposition 3.7,

$$\sum_{x \in T} I_x\big(\widetilde{A}_K, \widetilde{B}_{p/q}\big) = \sum_{s=0}^{1} \sum_{t=0}^{1} m_{[(-1)^s,(-1)^t,1]}(\widetilde{A}_K). \tag{18}$$

On the other hand, by Proposition 3.11,

$$\sum_{x \in S} I_x\big(\widetilde{A}_K, \widetilde{B}_{p/q}\big) = \deg(A_K) \cdot \deg(B_{p/q}) - 2w(p/q). \tag{19}$$

Combining (17), (18), and (19) gives:

$$2w(p/q) = \sum_{s=0}^{1} \sum_{t=0}^{1} m_{[(-1)^s,(-1)^t,1]}(\widetilde{A}_K). \tag{20}$$

The computation for the curves $\mathcal{V}(\widetilde{A}_K)$ and $\mathcal{V}(\widetilde{B}_{r/s})$ is slightly different since we can apply neither Lemma 3.5 nor Proposition 3.7. As above, partition $\mathcal{V}(\widetilde{A}_K)$ and $\mathcal{V}(\widetilde{B}_{r/s})$ into sets T and S. Now T need not be contained in $\{-1,1\} \times \{-1,1\} \times \{1\}$. However, since $I_x(\widetilde{A}_K, \widetilde{B}_{r/s}) \geqslant m_x(\widetilde{A}_K) \cdot m_x(\widetilde{B}_{r/s})$,

$$\sum_{x \in T} I_x\big(\widetilde{A}_K, \widetilde{B}_{r/s}\big) \geqslant \sum_{s=0}^{1} \sum_{t=0}^{1} m_{[(-1)^s,(-1)^t,1]}(\widetilde{A}_K). \tag{21}$$

Moreover, by Proposition 3.11,

$$\sum_{x \in S} I_x\big(\widetilde{A}_K, \widetilde{B}_{r/s}\big) \geqslant \deg(A_K) \cdot \deg(B_{r/s}) - 2w(r/s). \tag{22}$$

Summing (21) and (22), and applying Bézout's Theorem gives:

$$2w(r/s) \geqslant \sum_{s=0}^{1} \sum_{t=0}^{1} m_{[(-1)^s,(-1)^t,1]}(\widetilde{A}_K).$$

Therefore, $w(r/s) \geqslant w(p/q)$ follows from (20). □

Example 3.14. Suppose that both m and n are positive and odd. The A-polynomial of an (m,n) torus knot is $M^{2mn}L^2 - 1$. The only bad slope is mn. By Theorem 3.13, if p/q

and r/s are good cyclic surgery slopes, then $w(p/q) = w(r/s)$. Notice that $1/0$ is a good cyclic surgery slope, and $w(1/0) = 2$. Therefore, if p/q is a good cyclic surgery slope, then $w(p/q) = 2$. For $p/q \neq mn$, we have $w(p/q) = (2mn - 2(p/q))|q|$. Thus, for p/q to be a good cyclic surgery slope, it is necessary that $p = mnq \pm 1$. It is well known that these are all of the cyclic surgery slopes for an (m, n) torus knot.

As mentioned in Section 2, if a knot complement contains no closed essential surface, then its knot group has property NCIS$^-$. This leads to the following corollary of Theorem 3.13.

Corollary 3.15. *Let K be a knot in $\mathbb{S}^3$ with $A_K(L, M) \neq 1$, and suppose that X contains no closed essential surface. If p/q surgery on K is cyclic, then p/q is not the slope of a side of Newt(A_K). Moreover, $w(p/q)$ is the minimal value of w restricted to the set of slopes which are not the slope of a side of Newt(A_K).*

Proof. Since X contains no closed essential surface, it follows that $\pi_1(X)$ has property NCIS$^-$. Recall that a slope of a side of *Newt*(A_K) is a strict boundary slope of K. By Theorem 2.0.3 of [5], surgery along a strict boundary slope can be cyclic only if there is a closed essential surface in X. Hence, surgery along a slope of a side of *Newt*(A_K) cannot be cyclic.

