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Suzanne Larson
Loyola Marymount University, slarson@lmu.edu

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Finitely 1-convex f-rings

Suzanne Larson

Loyola Marymount University, Los Angeles, CA 90045, United States

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ABSTRACT

This paper investigates f-rings that can be constructed in a finite number of steps where every step consists of taking the fibre product of two f-rings, both being either a 1-convex f-ring or a fibre product obtained in an earlier step of the construction. These are the f-rings that satisfy the algebraic property that rings of continuous functions possess when the underlying topological space is finitely an F-space (i.e. has a Stone–Čech compactification that is a finite union of compact F-spaces). These f-rings are shown to be SV f-rings with bounded inversion and finite rank and, when constructed from semisimple f-rings, their maximal ideal space under the hull-kernel topology contains a dense open set of maximal ideals containing a unique minimal prime ideal. For a large class of these rings, the sum of prime, semiprime, primary and z-ideals are shown to be prime, semiprime, primary and z-ideals respectively.

1. Introduction

A commutative f-ring is 1-*convex* if for any $u, v \in A$ such that $0 \leqslant u \leqslant v$, there is a $w \in A$ such that $u = wv$. Given f-rings A_1, A_2, B and surjective ℓ-homomorphisms $\phi_1 : A_1 \to B$ and $\phi_2 : A_2 \to B$, the *fibre product* of A_1 and A_2, denoted $A_1 \times_B A_2$, is the sub-f-ring of $A_1 \times A_2$ given by $A_1 \times_B A_2 = \{(a_1, a_2)\colon \phi_1(a_1) = \phi_2(a_2)\}$. We say an f-ring (a ring) is a *finite fibre product of the f-rings (rings)* $A_1, A_2, \ldots, A_n$ if it can be constructed in a finite number of steps where every step consists of taking the fibre product of two f-rings (rings), both of these f-rings (rings) satisfying either the property that it is one of the A_i not used in a previous step, or it is a fibre product obtained in an earlier step of the construction. An f-ring A is *finitely* 1-*convex* if it is either a 1-convex f-ring or can be written as a finite fibre product of 1-convex f-rings. These are the f-rings that satisfy the algebraic property that rings of continuous functions possess when the underlying topological space is finitely an F-space. A topological space X is *finitely an F-space* if its Stone–Čech compactification is a union of finitely many closed F-spaces. An often used construction of a space that is finitely an F-space but not an F-space begins with n copies of a compact F-space X and a certain type of closed nowhere dense set $A \subseteq X$, and then for each $a \in A$, all n copies of a are identified as a single point. In [12], it is shown that for a normal space X, $C(X)$ is finitely 1-convex if and only if X is finitely an F-space.

An f-ring A is an *SV f-ring* if for every minimal prime ideal P of A, A/P is a valuation domain. A topological space is an *SV space* if $C(X)$ is an SV f-ring. Mel Henriksen and Richard Wilson initiated the study of SV rings and spaces with their 1992 papers (see [6,7]). A dozen or more papers have been written that study SV rings and spaces and related matters. Spaces that are finitely an F-space were introduced in [6] and are of interest because they are relatively easy to construct, their corresponding ring of continuous functions is an SV f-ring, and the corresponding space of prime ideals is relatively simple. In 2.9, [6] it is shown that if the space X is finitely an F-space then $C(X)$ is an SV f-ring and in [11], it is shown that the converse does not hold. Finitely 1-convex f-rings were later introduced in [12] largely because of their connection to spaces that are finitely an F-space and to SV f-rings. If the space X is compact and finitely an F-space, then, as shown

E-mail address: suzanne.larson@lmu.edu.

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in 5.16 of [5], X contains a dense open set of points of rank 1 (i.e. X contains a dense open set of points x for which the corresponding maximal ideal $M_x = \{f \in C(X): f(x) = 0\}$ contains a unique minimal prime ideal).

In this paper, we investigate commutative semiprime f-rings with identity element that are finitely 1-convex. We will look at the relationship of a finitely 1-convex f-ring to that of an SV f-ring and an f-ring with finite rank and at the bounded ring of elements of a finitely 1-convex f-ring. We then will investigate properties of maximal, minimal, prime, semiprime, primary and z-ideals in finitely 1-convex f-rings. We look at the maximal and minimal prime ideals in a finitely 1-convex f-ring and will show that for a commutative semiprime finitely 1-convex f-ring with identity element that is constructed from semisimple f-rings, the space of maximal ideals contains a dense open set of maximal ideals of rank 1 under the hull-kernel topology. This extends the known result that a compact space that is finitely an F-space has a dense open set of points for which the corresponding maximal ideal has rank 1. In the last section, we show that there is a large class of finitely 1-convex f-rings in which the sum of two prime, semiprime, primary and z-ideals is a prime, semiprime, primary and z-ideal respectively.

2. Preliminaries

Throughout this paper, all rings will be assumed to be commutative semiprime rings with identity element, and with the exception of Section 4, all rings will be f-rings as well.

An f-ring is a lattice ordered ring that is a subdirect product of totally ordered rings. For general information on f-rings see [1]. Given an f-ring A, we let $A^+ = \{a \in A: a \geqslant 0\}$, and for an element $a \in A$, we let $a^+ = a \vee 0$, $a^- = (-a) \vee 0$, and $|a| = a \vee (-a)$. If A is an f-ring with identity element, let $A^* = \{a \in A: |a| \leqslant n \cdot 1$ for some positive integer $n\}$. Then A^* is a sub-f-ring of A, and is called the *subring of bounded elements*. If A is an f-ring with identity element in which every element $a \geqslant 1$ is invertible, then A is said to be *closed under bounded inversion* or to have *bounded inversion*.

An *ℓ-homomorphism* $\phi: A \to B$ mapping an f-ring A to an f-ring B is a ring homomorphism such that for all $a, b \in A$, $\phi(a \vee b) = \phi(a) \vee \phi(b)$ and $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$. A ring ideal I of an f-ring is an *l-ideal* if $|a| \leqslant |b|$ and $b \in I$ implies $a \in I$, or equivalently, if it is the kernel of a lattice-preserving homomorphism (ℓ-homomorphism). Given any element a of an f-ring A, there is a smallest ℓ-ideal containing a, and we denote this by $\langle a \rangle$. Given an f-ring A and an ℓ-ideal I of A, the quotient ring A/I is in fact an f-ring under the usual ring operations on A/I and an order given by $a + I \leqslant b + I$ if there exists $i_1, i_2 \in I$ such that $a + i_1 \leqslant b + i_2$ in A.

Suppose A is an f-ring and I is an ideal of A. The ideal I is *semiprime* (resp. *prime*) if $J^2 \subseteq I$ (resp. $JK \subseteq I$) implies $J \subseteq I$ (resp. $J \subseteq I$ or $K \subseteq I$) for ideals J, K. An ℓ-ideal I of an f-ring is a semiprime (resp. prime) ideal if and only if $a^2 \in I$ implies $a \in I$ (resp. $ab \in I$ implies $a \in I$ or $b \in I$). The f-ring A is called semiprime (resp. prime) if $\{0\}$ is a semiprime (resp. prime) ideal. It is well known that in an f-ring, an ℓ-ideal I is a semiprime ideal if and only if it is an intersection of prime ℓ-ideals which are minimal with respect to containing I. If P is a prime ℓ-ideal of the f-ring A, then A/P is a totally ordered prime ring and all ℓ-ideals of A containing P form a chain. An ideal I of a commutative ring with identity element is *pseudoprime* if $ab = 0$ implies $a \in I$ or $b \in I$ and is *primary* if $ab \in I$ implies $a \in I$ or $b^n \in I$ for some natural number n. In a commutative and semiprime ring, a pseudoprime ideal contains a prime ideal.

For an element a of a ring A, we let $\mathcal{M}_A(a)$ denote the set of all maximal ideals of A containing a. An ideal I of a commutative ring with identity element is called a *z-ideal* if whenever $a, b \in A$ with $\mathcal{M}_A(a) = \mathcal{M}_A(b)$ and $a \in I$, then $b \in I$. Equivalently, I is a z-ideal if whenever $a, b \in A$ with $\mathcal{M}_A(a) \subseteq \mathcal{M}_A(b)$ and $a \in I$, then $b \in I$. It is easily seen that every z-ideal is a semiprime ideal. As is shown in Theorem 1.1 of [13], every minimal prime ideal of an f-ring is a z-ideal.

For any f-ring A, we let $\mathrm{Max}(A)$ denote the set of all maximal ideals of A. If $a \in A$, let $h^c(a) = \{M \in \mathrm{Max}(A): a \notin M\}$. The *hull-kernel topology* on $\mathrm{Max}(A)$ is the topology generated by $\{h^c(a): a \in A\}$. If A has an identity element and satisfies the bounded inversion property then $\mathrm{Max}(A)$, under the hull-kernel topology, will be a compact Hausdorff space (see [4]).

A commutative ring is a *valuation ring* if given any two elements, one divides the other. An f-ring A is an *SV f-ring* if for every minimal prime ideal P of A, A/P is a valuation domain. A commutative f-ring A is said to satisfy the *1st-convexity condition*, or to be 1-*convex* if for any $u, v \in A$ such that $0 \leqslant u \leqslant v$, there is a $w \in A$ such that $u = wv$. In a commutative f-ring A with identity element and satisfying the 1st-convexity condition, $u, v \in A$ with $0 \leqslant u \leqslant v$ implies that there is a $w \in A$ such that $0 \leqslant w \leqslant 1$ and $u = wv$. Every commutative f-ring with the 1st-convexity condition is an SV f-ring and the following lemma shows a further connection between SV f-rings and f-rings satisfying the 1st-convexity condition.

Lemma 1. *([12, Lemma 5.8]) Suppose A is a commutative f-ring with identity element and bounded inversion. Then A is an SV f-ring if and only if for every minimal prime ideal P of A, A/P is* 1*-convex.*

Suppose M is a maximal ℓ-ideal of an f-ring A. The *rank of M* is the number of minimal prime ideals contained in M if the set of all such minimal prime ideals is finite, and the rank of M is infinite otherwise. We let $\mathrm{rank}_A(M)$ denote the rank of the maximal ℓ-ideal M in the f-ring A. If A is an f-ring, then the *rank of A* is the supremum of the ranks of the maximal ℓ-ideals of A. The f-ring A is said to have *finite rank* if the rank of A is finite. A commutative semiprime 1-convex f-ring with identity element has rank 1 as was shown in Theorem 5.6 of [12] and also has the property that all ideals are ℓ-ideals (see [8]). As a result, the set of all prime ideals contained in a given maximal ideal of a commutative semiprime 1-convex f-ring with identity element form a chain.