Now assume that p/q surgery on K is cyclic. Then p/q is not the slope of a side of *Newt*(A_K), so it is a good slope. Let r/s be any slope that is not the slope of a side of *Newt*(A_K). It follows that r/s is a good slope. Hence, by Lemma 3.12, $\gcd(A_K, B_{r/s}) = 1$. Therefore, by Theorem 3.13, $w(p/q) \leqslant w(r/s)$. □

Example 3.16. The figure-eight knot satisfies the hypotheses of Corollary 3.15. Moreover, $1/0$ is a good cyclic surgery slope, and $w(1/0) = 2$. A quick calculation using the Newton polygon shows that $w(p/q) > 2$ if $p/q \neq 1/0$. Therefore, by Corollary 3.15, the only possible cyclic surgery slopes for the figure-eight knot (other than $1/0$) are the boundary slopes ± 4. However, neither of these slopes are cyclic by Theorem 2.0.3 of [5].

4. The Cyclic Surgery Theorem

One of the most celebrated results concerning cyclic surgery is the Cyclic Surgery Theorem of Culler et al. [5].

Theorem 4.1 (The Cyclic Surgery Theorem). *Let X be a compact, connected, irreducible three-manifold such that ∂X is a torus. Suppose that X is not a Seifert fibered space. If p/q and r/s surgeries are cyclic, then $\Delta(p/q, r/s) \leqslant 1$. Hence, there are at most three cyclic surgery slopes.*

Given a hyperbolic knot which contains no closed essential surface in its complement, we shall produce a reformulation of the Cyclic Surgery Theorem in terms of the p/q width.

The proof of this result will exploit the fact that the A-polynomial of a hyperbolic knot has a special factor.

Definition 4.2. Let r and s be co-prime non-negative integers. Let $G(L,M)$ be the product of all factors of $A_K(L,M)$ of the form $M^rL^s \pm 1$ or $M^r \pm L^s$. The polynomial $H_K(L,M) := A_K(L,M)/G(L,M)$ is called the *hyperbolic factor* of $A_K(L,M)$.

In [4], Cooper and Long prove that if K is hyperbolic, then $H_K(L,M) \neq 1$. Therefore, we have the following corollary to Theorem 3.13.

Corollary 4.3. *Suppose that K is a hyperbolic knot and X contains no closed essential surface. Let w denote the width function on Newt(H_K). If p/q surgery on K is cyclic, and if r/s is any slope, then $w(p/q) \leqslant w(r/s)$.*

Proof. We apply Theorem 3.13 with $H_K(L,M)$ used in place of $A_K(L,M)$. Thus, it suffices to show that the hypotheses of Theorem 3.13 are satisfied. The result of Cooper and Long implies that $H_K(L,M) \neq 1$. Since X contains no closed essential surface, we know that $\pi_1(X)$ has property NCIS$^-$. Moreover, since p/q is a cyclic surgery slope and X contains no closed essential surface, we know p/q is not the slope of a side of *Newt*(A_K) by Corollary 3.15. Hence, p/q is a good slope, and by Theorem 3.13, $w(p/q) \leqslant w(r/s)$ or $\gcd(H_K, B_{r/s}) \neq 1$. However, $\gcd(H_K, B_{r/s}) = 1$ for all r/s by the definition of $H_K(L,M)$. Therefore, $w(p/q) \leqslant w(r/s)$. □

The proof of the reformulation of the Cyclic Surgery Theorem for hyperbolic knots in $\mathbb{S}^3$ will depend on the following lemma.

Lemma 4.4. *Let N be a non-degenerate polygon in the plane whose vertices lie in the integer lattice, and let w denote the width function on N. Let p/q be a slope such that*:

(1) *p/q is not the slope of a side of N and*
(2) *$w(p/q)$ is the minimal value of w.*

If r/s is any slope with $w(r/s) = w(p/q)$, then $\Delta(p/q, r/s) \leqslant 1$. Hence, for such a polygon, there exist at most three slopes p/q satisfying (1) *and* (2).

Proof. After an integral change of basis, we may assume that $p/q = 1/0$. Moreover, since N has integral vertices and since w is invariant under integral translations, we may assume that N lies in the first quadrant and intersects both axes. Since $p/q = 1/0$, it follows that $\Delta(p/q, r/s) = \Delta(1/0, r/s) = |s|$. Therefore, in order to prove the lemma, it suffices to show that $|s| \leqslant 1$.

Assume $s \neq 0$, and consider the family consisting of all lines of slope r/s which intersect N. Let l_1 and l_2 be the lines in this family with the respective maximum and minimum M-intercepts. Similarly, consider the family of vertical lines intersecting N, and let l_3 and l_4 be the lines in this family with the respective maximum and minimum L-intercepts. Define P to be the parallelogram bounded by the lines l_1, l_2, l_3, and l_4. Notice that $N \subset P$.

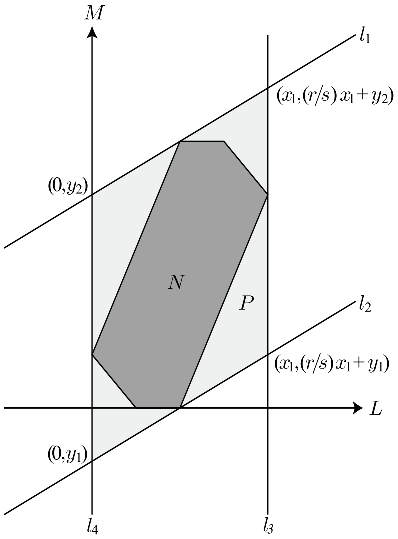

Fig. 2. The parallelogram P containing N.

So, by hypothesis, P is also non-degenerate. Starting at the South West vertex of P and moving clockwise, label the four vertices of P $(0, y_1)$, $(0, y_2)$, $(x_1, (r/s)x_1 + y_2)$, and $(x_1, (r/s)x_1 + y_1)$. By Lemma 3.8, $(y_2 - y_1)|s| = w(r/s)$ and $x_1 = w(1/0)$. (See Fig. 2.)

Define a new width function w_P on the parallelogram P as follows. If $b \neq 0$, then let α and β be the respective minimum and maximum M-intercepts of a line of slope a/b which intersects P. Define w_P by $w_P(a/b) := (\beta - \alpha)|b|$. If $b = 0$, then let β be the maximum L-intercept of a vertical line which intersects P, and define w_P by $w_P(1/0) := \beta$. If the vertices of P are integral, then by Lemma 3.8, w_P agrees with the width function from Definition 1.2. Therefore, since N is contained in P, and since N does have integral vertices, $w_P \geqslant w$ for all slopes. Moreover, $w_P(r/s) = w(r/s) = w(1/0) = w_P(1/0)$.

By way of contradiction, assume that $|s| \neq 1$. Hence, there exists an integer a such that:

$$a < \frac{r}{s} < a + 1.$$

We shall prove that $w_P(a) = w(a)$ and $w_P(a+1) = w(a+1)$. This leads to a contradiction as follows. Since $a < r/s$, $w_P(a) = w(a)$ if and only if the South West and North East vertices of P are in N. Similarly, since $r/s < a + 1$, we see that the North West and South East vertices of P are in N if $w_P(a + 1) = w(a + 1)$. Therefore, if $w_P(a) = w(a)$ and $w_P(a + 1) = w(a + 1)$, then $P = N$. This contradicts the hypothesis that $1/0$ is not the slope of a side of N.

We now proceed to show $w_P(a) = w(a)$. We shall assume that $r/s \geqslant 0$. The proof for $r/s < 0$ is similar. Define b so that:

$$a + \frac{b}{s} = \frac{r}{s};$$

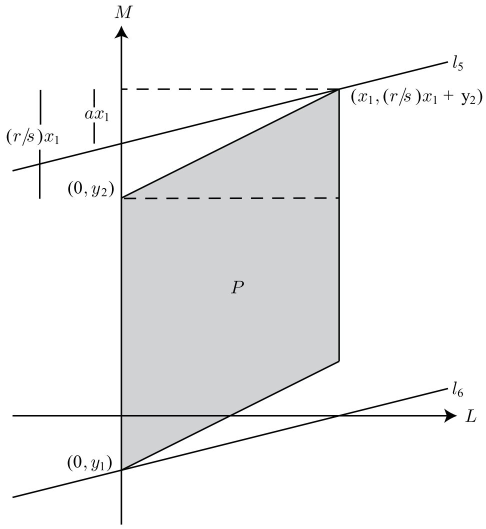

Fig. 3. The computation of $w_P(a)$.