An *F-space* is a (completely regular) topological space X such that in $C(X)$, the ring of all real-valued continuous functions defined on X, every finitely generated ideal is principal. A number of conditions, both topological conditions on X, and algebraic conditions on $C(X)$, are equivalent to X being an F-space and appear in 14.25 of [2], 1 of [15], and 2.4 of [8]. One particular equivalence we will make use of is that a topological space X is an F-space if and only if $C(X)$ is 1-convex. For a given function $f \in C(X)$, the *zeroset* of f is $Z(f) = \{x \in X\colon\ f(x) = 0\}$. A topological space X is *finitely an F-space* if its Stone–Čech compactification, βX, is a union of finitely many closed F-spaces. See [2] for more information on the Stone–Čech compactification of a space X.

3. Basic properties of finitely 1-convex f-rings

Given f-rings (rings) A_1, A_2, B and surjective ℓ-homomorphisms (homomorphisms) $\phi_1 : A_1 \to B$ and $\phi_2 : A_2 \to B$, recall that the *fibre product* of A_1 and A_2, denoted $A_1 \times_B A_2$, is the sub-f-ring (subring) of $A_1 \times A_2$ given by $A_1 \times_B A_2 = \{(a_1, a_2) \in A_1 \times A_2\colon\ \phi_1(a_1) = \phi_2(a_2)\}$. It is worth noting that if $A_1 = A_2 = B$ is an f-ring (ring) and the identity ℓ-homomorphisms (homomorphisms) are used, then the fibre product $A_1 \times_B A_2 = \{(a, a)\colon\ a \in A_1\} \cong A_1$. On the other hand, if A_1, A_2 are f-rings (rings) and $B = \{0\}$, then the fibre product $A_1 \times_B A_2 = \{(a_1, a_2)\colon\ a_1 \in A_1, a_2 \in A_2\} \cong A_1 \times A_2$ (the direct product of A_1, A_2).

We say an f-ring (a ring) is a *finite fibre product of the f-rings (rings) $A_1, A_2, \dots, A_n$* if it can be constructed in a finite number of steps where every step consists of taking the fibre product of two f-rings (rings), both of these f-rings (rings) satisfying either the property that it is one of the A_i not used in a previous step, or it is a fibre product obtained in an earlier step of the construction. Note that by including the requirement that the f-rings (rings) A_i not be used in more than one step of the construction, we simply require every time a ring is used that is not a fibre product obtained in an earlier step of the construction, that ring be included as an entry in the listing of the A_i's, even if it causes a repetition of rings in the listing. For example, we say $(A \times_{B_1} A') \times_{B_2} A$ is a finite fibre product of the rings A, A', A, and we say $A \times_A A$ is a finite fibre product of A, A.

Our definition of a finite fibre product of the f-rings (rings) $A_1, A_2, \dots, A_n$ allows for variations in the steps taken when constructing such a ring. We may assume that the steps in the construction make use of the f-rings (rings) $A_1, A_2, \dots, A_n$ in the order listed and that the first step of the construction yields an f-ring (ring) of the form $A_1 \times_{B_1} A_2$. Still, later steps could involve taking the fibre product of an f-ring (ring) resulting from an earlier step and the "next" A_i that has not been used in an earlier step, could involve taking the fibre product of two f-rings (rings) resulting from earlier steps, or could involve taking the fibre product of the "next" two A_i that have not been used in an earlier step. For example, the construction of a finite fibre product of the f-rings A_1, A_2, A_3, A_4 could result in an f-ring of the form $((A_1 \times_{B_1} A_2) \times_{B_2} A_3) \times_{B_3} A_4$ or of the form $(A_1 \times_{B_1} A_2) \times_{B_3} (A_3 \times_{B_2} A_4)$. As the next theorem will indicate, every finite fibre product constructed from the f-rings (rings) $A_1, A_2, \dots, A_n$ is isomorphic to a finite fibre product constructed first by taking the fibre product of A_1, A_2, then at each stage taking the fibre product of the f-ring (ring) that resulted from the previous step and the next f-ring (ring) in the list of the A_i's. First, however, we need a version of Goursat's lemma for rings. A similar lemma, given in a different context, appears in [16]. Recall also that a *subdirect product* of the rings $A_1, A_2, \dots, A_n$ is a subring of $A_1 \times A_2 \times \dots \times A_n$ for which each projection mapping onto A_i is surjective.

Lemma 2. *Suppose A, A_1, A_2 are commutative semiprime f-rings (rings) with identity element and $A \subseteq A_1 \times A_2$ is a subdirect product of A_1, A_2. Then there is a commutative f-ring (ring) B such that $A \cong A_1 \times_B A_2$.*

Proof. We prove the result for f-rings. Define $I_1 = \{a \in A_1\colon\ (a, 0) \in A\}$ and $I_2 = \{a \in A_2\colon\ (0, a) \in A\}$. Then I_1, I_2 are ℓ-ideals of A_1, A_2 respectively. We will show that $A_1/I_1 \cong A_2/I_2$. To do so, define $\psi : A_1/I_1 \to A_2/I_2$ by $\psi(a_1 + I_1) = a_2 + I_2$ where $a_2 \in A_2$ is chosen such that $(a_1, a_2) \in A$. First we will show that ψ is a well defined mapping. So suppose that $a_2, a_3 \in A_2$ such that $(a_1, a_2), (a_1, a_3) \in A$. Then $(a_1, a_2) - (a_1, a_3) = (0, a_2 - a_3) \in A$ and so $a_2 - a_3 \in I_2$. This implies that $a_2 + I_2 = a_3 + I_2$ and hence ψ is well defined. It is straightforward to see that ψ preserves the operations of addition, multiplication, and taking supremum. To see that ψ is injective, suppose that $\psi(a + I_1) = \psi(a' + I_1)$. Then $b + I_2 = b' + I_2$ for some $b, b' \in A_2$ with $(a, b), (a', b') \in A$. So $b - b' \in I_2$, and $(0, b - b') \in A$. Then $(a - a', 0) = (a, b) - (a', b') - (0, b - b') \in A$. It follows that $a - a' \in I_1$ and $a + I_1 = a' + I_1$. That ψ is surjective follows from the fact that A is a subdirect product of A_1, A_2. Thus ψ is an ℓ-homomorphism and $A_1/I_1 \cong A_2/I_2$.

Now let $B = A_2/I_2$, let ϕ_1 be ψ composed with the natural ℓ-homomorphism mapping A_1 to A_1/I_1, and let ϕ_2 be the natural ℓ-homomorphism mapping A_2 to A_2/I_2. Then it is straightforward to show that $A = A_1 \times_B A_2$. □

Note that any fibre product of the f-rings (rings) A_1, A_2 is a subdirect product of A_1, A_2.

Theorem 3. *Suppose A, $A_1, A_2, \dots, A_n$ are commutative semiprime f-rings (rings) with identity element and A is a finite fibre product of the f-rings (rings) $A_1, A_2, \dots, A_n$. Then there are f-rings (rings) $B_1, B_2, \dots, B_n$ such that A is ℓ-isomorphic (isomorphic) to*

$$\big(\big((A_1 \times_{B_1} A_2) \times_{B_2} A_3\big) \cdots \times_{B_{n-2}} A_{n-1}\big) \times_{B_{n-1}} A_n.$$

Proof. We will prove the result for f-rings. Suppose A is a finite fibre product of the f-rings $A_1, A_2, \dots, A_n$. We may suppose that the steps in the construction of A make use of the f-rings $A_1, A_2, \dots, A_n$ in the order listed and that the first step of the construction yields $A_1 \times_{B_1} A_2$ for some f-ring B_1. We proceed by induction on the number of f-rings used in the construction of A. If $n = 2$, $A = A_1 \times_{B_1} A_2$, and there is nothing we need prove. Note that if $n = 3$, then A is either $(A_1 \times_{B_1} A_2) \times_{B_2} A_3$ or $A_3 \times_{B_2} (A_1 \times_{B_1} A_2)$ for some f-ring B_2. Since the second of these is isomorphic to the first, the result holds when $n = 3$.

Now suppose that the desired result holds for any finite fibre product constructed from k f-rings, where $2 < k < n$. The final step of the construction of A is to take a fibre product of two f-rings, say K_0, K_1, where either (i) K_0 is a finite fibre product involving $A_1, A_2, \dots, A_{n-1}$ obtained in an earlier step of the construction and $K_1 = A_n$, or (ii) $K_0 = A_n$ and K_1 is a finite fibre product involving $A_1, A_2, \dots, A_{n-1}$ obtained in an earlier step of the construction, or (iii) K_0 is a finite fibre product involving $A_1, A_2, \dots, A_t$ obtained in an earlier step of the construction for some $t < n-1$ and K_1 is a finite fibre product involving $A_{t+1}, A_{t+2}, \dots, A_n$ obtained in an earlier step of the construction. If (i) holds, by our induction hypothesis $K_0 \cong ((A_1 \times_{B_1} A_2) \times_{B_2} A_3) \cdots \times_{B_{n-2}} A_{n-1}$ for some f-rings $B_1, B_2, \dots, B_{n-2}$. Then $A \cong (((A_1 \times_{B_1} A_2) \times_{B_2} A_3) \cdots \times_{B_{n-2}} A_{n-1}) \times_{B_{n-1}} A_n$ for some f-ring B_{n-1} and we are done. Similarly, if (ii) holds, $K_1 \cong ((A_1 \times_{B_1} A_2) \times_{B_2} A_3) \cdots \times_{B_{n-2}} A_{n-1}$ for some f-rings $B_1, B_2, \dots, B_{n-2}$, and $A \cong A_n \times_{B_{n-1}} (((A_1 \times_{B_1} A_2) \times_{B_2} A_3) \cdots \times_{B_{n-2}} A_{n-1}) \cong (((A_1 \times_{B_1} A_2) \times_{B_2} A_3) \cdots \times_{B_{n-2}} A_{n-1}) \times_{B_{n-1}} A_n$ for some f-ring B_{n-1}. If (iii) holds, then by our induction hypothesis $K_0 \cong ((A_1 \times_{B_1} A_2) \times_{B_2} A_3) \cdots \times_{B_{t-1}} A_t$ for some f-rings $B_1, B_2, \dots, B_{t-1}$ and $K_1 = (((A_{t+1} \times_{C_{t+1}} A_{t+2}) \times_{C_{t+2}} A_{t+3}) \cdots \times_{C_{n-1}} A_n)$ for some f-rings $C_{t+1}, C_{t+2}, \dots, C_{n-1}$. Let $K_t = K_0$ and define recursively, for $i = t+1, \dots, n-1$,

$$K_i = \big\{\big(\big(\big((a_1, a_2), a_3\big), \dots, a_{i-1}\big), a_i\big)\colon \big(\big((a_1, a_2), a_3\big), \dots, a_{i-1}\big) \in K_{i-1}, \text{ and there exists } a_{i+1}, a_{i+2}, \dots, a_n \text{ such that } \big(\big((a_{t+1}, a_{t+2}), a_{t+3}\big), \dots, a_n\big) \in K_1\big\}.$$