therefore, $0 < b < s$. Consider the family of lines with slope a which intersect P. Let l_5 and l_6 be the lines in this family with the respective maximum and minimum M-intercepts. Notice that the M-intercept of l_5 is $y_2 + (r/s)x_1 - ax_1$. (See Fig. 3.) So, by definition of w_P,

$$\begin{aligned} w_P(a) &= \left(\frac{r}{s}x_1 + y_2 - ax_1\right) - y_1 \\ &= \left(\frac{r}{s} - a\right)x_1 + (y_2 - y_1) \\ &= \left(\frac{r}{s} - a\right)w(1/0) + \frac{w(1/0)}{s} \\ &= \left(\frac{b+1}{s}\right)w(1/0). \end{aligned} \tag{23}$$

Since $w_P \geqslant w$ and $w(1/0)$ is minimal, it follows that $w_P(a) \geqslant w(a) \geqslant w(1/0)$. Hence, from (23), $b + 1 \geqslant s$. On the other hand, $0 < b < s$. Therefore, $b + 1 = s$, and this implies that $w_P(a) = w(a) = w(1/0)$.

The proof that $w_P(a+1) = w(a+1)$ is essentially the same. Once again we assume that $r/s \geqslant 0$. Consider the family of lines with slope $a+1$ which intersect P. Let l_7 and l_8 be the lines in this family with the respective maximum and minimum M-intercepts. The M-intercept of l_8 is $y_1 + (r/s)x_1 - (a+1)x_1$. (See Fig. 4.) Therefore,

$$\begin{aligned} w_P(a+1) &= y_2 - \left(y_1 + \frac{r}{s}x_1 - (a+1)x_1\right) \\ &= (y_2 - y_1) + \left(a + 1 - \frac{r}{s}\right)x_1 \end{aligned}$$

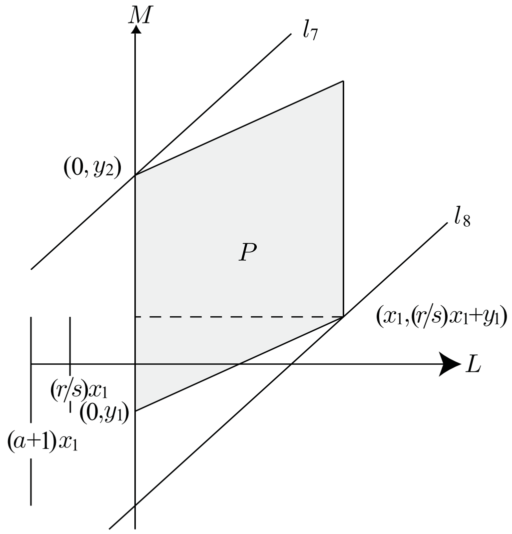

Fig. 4. The computation of $w_P(a+1)$.

$$= \frac{w(1/0)}{s} + \left(a + 1 - \frac{r}{s}\right) w(1/0)$$
$$= \left(\frac{1+s-b}{s}\right) w(1/0). \tag{24}$$

Since $w_P(a+1) \geqslant w(a+1) \geqslant w(1/0)$, it follows from (24) that $1+s-b \geqslant s$. Hence, $b = 1$ because $0 < b < s$. Therefore, $w_P(a+1) = w(a+1) = w(1/0)$. □

Combining Corollary 4.3 and Lemma 4.4 we produce the following reformulation of the Cyclic Surgery Theorem for hyperbolic knots in $\mathbb{S}^3$.