Then $K_{t+1} \subseteq K_t \times A_{t+1}$ is a subdirect product of K_t and A_{t+1}, and so by the previous lemma, there is an f-ring B_t such that $K_{t+1} \cong K_t \times_{B_t} A_{t+1} \cong (((A_1 \times_{B_1} A_2) \times_{B_2} A_3) \cdots \times_{B_{t-1}} A_t) \times_{B_t} A_{t+1}$. Repeating this argument for $K_{t+2}, \dots, K_{n-1}$ results in $K_{n-1} \cong ((A_1 \times_{B_1} A_2) \times_{B_2} A_3) \cdots \times_{B_{n-2}} A_{n-1}$. Finally, A is ℓ-isomorphic to a subdirect product of K_{n-1} and A_n, and so by the previous lemma, $A \cong K_{n-1} \times_{B_{n-1}} A_n$ for some f-ring B_{n-1}. Hence $A \cong K_{n-1} \times_{B_{n-1}} A_n \cong (((A_1 \times_{B_1} A_2) \times_{B_2} A_3) \cdots \times_{B_{n-2}} A_{n-1}) \times_{B_{n-1}} A_n$ for f-rings $B_1, B_2, \dots, B_{n-1}$. □

Every finite fibre product of the f-rings $A_1, A_2, \dots, A_n$ is ℓ-isomorphic to a sub-f-ring of $A_1 \times A_2 \times \cdots \times A_n$. We let $\psi : A \to A_1 \times A_2 \times \cdots \times A_n$ denote an ℓ-embedding. To aid in our investigation, we adopt the following notational convention: for $i = 1, 2, \dots, n$,

$\pi_i : A \to A_i$ will denote the projection mapping of $\psi(A)$ onto A_i composed with ψ.

Note that for each i, π_i is surjective.

Definition 4. An f-ring A is *finitely* 1*-convex* if it is either a 1-convex f-ring or can be written as a finite fibre product of 1-convex f-rings.

Next we give an example of a finitely 1-convex f-ring that we will make use of several times.

Example 5. Let $\mathbf{R}[x]$ denote the ring of polynomials over the reals in one indeterminate. Totally order $\mathbf{R}[x]$ lexicographically, so that $1 \gg x \gg x^2 \gg \cdots$. Now let $A_1 = \{\frac{p}{q}\colon\ p, q \in \mathbf{R}[x],\ q \geqslant 1\}$ under the usual addition and multiplication of quotients of polynomials and under the order induced by the order on $\mathbf{R}[x]$. That is, $\frac{p_1}{q_1} \leqslant \frac{p_2}{q_2}$ if and only if $p_1 q_2 \leqslant p_2 q_1$. Then A_1 is a totally ordered 1-convex f-ring. Let $A_2 = \{f \in C(\mathbf{N})\colon\ \exists n_0 \in \mathbf{N},\ r \in \mathbf{R}$ such that $f(n) = r\ \forall n \geqslant n_0\}$ under the usual addition, multiplication, and partial order of functions. Then A_2 is also a 1-convex f-ring. Define $\phi_1 : A_1 \to \mathbf{R}$ by $\phi_1(\frac{p}{q}) = \frac{p(0)}{q(0)}$ and $\phi_2 : A_2 \to \mathbf{R}$ by $\phi_2(f) = r$ where there exists $n_0 \in \mathbf{N}$ such that $f(n) = r$ for all $n \geqslant n_0$. Both ϕ_1, ϕ_2 are surjective ℓ-homomorphisms. Then the f-ring $A_1 \times_{\mathbf{R}} A_2 = \{(\frac{p}{q}, f) \in A_1 \times A_2\colon\ \frac{p(0)}{q(0)} = f(n_0)$, where $f(n) = f(n_0)$ for all $n \geqslant n_0\}$ is finitely 1-convex.

A topological space X is *finitely an F-space* if its Stone–Čech compactification is a union of finitely many closed F-spaces. Suppose X is a compact space that is finitely an F-space. Then, as shown in Theorem 5.3 of [12], $C(X)$ is a finitely 1-convex f-ring. In fact, if $X = X_1 \cup X_2$ for some compact F-spaces X_1, X_2, then

$$C(X) \cong C(X_1) \times_{C(X_1 \cap X_2)} C(X_2)$$

where the required ℓ-homomorphisms are the restriction mappings of the form $f \to f|_{X_1 \cap X_2}$. An inductive argument can be employed to show that if $X = \bigcup_{i=1}^n X_i$ for compact F-spaces $X_1, X_2, \dots, X_n$ then $C(X)$ is finitely 1-convex. Conversely, if $C(X)$ is finitely 1-convex (and X is compact), then X is finitely an F-space as is also shown in Theorem 5.3 of [12].

Suppose B is a 1-convex f-ring and Q a semiprime ℓ-ideal of B. One type of finitely 1-convex f-ring that is particularly nice to work with can be constructed as the sub-f-ring of $\prod_{i=1}^n B$ given by $A = \{(b_1, b_2, \dots, b_n)\colon\ b_i - b_j \in Q$ for all $i, j\}$.

Indeed, the f-ring A could be written as a finite fibre product of n copies of B in the form $([(B\times_{B/Q}B)\times_{B/Q}B]\cdots\times_{B/Q}B)$. We will say that an f-ring (ring) A is a *homogeneously finite fibre product* if there is an f-ring (ring) B and a semiprime ℓ-ideal (ideal) Q of B such that A is ℓ-isomorphic to the sub-f-ring (subring) of $B\times B\times\cdots\times B$ given by $\{(b_1,b_2,\dots,b_n)\colon b_i-b_j\in Q$ for all $i,j=1,2,\dots,n\}$.

Definition 6. An f-ring A is *homogeneously finitely* 1*-convex* if there is a 1-convex f-ring B and semiprime ℓ-ideal Q of B such that A is ℓ-isomorphic to the sub-f-ring of $B\times B\times\cdots\times B$ given by $\{(b_1,b_2,\dots,b_n)\colon b_i-b_j\in Q$ for all $i,j=1,2,\dots,n\}$.

Because finitely 1-convex f-rings were first introduced in connection with the study of SV f-rings and f-rings of finite rank, it is appropriate to begin with a theorem that helps to show the relationship of a finitely 1-convex f-ring to that of an SV f-ring and an f-ring with finite rank. The proof of this theorem will make use of the fact that a finitely 1-convex f-ring necessarily satisfies the bounded inversion property as was noted in [12], and the following well-known result that follows directly from Proposition 3 of [3].

Theorem 7. *Suppose A is a commutative semiprime f-ring with identity element. If M is a maximal ideal of A and a is an element in the intersection of all the minimal prime ideals contained within M, there is an element $b\notin M$ such that $ab=0$.*

Theorem 8. *Suppose A is a commutative semiprime f-ring with identity element. Then* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

(1) *A is a finitely* 1*-convex f-ring.*
(2) *A is an SV f-ring with finite rank and bounded inversion.*
(3) *For every u,v such that $0\leqslant u\leqslant v$, there are finitely many elements $w_1,w_2,\dots,w_n\in A$ such that $0=(u-w_1v)(u-w_2v)\cdots(u-w_nv)$.*
(4) *A is an SV f-ring.*

Proof. (1) $\Rightarrow$ (2) appears in Theorems 5.5 and 5.7 of [12].

(2) $\Rightarrow$ (3): Suppose $0\leqslant u\leqslant v$. We first show that for any maximal ideal M, there exist finitely many elements $w_{M,1},w_{M,2},\dots,w_{M,t_M}$ and an element $a_M\in M^+$ such that $(u-w_{M,1}v)(u-w_{M,2}v)\cdots(u-w_{M,t_M}v)a_M=0$. Let M be a maximal ideal and let $t_M=\mathrm{rank}_A(M)$. Suppose the minimal prime ideals contained in M are $P_{M,1},P_{M,2},\dots,P_{M,t_M}$. Since A is an SV f-ring, each factor f-ring $A/P_{M,i}$ is 1-convex by Lemma 1. So for each $i=1,2,\dots,t_M$, there is a $w_{M,i}\in A$ such that $u+P_{M,i}=(w_{M,i}+P_{M,i})(v+P_{M,i})$ in $A/P_{M,i}$. Then $u-w_{M,i}v\in P_{M,i}$ for each i. This implies $(u-w_{M,1}v)(u-w_{M,2}v)\cdots(u-w_{M,t_M}v)\in\bigcap_{i=1}^{t_M}P_{M,i}$. So by the previous theorem there is an $a_M\geqslant 0$ such that $a_M\notin M$ and $(u-w_{M,1}v)(u-w_{M,2}v)\cdots(u-w_{M,t_M}v)a_M=0$.

Now the collection $\{h^c(a_M)\colon M\in\mathrm{Max}(A)\}$ is an open cover of $\mathrm{Max}(A)$. Because A has bounded inversion, $\mathrm{Max}(A)$ is compact and so there is a finite subcover of $\mathrm{Max}(A)$. We will denote the subcover by $\{h^c(a_{M_1}),h^c(a_{M_2}),\dots,h^c(a_{M_n})\}$. Then $a_{M_j}\prod_{i=1}^n(u-w_{M_i,1}v)(u-w_{M_i,2}v)\cdots(u-w_{M_i,t_{M_i}}v)=0$ for each $j=1,2,\dots,n$. So

$$\left[\sum_{j=1}^n a_{M_j}\right]\left[\prod_{i=1}^n(u-w_{M_i,1}v)(u-w_{M_i,2}v)\cdots(u-w_{M_i,t_{M_i}}v)\right]$$
$$=\sum_{j=1}^n\left[a_{M_j}\prod_{i=1}^n(u-w_{M_i,1}v)(u-w_{M_i,2}v)\cdots(u-w_{M_i,t_{M_i}}v)\right]$$
$$=0.$$

Now $\sum_{j=1}^n a_{M_j}$ is not contained in any maximal ideal since $\sum_{j=1}^n a_{M_j}\geqslant a_{M_i}\geqslant 0$ for each i, and since A has bounded inversion, every maximal ideal is an ℓ-ideal and does not contain (at least) one of the a_{M_i}. Because A has bounded inversion, then $\sum_{j=1}^n a_{M_j}$ is a unit of A. This implies that $\prod_{i=1}^n(u-w_{M_i,1}v)(u-w_{M_i,2}v)\cdots(u-w_{M_i,t_{M_i}}v)=0$.

(3) $\Rightarrow$ (4): Suppose $u,v\in A$, P is a minimal prime ideal of A, and $0\leqslant u+P\leqslant v+P$ in A/P. Then there are p_1,p_2 such that $0\leqslant u+p_1\leqslant v+p_2$ in A. Let $u'=u+p_1$, $v'=v+p_2$. By (3), there is a finite number of $w_i\in A$ such that $(u'-w_1v')(u'-w_2v')\cdots(u'-w_nv')=0\in P$. Since P is a prime ideal, there is an i such that $u'-w_iv'\in P$. Then $u+p_1-w_i(v+p_2)\in P$ and it follows that $u-w_iv\in P$. So $u+P=(w_i+P)(v+P)$. This shows A/P is 1-convex and hence by Lemma 1, A is an SV f-ring. □

Property (2) does not imply property (1) in the previous theorem. In [11], an example of a normal topological space X is constructed such that $C(X)$ is an SV f-ring of finite rank, while X is not finitely an F-space. Since a normal topological space is finitely an F-space if and only if its corresponding ring of continuous functions is finitely 1-convex, this $C(X)$ provides an example of a commutative semiprime SV f-ring with identity element and bounded inversion that has finite rank and bounded inversion and yet is not finitely 1-convex.

Note that for a $C(X)$ for X a compact space, properties (2), (3), and (4) of the previous theorem are equivalent since every SV $C(X)$ has finite rank and bounded inversion. (See 4.1 of [5].) However, in general f-rings, property (2) is neither equivalent to property (3) nor to property (4) and we do not know if properties (3) and (4) are equivalent. The next example demonstrates that neither property (3) nor property (4) of the previous theorem implies property (2).

Example 9. Let $\beta\mathbf{N}$ denote the Stone–Čech compactification of the natural numbers $\mathbf{N}$. Let $\alpha \in \beta\mathbf{N} - \mathbf{N}$ be a point for which there is a G_δ set containing α that fails to be a neighborhood of α (i.e. let α be a non-P-point). Let Y denote the topological space $\mathbf{N} \cup \{\alpha\}$ under the subspace topology relative to $\beta\mathbf{N}$. Note that Y is an F-space and so $C(Y)$ is 1-convex. For each $n \in \mathbf{N}$, let X_n denote the topological space obtained by taking n copies of Y and pasting them together at α. We will call the identified point α_n. It is straightforward to show that for each n, $C(X_n)$ is an SV f-ring of rank n. Then by the previous theorem, property (3) holds in each $C(X_n)$. Now let $X = \bigcup_{n=1}^{\infty} X_n$.

We define the f-ring A as follows. Let $A = \{f \in C(X)\colon\ f = m + g$, where $g \in C(X)$, $g|_{X_n} = 0$ for all but finitely many n and $m \in C(X)$ is a constant function $\}$. It is not difficult to see that A is a sub-f-ring of $C(X)$. We will show that property (3) of the previous theorem holds in A. So suppose $0 \leqslant u \leqslant v$ in A, where $u = m_1 + g_1$, $v = m_2 + g_2$, m_1, m_2 are constant functions on X and $g_1, g_2 \in C(X)$ with $g_1|_{X_n}, g_2|_{X_n} = 0$ for all but finitely many n. Let $t \in \mathbf{N}$ be such that $g_1|_{X_n}, g_2|_{X_n} = 0$ for all $n > t$. First assume that $m_2 = 0$. Then $m_1 = 0$ and $u|_{X_n} = v|_{X_n} = 0$ for all $n > t$. Putting this together with the fact that for $n = 1, 2, \ldots, t$, $C(X_n)$ satisfies property (3), it follows easily that A satisfies property (3). Now assume $m_2 \neq 0$. Then $u|_{X_n} = m_1$, $v|_{X_n} = m_2$ for all $n > t$. So $(u - \frac{m_1}{m_2} v)|_{X_n} = 0$ for all $n > t$. This, together with the fact that for $n = 1, 2, \ldots, t$, $C(X_n)$ satisfies property (3), implies that A satisfies property (3). By the previous theorem, then A also satisfies property (4).

Now for each $n \in \mathbf{N}$, a maximal ideal of A is $M_n = \{f \in A\colon\ f(\alpha_n) = 0\}$. For each n, let $Y_{n,i}$ denote the ith copy of Y used in the construction of X_n. Then let $P_{n,i} = \{f \in A\colon\ f(U \cap Y_{n,i}) = 0$ for some neighborhood U of $\alpha_n\}$. For $i = 1, 2, \ldots, n$, $P_{n,i}$ is a minimal prime ideal contained in M_n. This shows that for each $n \in \mathbf{N}$, M_n has rank at least n. So A has infinite rank, and property (2) does not hold in A. Also, note A satisfies the bounded inversion property since if $f \geqslant 1$ and $f = m + g$ where $g \in C(X)$, $g|_{X_n} = 0$ for all but finitely many n and $m \in C(X)$ is a constant function, then $\frac{1}{f} - \frac{1}{m} \in C(X)$ with $(\frac{1}{f} - \frac{1}{m})|_{X_n} = 0$ for all but finitely many n and $f^{-1} = \frac{1}{m} + (\frac{1}{f} - \frac{1}{m}) \in A$.

If A is a commutative semiprime f-ring with identity element and bounded inversion, then A has finite rank if and only if A^* has finite rank as shown in Proposition 3.2 of [5]. A commutative semiprime f-ring A with identity element and bounded inversion is SV if and only if A^* is SV. For a commutative semiprime finitely 1-convex f-ring with identity element, we can show that the sub-f-ring of bounded elements is also finitely 1-convex.

Theorem 10. *Suppose A is a commutative semiprime f-ring with identity element. If A is finitely 1-convex then A^* is also finitely 1-convex.*

Proof. It will be sufficient to establish that (i) if A is 1-convex, then A^* is a 1-convex f-ring and (ii) if $A = A_1 \times_B A_2$ where A_1, A_2 are finitely 1-convex f-rings, A_1^*, A_2^* are finitely 1-convex f-rings, and $\phi_1 : A_1 \to B$, $\phi_2 : A_2 \to B$ are surjective ℓ-homomorphisms then A^* is finitely 1-convex. An induction argument would then show the result holds for all finitely 1-convex f-rings. So, first assume that A is 1-convex and that $0 \leqslant u \leqslant v$ in A^*. Since A is 1-convex, there is a $w \in A$ such that $u = wv$. Then $w \wedge 1 \in A^*$ and $u = (w \wedge 1)v$. Hence A^* is 1-convex.

Next suppose that $A = A_1 \times_B A_2$ where A_1, A_2 are finitely 1-convex f-rings, A_1^*, A_2^* are finitely 1-convex f-rings, and $\phi_1 : A_1 \to B$, $\phi_2 : A_2 \to B$ are surjective ℓ-homomorphisms. For $i = 1, 2$, define $\phi_i^* : A_i^* \to B^*$ to be the restriction mapping $\phi_i|_{A_i^*}$. It is not hard to see that ϕ_i^* preserves the ring and lattice operations. So to show that ϕ_i^* is a surjective ℓ-homomorphism mapping A_i^* onto B^* we need only show that it is surjective. Suppose that $b \in B^*$. Then there is an $m \in \mathbf{N}$ such that $b \leqslant m \cdot 1$. Since ϕ_i is surjective, there exists an $a \in A_i$ such that $\phi_i(a) = b$. Then $a \wedge m \cdot 1 \in A_i^*$ and $\phi_i^*(a \wedge m \cdot 1) = \phi_i(a \wedge m \cdot 1) = \phi_i(a) \wedge \phi_i(m \cdot 1) = b \wedge m \cdot 1 = b$. Hence ϕ_i^* is surjective. It is now straightforward to show that $A^* = A_1^* \times_{B^*} A_2^*$. Hence A^* is finitely 1-convex. □

When the f-ring A does not have the bounded inversion property, it is possible for A^* to be finitely 1-convex, while A is not finitely 1-convex. For example, the f-ring $\mathbf{R}[x]$ of polynomials with real coefficients under the total ordering in which $1 \ll x \ll x^2 \ll x^3 \ll \cdots$ has the property that $A^* = \mathbf{R}$ is a 1-convex f-ring, while A is not finitely 1-convex (or even SV).

4. Ideals in fibre products of (unordered) rings

Some of the basic properties of ideals in finitely 1-convex f-rings that we will use do not depend on the existence of a partial order on the ring. These properties hold in finite fibre products of commutative, but not necessarily partially ordered rings. The purpose of this section is to gather together these basic properties that do not depend on the existence of a partial order.

The following lemma provides us with a means for constructing ideals of various types in a finite fibre product of commutative rings. Its proof is straightforward, and omitted.

Lemma 11. *Let A be a commutative semiprime ring with identity element. Suppose that A is a finite fibre product constructed from the rings $A_1, A_2, \dots, A_n$. Let $j \in \{1, 2, \dots, n\}$.*

(1) *If M_j is a maximal ideal of A_j then $\pi_j^{-1}(M_j)$ is a maximal ideal of A.*
(2) *If P_j is a prime ideal of A_j then $\pi_j^{-1}(P_j)$ is a prime ideal of A.*
(3) *If P_j is a pseudoprime ideal of A_j then $\pi_j^{-1}(P_j)$ is a pseudoprime ideal of A.*
(4) *If P_j is a semiprime ideal of A_j then $\pi_j^{-1}(P_j)$ is a semiprime ideal of A.*
(5) *If P_j is primary and pseudoprime ideal of A_j then $\pi_j^{-1}(P_j)$ is a primary and pseudoprime ideal of A.*
(6) *If P_j is z-ideal of A_j then $\pi_j^{-1}(P_j)$ is a z-ideal of A.*

In fact, for several types of ideals, every ideal of that type is of the form given in the previous lemma, as our next theorem indicates.