Theorem 4.5. *Let K be a hyperbolic knot in $\mathbb{S}^3$ with no closed essential surface in its complement. Let w denote the width function on Newt(H_K). Then there are at most three slopes p/q such that p/q is not a slope of a side of Newt(H_K) and $w(p/q)$ is minimal. Hence, there are at most three cyclic surgery slopes.*

Proof. It suffices to show that the hypotheses of Lemma 4.4 are satisfied by the polygon *Newt*(H_K). Since K is hyperbolic, it follows by Theorem 6.3 of [4] that *Newt*(H_K) is non-degenerate. Moreover, since K is a knot in $\mathbb{S}^3$, $1/0$ is a cyclic surgery slope. So, by Corollary 4.3, $w(1/0)$ is minimal. Furthermore, by Theorem 2.0.3 of [5], $1/0$ is not strict boundary slope. Hence, $1/0$ is not the slope of a side of *Newt*(H_K). Therefore, by Lemma 4.4, there are at most three slopes p/q such that p/q is not a slope of a side of *Newt*(H_K) and $w(p/q)$ is minimal. The last remark of the Theorem follows from Corollary 4.3. □

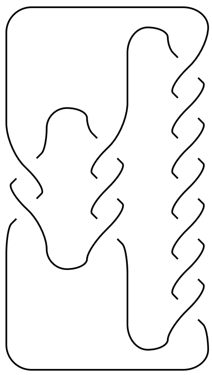

Fig. 5. The $(-2, 3, 7)$ pretzel knot.

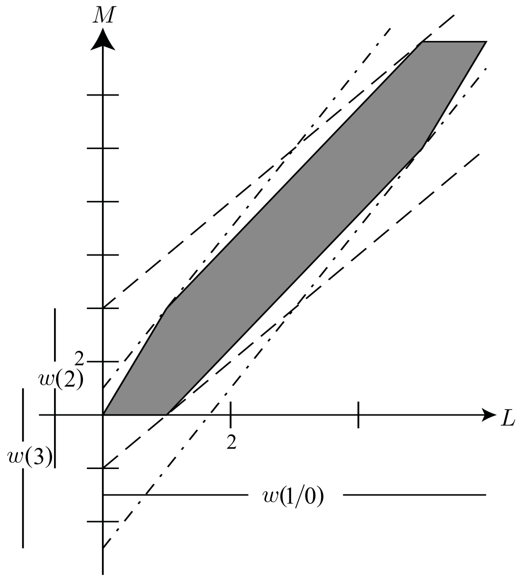

Fig. 6. The sheared Newton polygon of the A-polynomial of the $(-2, 3, 7)$ pretzel knot.

Using the p/q width, one can easily compute a list of possible cyclic surgery slopes from the Newton polygon. If the knot is hyperbolic and contains no closed essential surface, then the list we compute will contain at most three slopes.

Example 4.6. The best known example of a hyperbolic knot with three cyclic surgeries is the $(-2, 3, 7)$ pretzel knot shown in Fig. 5. By work of Oertel [12], this knot complement contains no closed essential surface. The A-polynomial of this knot is:

$$A_K(L, M) = -1 + LM^{16} - 2LM^{18} + LM^{20} + 2L^2M^{36} + L^2M^{38} - L^4M^{72} - 2L^4M^{74} - L^5M^{90} + 2L^5M^{92} - L^5M^{94} + L^6M^{110}.$$

The shape of the Newton polygon for this polynomial is less obscure after applying the linear shear map $\left[\begin{smallmatrix} 1 & 0 \\ -16 & 1 \end{smallmatrix}\right]$. (See Fig. 6.) The width function on the sheared polygon evaluated on slope p/q is equal to the width function on the original Newton polygon evaluated at slope $(p+16q)/q$. Since the width function on the sheared polygon takes on minimal values when p/q = 2, 3, and $1/0$, the width function on the original Newton polygon takes on minimal values when p/q = 18, 19, and $1/0$. It was shown by Fintushel and Stern [7] that all of these slopes are cyclic surgery slopes for the $(-2, 3, 7)$ pretzel knot.