Theorem 12. *Let A be a commutative semiprime ring with identity element. Suppose that A is a finite fibre product constructed from the rings $A_1, A_2, \dots, A_n$.*

(1) *If I is an ideal of A and $j \in \{1, 2, \dots, n\}$ then $\pi_j^{-1}(\{0\}) \subseteq I$ if and only if $I = \pi_j^{-1}(\pi_j(I))$.*
(2) *Every prime ideal of A has the form $\pi_j^{-1}(P_j)$ for some $j \in \{1, 2, \dots, n\}$ and prime ideal P_j of A_j.*
(3) *Every minimal prime ideal of A has the form $\pi_j^{-1}(P_j)$ for some $j \in \{1, 2, \dots, n\}$ and minimal prime ideal P_j of A_j.*
(4) *Every maximal ideal of A has the form $\pi_j^{-1}(M_j)$ for some $j \in \{1, 2, \dots, n\}$ and maximal ideal M_j of A_j.*
(5) *Every pseudoprime ideal of A has the form $\pi_j^{-1}(I_j)$ for some $j \in \{1, 2, \dots, n\}$ and pseudoprime ideal I_j of A_j.*
(6) *Every semiprime ideal of A is an intersection of finitely many semiprime ideals of the form $\pi_j^{-1}(I_j)$ for some $j \in \{1, 2, \dots, n\}$ and semiprime ideal I_j of A_j.*
(7) *Every primary and pseudoprime ideal of A has the form $\pi_j^{-1}(I_j)$ for some $j \in \{1, 2, \dots, n\}$ and primary and pseudoprime ideal I_j of A_j.*
(8) *Every z-ideal of A is an intersection of finitely many z-ideals of the form $\pi_j^{-1}(I_j)$ for some $j \in \{1, 2, \dots, n\}$ and z-ideal I_j of A_j.*

Proof. (1): This follows from the fourth isomorphism theorem.

(2), (3): Let P be a prime ideal of A. Note that $\pi_i^{-1}(\{0\})$ is an ideal for $i = 1, 2, \dots, n$, and that $\pi_1^{-1}(\{0\}) \cdot \pi_2^{-1}(\{0\}) \cdots \pi_n^{-1}(\{0\}) = \{0\} \subseteq P$. Since P is prime, $\pi_j^{-1}(\{0\}) \subseteq P$ for some j. Now $\pi_j(P)$ is an ideal of A_j that we will show is prime. Suppose $a_j, b_j \in A_j$ and $a_j b_j \in \pi_j(P)$. There exists $a, b, c \in A$ such that $\pi_j(a) = a_j$, $\pi_j(b) = b_j$, $\pi_j(c) = a_j b_j$, and $c \in P$. Then $c - ab \in \pi_j^{-1}(\{0\}) \subseteq P$. Since $c \in P$, then $ab \in P$. Since P is prime, $a \in P$ or $b \in P$. This implies $a_j \in \pi_j(P)$ or $b_j \in \pi_j(P)$. Hence $\pi_j(P)$ is prime. By (1), $P = \pi_j^{-1}(\pi_j(P))$. Next, we note that if P is a minimal prime ideal, then $\pi_j(P)$ must also be a minimal prime ideal; for if not, there is a prime ideal Q of A_j that is a proper subset of $\pi_j(P)$, which then would imply $\pi_j^{-1}(Q)$ is a prime ideal properly contained in $\pi_j^{-1}(\pi_j(P)) = P$, contrary to P being a minimal prime ideal of A.

(4): Let M be a maximal ideal of A. First suppose for each j that $\pi_j(M) = A_j$. Then for each j, there is a $p_j \in M^+$ such that $\pi_j(p_j) = \pi_j(1)$. Since in a commutative ring with identity element, every maximal ideal is a prime ideal, M is a prime ideal. Then $(1 - p_1)(1 - p_2) \cdots (1 - p_n) = 0$ implies $1 - p_j \in M$ for some j. But then $p_j \in M$ implies $1 \in M$, a contradiction. Hence there must exist a j such that $\pi_j(M)$ is a proper subset of A_j. Define $M_j = \pi_j(M)$. Then M_j is an ideal of A_j. Now $M \subseteq \pi_j^{-1}(M_j)$ and since M is a maximal ideal of A, $M = \pi_j^{-1}(M_j)$.

(5): Suppose I is a pseudoprime ideal of A. Then by (2), there is a prime ideal of the form $\pi_j^{-1}(P_j)$ (for some j and prime ideal P_j of A_j) contained in I. Because $\pi_j^{-1}(\{0\}) \subseteq \pi_j^{-1}(P_j) \subseteq I$, (1) implies $I = \pi_j^{-1}(\pi_j(I))$. Now $\pi_j(I)$ is a pseudoprime ideal of A_j since $P_j \subseteq \pi_j(I)$.

(6): Let I be a semiprime ideal of A. Then I is an intersection of prime ideals of A, and by (2), each of these prime ideals has the form $\pi_j^{-1}(P)$ for some j and prime ideal P of A_j. For $j = 1, 2, \dots, n$, let $\{P_{\alpha_j}\colon \alpha_j \in \Lambda_j\}$ denote the collection of prime ideals of A_j used in forming this intersection. For $j = 1, 2, \dots, n$ let $I_j = \bigcap_{\alpha_j} P_{\alpha_j}$. Then each I_j is a semiprime ideal of A_j and $I = \bigcap_{j=1}^{n} \pi_j^{-1}(I_j)$.

(7): Suppose I is a primary and pseudoprime ideal of A. Since I is pseudoprime, (5) implies $I = \pi_j^{-1}(I_j)$ for some pseudoprime ideal I_j of A_j. We will show that I_j is also primary. Suppose that $a'b' \in I_j$. Let $a, b \in A$ such that $\pi_j(a) = a'$ and $\pi_j(b) = b'$. Then $ab \in \pi_j^{-1}(I_j) = I$. Since I is primary, $a \in I$ or $b^m \in I$ for some m. Then $\pi_j(a) = a' \in I_j$ or $(\pi_j(b))^m = b'^m \in I_j$. Hence I_j is pseudoprime and primary in A_j.

(8): Let I be a z-ideal of A. Since any z-ideal is semiprime, I is the intersection of the prime ideals minimal with respect to containing I, and by (2), each of these prime ideals has the form $\pi_j^{-1}(P)$ for some j and prime ideal P of A_j. Now each of these prime ideals is a z-ideal by Theorem 1.1 of [13], which states that in a commutative ring every minimal ideal in the class of prime ideals containing a z-ideal is a z-ideal. For $j = 1, 2, \dots, n$, let $\{P_{\alpha_j}\colon \alpha_j \in \Lambda_j\}$ denote the collection of prime ideals of A_j used in forming this intersection. For $j = 1, 2, \dots, n$ let $I_j = \bigcap_{\alpha_j} P_{\alpha_j}$. Then each I_j is an intersection of z-ideals and hence is a z-ideal of A_j. Then $I = \bigcap_{j=1}^n \pi_j^{-1}(I_j)$. □

The next example demonstrates that we must take care when using these methods to construct a minimal prime ideal of a finite fibre product ring. In fact, when A is a finite fibre product constructed from the rings $A_1, A_2, \dots, A_n$ and P_k is a minimal prime ideal of A_k, the prime ideal $\pi_k^{-1}(P_k)$ is not necessarily a minimal prime ideal of A. The example we present is in fact an f-ring, demonstrating that even the addition of a partial order structure does not guarantee that for a minimal prime ideal P_k of a coordinate ring A_k, the ideal $\pi_k^{-1}(P_k)$ is a minimal prime ideal.

Example 13. Let A be the finitely 1-convex f-ring defined in Example 5. Let $P_1 = \{0\}$ in A_1 and $P_2 = \{f \in A_2\colon \exists n_0 \in \mathbf{N}$ such that $f(n) = 0\ \forall n \geqslant n_0\}$ in A_2. Then P_1, P_2 are minimal prime ideals of A_1, A_2 respectively. Then $\pi_1^{-1}(P_1) = \{(a_1, a_2) \in A\colon a_1 = 0,\ a_2$ is eventually $0\}$ is a prime ideal of A and $\pi_2^{-1}(P_2) = \{(a_1, a_2) \in A\colon a_1 \in \langle x\rangle,\ a_2$ is eventually $0\}$ is also a prime ideal of A. However, $\pi_1^{-1}(P_1)$ is a proper subset of $\pi_2^{-1}(P_2)$ and so $\pi_2^{-1}(P_2)$ is a prime ideal, but not a minimal prime ideal of A.

In certain homogeneously finite fibre product rings, we can characterize all minimal prime ideals in terms of the minimal prime ideals of the coordinate rings. In the proof of the next theorem we will make use of the fact that in a commutative ring with identity element, a prime ideal P is a minimal prime ideal if and only if for every $p \in P$, there exists $q \notin P$ such that $pq = 0$. This is a re-statement of the fact that a prime ideal P in the commutative ring A with identity element is a minimal prime ideal if and only if $A - P$ is a maximal multiplicative system. (A multiplicative system of a commutative ring A is a set of elements closed under multiplication.) See Chapter V of [14] for more detail.