5. An application to mutant knots

Assume that K is a knot in $\mathbb{S}^3$ and F is a 2-sphere in $\mathbb{S}^3$ with the following properties:

(1) $F \cap \partial X$ = four copies of μ,

(2) $F \cap X$ is essential in X.

We may identify F with the unit sphere so that the punctures of $F \cap K$ are the points $\{(1,0,0), (-1,0,0), (0,0,1), (0,0,-1)\}$, and, inside F, K connects the punctures $(1,0,0)$ to $(-1,0,0)$ and $(0,0,1)$ to $(0,0,-1)$. There are four involutions of F that are central in the mapping class group: the identity, rotation by π about the x-axis, rotation by π about the z-axis, and the product of the last two maps. If we cut along F, apply one of the four involutions, then glue back in, we form one of four knots called *mutants* of K (one of which is K itself).

If K is a hyperbolic knot, then Theorem 7.3 of [4] implies that there is a common non-trivial factor, $C(L, M)$, of the A-polynomial's of K and any mutant of K. Moreover, this factor divides the hyperbolic factor. This leads to the following theorem.

Theorem 5.1. *Let K and K' be hyperbolic mutant knots in $\mathbb{S}^3$. Suppose that both knot groups have property NCIS$^-$. If p/q and r/s are slopes such that*:

(1) *p/q surgery on K is cyclic,*

(2) *r/s surgery on K' is cyclic, and*

(3) *neither p/q nor r/s is a strict boundary slope,*

then $\Delta(p/q, r/s) \leqslant 1$.

Proof. Let $C(L, M)$ be the common factor of the A-polynomials of K and K'. Let w denote the width function on *Newt*(C). Since neither p/q nor r/s is strict boundary slope, neither can be the slope of a side of *Newt*(C). So, both p/q and r/s are good slopes. Since $C(L, M)$ is non-trivial and divides the hyperbolic factor, $\gcd(C, B_{m/n}) = 1$ for all slopes m/n. Hence, by Theorem 3.13, $w(p/q) = w(r/s)$ is the minimal value of w. Moreover, *Newt*(C) is non-degenerate because $C(L, M)$ is non-trivial and divides the hyperbolic factor. Therefore, by Lemma 4.4, $\Delta(p/q, r/s) \leqslant 1$. □

We conclude with two questions.

Question 5.2. By Theorem 3.13, if p/q surgery is cyclic then $w(p/q)$ must be minimal. Examples 3.14 and 4.6 lead one to wonder if the converse is also true.

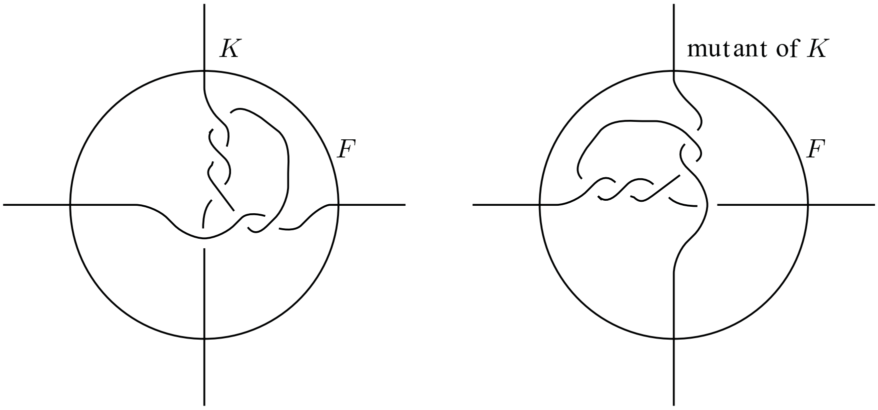

Fig. 7. A mutant of K.

Question 5.3. A knot has *Property-P* if the only surgery which produces a manifold with trivial fundamental group is $1/0$ surgery. If K is a hyperbolic knot with no closed essential surface in its complement, then it is necessary that either $w(1)$ or $w(-1)$ be minimal in order for K to fail to have Property-P. Can one show that neither $w(1)$ nor $w(-1)$ can be minimal for such a knot?

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