Theorem 14. *Suppose A is a commutative semiprime ring with identity element. Suppose A is a homogeneously finite fibre product constructed from n copies of the ring B and the semiprime ideal Q. If P is a minimal prime ideal of B, then $\pi_i^{-1}(P)$ is a minimal prime ideal of A for $i = 1, 2, \dots, n$.*

Proof. Suppose P is a minimal prime ideal of B. By Lemma 11, $\pi_i^{-1}(P)$ is a prime ideal of A. We need only show that $\pi_i^{-1}(P)$ is minimal. We do so by showing for any $p \in \pi_i^{-1}(P)$ there is a $q \in A - \pi_i^{-1}(P)$ such that $pq = 0$. Let $p \in \pi_i^{-1}(P)$. Then $\pi_i(p) \in P$ and since P is a minimal prime ideal of B, there is a $b_i \in B - P$ such that $b_i\pi_i(p) = 0$. Consider the case where $Q \nsubseteq P$. Then there is a $q' \in Q - P$. Let q denote the element of A such that $\pi_i(q) = b_i q'$ and for all $j \neq i$, $\pi_j(q) = 0$. Then $q \in A - \pi_i^{-1}(P)$ and $pq = 0$. Next, consider the case where $Q \subseteq P$. Then for each $j \neq i$, $\pi_j(p) = \pi_i(p) + q_j$ for some $q_j \in Q$. For each $j \neq i$, $q_j \in P$ and so there exists $r_j \in B - P$ such that $r_j q_j = 0$. Let $r = b_i \prod_{j\neq i} r_j$. Then $r \in B - P$ and the element q defined by $\pi_k(q) = r$ for every k satisfies $pq = 0$, while $q \in A - \pi_i^{-1}(P)$. □

Suppose A is a commutative semiprime ring with identity element that is a finite fibre product constructed from the rings $A_1, A_2, \dots, A_n$. It should be noted that while part (7) of Theorem 12 asserts that every primary and pseudoprime ideal of A has the form $\pi_j^{-1}(I_j)$ for some $j \in \{1, 2, \dots, n\}$ and primary and pseudoprime ideal I_j of A_j, not every primary ideal of A need be pseudoprime and not every primary ideal of A need be of the form $\pi_j^{-1}(I_j)$ for a primary ideal I_j of A_j. This is the case even when A has a partial ordering; more specifically this is the case even when A is a finitely 1-convex f-ring constructed from 1-convex f-rings as our next example demonstrates.

Example 15. Let $\mathbf{R}[x]$ denote the ring of polynomials over the reals in one indeterminate. Totally order $\mathbf{R}[x]$ lexicographically, so that $1 \gg x \gg x^2 \gg \cdots$. Now let $B = \{\frac{p(x)}{q(x)}\colon p(x), q(x) \in \mathbf{R}[x],\ q(0) \neq 0$ and $q > 0\}$ under the usual addition and multiplication of quotients of polynomials and under the order induced by the order on $\mathbf{R}[x]$. That is, $\frac{p_1}{q_1} \leqslant \frac{p_2}{q_2}$ if and only if $p_1 q_2 \leqslant p_2 q_1$. Then B is a totally ordered 1-convex f-ring. Let $Q = \langle x\rangle$ in B, and let $A = \{(a, b) \in B \times B\colon a - b \in Q\}$. Then A is (homogeneously) finitely 1-convex. In A, consider the ℓ-ideal $I = \{(a, b) \in A\colon a, b \leqslant nx^2$ for some natural number $n\}$. Then I is an ℓ-ideal of A and we will show that I is primary. Suppose $(f, g)(h, k) \in I$, and $(f, g) \notin I$. Then either $f \nleqslant nx^2$ or $g \nleqslant nx^2$ for all natural numbers n. We may suppose that $f \nleqslant nx^2$ for all natural numbers n. Then h must be in Q and by the definition of A, $k \in Q$. So, $(h, k)^N \in I$ for some natural number N. Thus I is a primary ideal. Also, I is not pseudoprime since $(x, 0)(0, x) = (0, 0)$, while $(x, 0) \notin I$ and $(0, x) \notin I$. Now we will show I cannot be written in the form $\pi_1^{-1}(I_1)$ or $\pi_2^{-1}(I_1)$ for some primary ideal I_1 of B. Suppose there is a primary ideal I_1 of B such that $I = \pi_1^{-1}(I_1)$. Then $I_1 = \pi_1(I) = \{a \in B\colon a \leqslant nx^2$ for some $n\}$. Then $\pi_1^{-1}(I_1) = \{(a, b) \in A\colon a \leqslant nx^2,\ b \leqslant nx$ for some $n\} \neq I$. Similarly, there is no primary ideal I_1 of B such that $I = \pi_2^{-1}(I_1)$.

5. Maximal and minimal prime ideals in finitely 1-convex f-rings

Our last two sections focus on several types of ideals in finitely 1-convex f-rings. As is the case with all f-rings, every ℓ-ideal of a finitely 1-convex f-ring is an ideal, but not every ideal in a finitely 1-convex f-ring is an ℓ-ideal. However, there are several classes of ideals of a finitely 1-convex f-ring that are necessarily ℓ-ideals. These include maximal, prime, semiprime, and z-ideals. As Theorem 8 indicates, finitely 1-convex f-rings are SV f-rings and have the bounded inversion property, and it is well known that maximal ideals in f-rings with the bounded inversion property are ℓ-ideals (see [4]). By Theorem 5.9 of [12], every prime and every pseudoprime ideal of a semiprime SV f-ring with bounded inversion is an ℓ-ideal. Since semiprime ideals and z-ideals are intersections of prime ideals and intersections of ℓ-ideals are ℓ-ideals, it then follows that semiprime and z-ideals of a finitely 1-convex f-ring are also ℓ-ideals.

In this section we focus our attention on maximal ideals and minimal prime ideals in finitely 1-convex f-rings. We will show that there are many maximal ideals in a finitely 1-convex f-ring that contain just one minimal prime ideal. That is, for a large class of finitely 1-convex f-rings there is a dense open set of maximal ideals of rank 1 in $\mathrm{Max}(A)$ under the hull-kernel topology.

First, we present a lemma to demonstrate that the ideals of the form $\pi_k^{-1}(M_k)$ need not all be distinct in a finitely 1-convex f-ring.

Lemma 16. *Suppose A is a commutative semiprime f-ring with identity element. Suppose A is a finitely 1-convex f-ring constructed from the 1-convex f-rings $A_1, A_2, \ldots, A_n$ such that A is ℓ-isomorphic to a sub-f-ring of $A_1 \times A_2 \times \cdots \times A_n$ and M_i is a maximal ideal of A_i. Then for $j \neq i$, $\pi_i(\ker(\pi_j)) \subseteq M_i$ if and only if $M_j = \pi_j(\pi_i^{-1}(M_i))$ is a maximal ideal of A_j and $\pi_i^{-1}(M_i) = \pi_j^{-1}(M_j)$.*

Proof. $\Rightarrow$ Suppose $j \neq i$ and $\pi_i(\ker(\pi_j)) \subseteq M_i$. Define $M_j = \pi_j(\pi_i^{-1}(M_i))$. Then M_j is an ideal. We will show M_j is a maximal ideal of A_j. Suppose $p_j \in A_j - M_j$. Let $p \in A$ such that $\pi_j(p) = p_j$. It follows from the definition of M_j that $\pi_i(p) \notin M_i$. Since M_i is a maximal ideal of A_i, there exists $r_i \in A_i$ and $m_i \in M_i$ such that $r_i\pi_i(p) + m_i = \pi_i(1)$. Let $r, m \in A$ such that $\pi_i(r) = r_i$, $\pi_i(m) = m_i$. Then $\pi_i(1 - rp - m) = 0 \in M_i$, which implies $\pi_j(1 - rp - m) \in M_j$. Since $\pi_j(m)$ is also in M_j, this tells us that $\pi_j(1)$ is in the ideal generated by $\pi_j(p)$ and M_j. Hence M_j is a maximal ideal of A_j.

Because A_j is 1-convex, there is a unique minimal prime ideal – call it P_j – of A_j contained in M_j. Then $\pi_j^{-1}(P_j)$ is a prime ideal of A. Next we will show $\pi_j^{-1}(P_j) \subseteq \pi_i^{-1}(M_i)$. Let $p \in \pi_j^{-1}(P_j)$ and suppose $\pi_i(p) \notin M_i$. It follows that there is an $r, m \in A$ such that $\pi_i(m) \in M_i$ and $\pi_i(r)\pi_i(p) + \pi_i(m) = \pi_i(1)$. Then $\pi_i(1 - rp - m) = 0 \in M_i$ implies $\pi_j(1 - rp - m) \in M_j$. But because $p \in \pi_j^{-1}(P_j)$ and $\pi_i(m) \in M_i$ implies $\pi_j(m) \in M_j$, this shows $\pi_j(1) \in M_j$, a contradiction. So $\pi_i(p) \in M_i$ and $\pi_j^{-1}(P_j) \subseteq \pi_i^{-1}(M_i)$. Since the prime ℓ-ideals containing $\pi_j^{-1}(P_j)$ form a chain, and $\pi_i^{-1}(M_i)$, $\pi_j^{-1}(M_j)$ are both maximal ℓ-ideals containing $\pi_j^{-1}(P_j)$, it must be that $\pi_i^{-1}(M_i) = \pi_j^{-1}(M_j)$.

$\Leftarrow$ Suppose for some $j \neq i$, that $M_j = \pi_j(\pi_i^{-1}(M_i))$ and $\pi_i^{-1}(M_i) = \pi_j^{-1}(M_j)$. Let $q_i \in \pi_i(\ker(\pi_j))$. There is a $q \in \ker(\pi_j)$ such that $\pi_i(q) = q_i$. Then $\pi_j(q) = 0 \in M_j$ and hence $q \in \pi_j^{-1}(M_j) = \pi_i^{-1}(M_i)$. Then $q_i = \pi_i(q) \in M_i$. Thus $\pi_i(\ker(\pi_j)) \subseteq M_i$. $\square$

Our next theorem gives a condition under which a maximal ideal of a finitely 1-convex f-ring will have rank 1.

Theorem 17. *Suppose A is a commutative semiprime f-ring with identity element. Suppose A is a finitely 1-convex f-ring constructed from the 1-convex f-rings $A_1, A_2, \ldots, A_n$ such that A is ℓ-isomorphic to a sub-f-ring of $A_1 \times A_2 \times \cdots \times A_n$ and for some i that P_i is the minimal prime ideal contained in the maximal ideal M_i of A_i. If for every $j \neq i$ one of the following two conditions are met, then $\mathrm{rank}_A(\pi_i^{-1}(M_i)) = 1$:*

(1) $\pi_i(\ker(\pi_j)) \nsubseteq M_i$.
(2) $\pi_i(\ker(\pi_j)) \subseteq P_i$ *and* $\pi_j(\ker(\pi_i)) \subseteq R_j$, *where R_j is the minimal prime ideal contained in* $M_j = \pi_j(\pi_i^{-1}(M_i))$.

Proof. We will show every minimal prime ideal contained in $\pi_i^{-1}(M_i)$ is equal to $\pi_i^{-1}(P_i)$. Suppose first that $\pi_i^{-1}(R_i)$ is a minimal prime ideal contained in $\pi_i^{-1}(M_i)$ for some minimal prime ideal R_i of A_i. Then $R_i \subseteq M_i$ and since P_i is the unique minimal prime ideal contained in M_i, $R_i = P_i$ and $\pi_i^{-1}(R_i) = \pi_i^{-1}(P_i)$.

Suppose next that $\pi_j^{-1}(R_j)$ is a minimal prime ideal contained in $\pi_i^{-1}(M_i)$ for some $j \neq i$ and minimal prime ideal R_j of A_j. Note that $\pi_i(\ker(\pi_j)) \subseteq M_i$, since if $q_i \in \pi_i(\ker(\pi_j))$, there exists $q \in A$ such that $\pi_i(q) = q_i$ and $\pi_j(q) = 0 \in R_j$, which implies $q \in \pi_j^{-1}(R_j) \subseteq \pi_i^{-1}(M_i)$ and $\pi_i(q) = q_i \in M_i$. This implies that condition (2) in the statement of the theorem must hold. So $\pi_i(\ker(\pi_j)) \subseteq P_i$. By Lemma 16, $\pi_i^{-1}(M_i) = \pi_j^{-1}(M_j)$ where $M_j = \pi_j(\pi_i^{-1}(M_i))$ is a maximal ideal of A_j. Thus $\pi_j^{-1}(R_j) \subseteq \pi_i^{-1}(M_i) = \pi_j^{-1}(M_j)$ and it follows that $R_j \subseteq M_j$ and R_j is the minimal prime ideal contained in M_j.

Define $R_i = \pi_i(\pi_j^{-1}(R_j))$ and $P_j = \pi_j(\pi_i^{-1}(P_i))$. It is easy to see that R_i is an ideal; we now show it is a prime ideal. So suppose $a_i, b_i \in A_i$ and $a_ib_i \in R_i$. Then there exists a $c \in \pi_j^{-1}(R_j)$ such that $\pi_i(c) = a_ib_i$ and there exists $a, b \in A$ such that $\pi_i(a) = a_i$, $\pi_i(b) = b_i$. Then $c - ab \in \ker(\pi_i)$ and so by condition (2), $\pi_j(c - ab) \in R_j$. Since $\pi_j(c) \in R_j$, it follows that $\pi_j(ab) = \pi_j(a)\pi_j(b) \in R_j$. Therefore $\pi_j(a) \in R_j$ or $\pi_j(b) \in R_j$. This implies $a \in \pi_j^{-1}(R_j)$ or $b \in \pi_j^{-1}(R_j)$ and then $\pi_i(a) = a_i \in R_i$ or $\pi_i(b) = b_i \in R_i$. A similar argument shows P_j is also a prime ideal of A_j.

Since P_i is the unique minimal prime ideal contained in M_i and $R_i \subseteq M_i$, we have $P_i \subseteq R_i$. Suppose now that $P_i \neq R_i$. Then there is a $p_i \in R_i - P_i$ and there is a $p \in A$ such that $\pi_i(p) = p_i$ and $\pi_j(p) \in R_j$. If $\pi_j(p) \in P_j$, then there is an $a \in A$ such that $\pi_j(a) = \pi_j(p)$ and $\pi_i(a) \in P_i$. Then $\pi_j(a - p) = 0$ and $\pi_i(a - p) \in \pi_i(\ker(\pi_j)) \subseteq P_i$. But since $\pi_i(a) \in P_i$, this would imply $\pi_i(p) = p_i \in P_i$, a contradiction. So, $\pi_j(p) \notin P_j$. This implies $P_j \neq R_j$, but $P_j \subseteq M_j$. Since R_j is the unique minimal prime ideal contained in M_j, we have $R_j \subseteq P_j$, contrary to $\pi_j(p) \in R_j - P_j$. Therefore, $P_i = R_i$.

Next we show that $\pi_i^{-1}(R_i) = \pi_j^{-1}(R_j)$. Let $r \in \pi_i^{-1}(R_i)$. Then $\pi_i(r) \in R_i$ and so there is an $s \in A$ such that $\pi_i(s) = \pi_i(r)$ and $\pi_j(s) \in R_j$. Then $\pi_j(r - s) \in \pi_j(\ker(\pi_i)) \subseteq R_j$. Since $\pi_j(s) \in R_j$, then $\pi_j(r) \in R_j$. So $r \in \pi_j^{-1}(R_j)$ and $\pi_i^{-1}(R_i) \subseteq \pi_j^{-1}(R_j)$. Now let $r \in \pi_j^{-1}(R_j)$. By the definition of R_i, $\pi_i(r) \in R_i$ and $r \in \pi_i^{-1}(R_i)$. So $\pi_i^{-1}(R_i) = \pi_j^{-1}(R_j)$. We now have $\pi_j^{-1}(R_j) = \pi_i^{-1}(R_i) = \pi_i^{-1}(P_i)$.

By Theorem 12, every minimal prime ideal contained in $\pi_i^{-1}(M_i)$ is of the form $\pi_k^{-1}(P_k)$ for some k and some minimal prime ideal P_k of A_k, and hence we have shown that any minimal prime ideal contained in $\pi_i^{-1}(M_i)$ is equal to $\pi_i^{-1}(P_i)$. Thus, $\mathrm{rank}_A(\pi_i^{-1}(M_i)) = 1$. □

The previous theorem allows us to characterize the rank of every maximal ideal in a homogeneously finitely 1-convex f-ring.

Corollary 18. *Suppose A is a commutative semiprime f-ring with identity element. Suppose A is a homogeneously finitely 1-convex f-ring constructed from n copies of the 1-convex f-ring B and the semiprime ideal Q and that M is a maximal ideal of B.*

(1) *If $Q \nsubseteq M$, then* $\mathrm{rank}_A(\pi_i^{-1}(M)) = 1$.
(2) *If P is the minimal prime ideal contained in M and $Q \subseteq P$ then* $\mathrm{rank}_A(\pi_i^{-1}(M)) = 1$.
(3) *If $Q \subseteq M$, P is the minimal prime ideal contained in M, and $Q \nsubseteq P$ then* $\mathrm{rank}_A(\pi_i^{-1}(M)) = n$.

Proof. (1) is a direct consequence of the previous theorem since for all $j \neq i$, $\pi_i(\ker(\pi_j)) = Q$.

(2): Suppose $Q \subseteq P$. We will show that for all $j \neq i$, condition (2) of the previous theorem is satisfied. If $j \neq i$, then $\pi_i(\ker(\pi_j)) = Q \subseteq P \subseteq M$. Now by Lemma 16, $M_j = \pi_j(\pi_i^{-1}(M))$ is a maximal ideal of B. Then it is easy to see that $M_j = M$ (in B). Then P is the minimal prime ideal contained in M_j and $\pi_j(\ker(\pi_i)) = Q \subseteq P$. Thus for all $j \neq i$, condition (2) of the previous theorem is satisfied and hence $\mathrm{rank}_A(\pi_i^{-1}(M)) = 1$.

(3): If P' is any minimal prime ideal of B different from P, then since B is 1-convex, $P' \nsubseteq M$. Letting $p' \in P' - M$, we see the element p such that $\pi_k(p) = p'$ for all k is contained in $\pi_j^{-1}(P') - \pi_i^{-1}(M)$ and hence that $\pi_j^{-1}(P') \nsubseteq \pi_i^{-1}(M)$ for all j. So the only possible ideals that could be minimal prime ideals contained in $\pi_i^{-1}(M)$ are of the form $\pi_j^{-1}(P)$, where $j = 1, 2, \ldots, n$. By Theorem 14, each $\pi_j^{-1}(P)$ is a minimal prime ideal. Next, note that each of the $\pi_j^{-1}(P)$ is contained in $\pi_i^{-1}(M)$. For if $p \in \pi_j^{-1}(P)$, then $\pi_i(p) - \pi_j(p) \in Q \subseteq M$ and $\pi_j(p) \in P \subseteq M$ implies that $\pi_i(p) \in M$ and therefore that $p \in \pi_i^{-1}(M)$. If $j_1 \neq j_2$ and $q' \in Q - P$, then the element q of A such that $\pi_j(q) = q'$ when $j = j_1$ and $\pi_j(q) = 0$ when $j \neq j_1$, is contained in $\pi_{j_2}^{-1}(P) - \pi_{j_1}^{-1}(P)$. This shows that $\pi_{j_1}^{-1}(P) \neq \pi_{j_2}^{-1}(P)$ when $j_1 \neq j_2$ and the ideals of the form $\pi_j^{-1}(P)$, for $j = 1, 2, \ldots, n$ are all distinct. We have found that there are exactly n minimal prime ideals of A contained in $\pi_i^{-1}(M)$. Hence $\mathrm{rank}_A(\pi_i^{-1}(M)) = n$. □

When X is a compact space that is finitely an F-space, there is a dense open set of points of X of rank 1 (see 5.16 in [5]). That is to say, there is a dense open set of points of X for which the associated maximal ideal $M_x = \{f \in C(X)\colon f(x) = 0\}$ has rank 1. In a compact space, every maximal ideal of $C(X)$ is of the form M_x for some $x \in X$ and there is a natural homeomorphism between the maximal ideal space $\mathrm{Max}(C(X))$ and X. Since a compact space X is finitely an F-space if and only if $C(X)$ is finitely 1-convex, the following theorem extends the result that in a compact space that is finitely an F-space, there is a dense open set of points of rank 1. Recall that a commutative f-ring with identity element is *semisimple* if the intersection of all its maximal ideals is $\{0\}$ and that every $C(X)$ is semisimple.

Theorem 19. *Suppose A is a commutative semiprime f-ring with identity element. Suppose A is a finitely 1-convex f-ring constructed from the 1-convex f-rings $A_1, A_2, \ldots, A_n$ such that A is ℓ-isomorphic to a sub-f-ring of $A_1 \times A_2 \times \cdots \times A_n$ and that each of the A_i is semisimple. Then there is a dense open set of maximal ideals of rank 1 in* $\mathrm{Max}(A)$ (*under the hull-kernel topology*).

Proof. Suppose A is finitely 1-convex and that each of the A_i are semisimple. Let V denote the set of maximal ideals of rank 1. We will show that $\mathrm{int}(V)$, the interior of V, is dense in $\mathrm{Max}(A)$ by showing that for each $a \in A$ with $a \neq 0$, $h^c(a)$

meets $\operatorname{int}(V)$. So let $a \in A$ with $a \neq 0$. Let $(a)^d$ denote the annihilator of the principal ideal (a). For ease of notation, for any $i, j \in \{1, 2, 3, \ldots, n\}$ we let Q_{ij} denote the ideal $Q_{ij} = \pi_i(\ker(\pi_j))$.

Suppose first that for all i, j with $i \neq j$, $\pi_i^{-1}(Q_{ij}) \subseteq (a)^d$. Since $a \neq 0$, there is a $k_1 \in \{1, 2, \ldots, n\}$ and a maximal ideal M_{k_1} of A_{k_1} such that $\pi_{k_1}(a) \notin M_{k_1}$. Hence $a \notin \pi_{k_1}^{-1}(M_{k_1})$ and $h^c(a) \neq \emptyset$. We will show $h^c(a) \subseteq V$. Suppose $\pi_k^{-1}(M_k) \in h^c(a)$, where M_k is a maximal ideal of A_k. Then $a \notin \pi_k^{-1}(M_k)$. To show that $\pi_k^{-1}(M_k) \in V$, we show condition (2) of Theorem 17 is satisfied. Let P_k denote the minimal prime ideal contained in M_k. Then for all $j \neq k$, $\pi_k^{-1}(Q_{kj}) \cdot (a) = \{0\} \subseteq \pi_k^{-1}(P_k)$. Since $(a) \nsubseteq \pi_k^{-1}(P_k)$, it must be that $\pi_k^{-1}(Q_{kj}) \subseteq \pi_k^{-1}(P_k)$ and hence $Q_{kj} \subseteq P_k$. If $M_j = \pi_j(\pi_k^{-1}(M_k))$ then M_j is a maximal ideal of A_j by Lemma 16 and $\pi_j^{-1}(Q_{jk}) \cdot (a) = \{0\} \subseteq \pi_j^{-1}(R_j)$ where R_j is the minimal prime ideal contained in M_j. Now if $a \in \pi_j^{-1}(R_j)$, then $\pi_j(a) \in M_j$ and so there would exist $b \in A$ such that $\pi_k(b) \in M_k$ and $\pi_j(b) = \pi_j(a)$. This would imply $a - b \in \pi_k^{-1}(Q_{kj}) \subseteq (a)^d$ and $(a-b)a = 0$. Then since $a \notin \pi_k^{-1}(M_k)$ and $\pi_k^{-1}(M_k)$ is prime, we would have $a - b \in \pi_k^{-1}(M_k)$, a contradiction to the fact that $b \in \pi_k^{-1}(M_k)$ and $a \notin \pi_k^{-1}(M_k)$. Thus $a \notin \pi_j^{-1}(R_j)$. Because $\pi_j^{-1}(Q_{jk}) \cdot (a) = \{0\} \subseteq \pi_j^{-1}(R_j)$ and $a \notin \pi_j^{-1}(R_j)$, it follows that $Q_{jk} \subseteq R_j$. We have shown for every $j \neq k$ that $Q_{kj} = \pi_k(\ker(\pi_j)) \subseteq P_k$ and $Q_{jk} = \pi_j(\ker(\pi_k)) \subseteq R_j$; that is, we have shown that condition (2) of Theorem 17 is satisfied, and therefore $\pi_k^{-1}(M_k) \in V$. So $h^c(a) \subseteq V$ and because $h^c(a)$ is open, $h^c(a) \subseteq \operatorname{int}(V)$.

Next suppose that there is an i, j with $i \neq j$ and $\pi_i^{-1}(Q_{ij}) \nsubseteq (a)^d$. Let $\mathcal{B} = \{Q_{i_1 j_1}, Q_{i_2 j_2}, \ldots, Q_{i_m j_m}\}$ denote a maximal set of the Q_{ij} such that $\pi_{i_1}^{-1}(Q_{i_1 j_1}) \cdot \pi_{i_2}^{-1}(Q_{i_2 j_2}) \cdot \pi_{i_3}^{-1}(Q_{i_3 j_3}) \cdots \pi_{i_m}^{-1}(Q_{i_m j_m}) \cdot (a) \neq \{0\}$. Let $z \in \pi_{i_1}^{-1}(Q_{i_1 j_1}) \cdot \pi_{i_2}^{-1}(Q_{i_2 j_2}) \cdot \pi_{i_3}^{-1}(Q_{i_3 j_3}) \cdots \pi_{i_m}^{-1}(Q_{i_m j_m}) \cdot (a)$ with $z \neq 0$. It follows from the hypothesis that each A_i is semisimple and $z \neq 0$ that $h^c(z) \neq \emptyset$. We will show $h^c(z) \subseteq V$. Suppose $\pi_k^{-1}(M_k) \in h^c(z)$ for some k and maximal ideal M_k of A_k. To show that $\pi_k^{-1}(M_k) \in V$, we show one of the two conditions of Theorem 17 is satisfied. Let P_k denote the minimal prime ideal contained in M_k. Then for all $j \neq k$ such that $Q_{kj} \in \mathcal{B}$, we have $Q_{kj} \nsubseteq M_k$ since $z \in \pi_k^{-1}(Q_{kj}) - \pi_k^{-1}(M_k)$. So for all $j \neq k$ such that $Q_{kj} \in \mathcal{B}$, the first condition of Theorem 17 is satisfied. Suppose now that $j \neq k$ and $Q_{kj} \notin \mathcal{B}$. By our choice of $\mathcal{B}$, we have $\pi_k^{-1}(Q_{kj}) \cdot (\pi_{i_1}^{-1}(Q_{i_1 j_1}) \cdot \pi_{i_2}^{-1}(Q_{i_2 j_2}) \cdot \pi_{i_3}^{-1}(Q_{i_3 j_3}) \cdots \pi_{i_m}^{-1}(Q_{i_m j_m}) \cdot (a)) = \{0\} \subseteq \pi_k^{-1}(P_k)$. Since $\pi_k^{-1}(P_k)$ is a prime ideal and $\pi_{i_1}^{-1}(Q_{i_1 j_1}) \cdot \pi_{i_2}^{-1}(Q_{i_2 j_2}) \cdot \pi_{i_3}^{-1}(Q_{i_3 j_3}) \cdots \pi_{i_m}^{-1}(Q_{i_m j_m}) \cdot (a) \nsubseteq \pi_k^{-1}(M_k)$, it must be that $\pi_k^{-1}(Q_{kj}) \subseteq \pi_k^{-1}(P_k)$ and hence $Q_{kj} \subseteq P_k \subseteq M_k$. Now suppose $M_j = \pi_j(\pi_i^{-1}(M_i))$ and R_j is the minimal prime ideal contained in M_j. By Lemma 16, $\pi_k^{-1}(M_k) = \pi_j^{-1}(M_j)$. Now if $q \in \pi_j^{-1}(Q_{jk})$ then $\pi_j(q) \in Q_{jk}$ and so there is an $r \in A$ such that $\pi_j(r) = \pi_j(q)$ and $\pi_k(r) = 0$. Then $q - r \in \pi_j^{-1}(M_j) = \pi_k^{-1}(M_k)$ and since $r \in \pi_k^{-1}(M_k)$, we must have $q \in \pi_k^{-1}(M_k)$. Thus $\pi_j^{-1}(Q_{jk}) \subseteq \pi_k^{-1}(M_k)$. This means that $Q_{jk} \notin \mathcal{B}$, for if Q_{jk} were in $\mathcal{B}$, we would have $z \in \pi_j^{-1}(Q_{jk}) - \pi_k^{-1}(M_k)$ which would imply $\pi_j^{-1}(Q_{jk}) \nsubseteq \pi_k^{-1}(M_k) = \pi_j^{-1}(M_j)$, a contradiction. Then $\pi_j^{-1}(Q_{jk}) \cdot (\pi_{i_1}^{-1}(Q_{i_1 j_1}) \cdot \pi_{i_2}^{-1}(Q_{i_2 j_2}) \cdot \pi_{i_3}^{-1}(Q_{i_3 j_3}) \cdots \pi_{i_m}^{-1}(Q_{i_m j_m}) \cdot (a)) = \{0\} \subseteq \pi_j^{-1}(R_j)$. But since $\pi_j^{-1}(R_j) \subseteq \pi_j^{-1}(M_j) = \pi_k^{-1}(M_k)$ and $\pi_{i_1}^{-1}(Q_{i_1 j_1}) \cdot \pi_{i_2}^{-1}(Q_{i_2 j_2}) \cdot \pi_{i_3}^{-1}(Q_{i_3 j_3}) \cdots \pi_{i_m}^{-1}(Q_{i_m j_m}) \cdot (a) \nsubseteq \pi_k^{-1}(M_k)$, we have $\pi_j^{-1}(Q_{jk}) \subseteq \pi_j^{-1}(R_j)$ and $Q_{jk} \subseteq R_j$. So for all $j \neq k$ such that $Q_{kj} \notin \mathcal{B}$, we have shown that $Q_{kj} = \pi_k(\ker(\pi_j)) \subseteq P_k$ and $Q_{jk} = \pi_j(\ker(\pi_k)) \subseteq R_j$; that is we have shown condition (2) of Theorem 17 is satisfied. Now for all $j \neq k$, we have shown that one of the conditions of Theorem 17 is satisfied. Therefore $\pi_k^{-1}(M_k) \in V$ and $h^c(z) \subseteq V$. Since $h^c(z)$ is open, $h^c(z) \subseteq \operatorname{int}(V)$. By our choice of z, $h^c(z) \subseteq h^c(a)$. It follows that $h^c(z) \subseteq \operatorname{int}(V) \cap h^c(a)$; so $h^c(a)$ meets $\operatorname{int}(V)$. □

We conclude this section with an example to demonstrate that the hypothesis that the A_i be semisimple cannot be left out of the previous theorem.

Example 20. Let A be the finitely 1-convex f-ring defined in Example 15. The f-ring A satisfies all of the hypotheses of the previous theorem except the hypothesis that B is semisimple. Then $\langle x \rangle$ is the unique maximal ideal of B and $M = \langle x \rangle \times \langle x \rangle$ is the unique maximal ideal of A. The maximal ideal M contains two minimal prime ideals: $\{0\} \times \langle x \rangle$ and $\langle x \rangle \times \{0\}$. So the maximal ideal space of A consists of a single maximal ideal and has no element of rank 1.

6. Sums of semiprime, prime, primary, and z-ideals

We now turn to look at the sums of several types of ideals.

A commutative f-ring satisfies the 2nd-convexity property if for any $u, v \in A$ such that $v \geqslant 0$ and $0 \leqslant u \leqslant v^2$, there exists a $w \in A$ such that $u = wv$.

Theorem 21. *Let A be a commutative semiprime f-ring with identity element. Suppose A is a finitely 1-convex f-ring constructed such that at each stage of the construction, the surjective ℓ-homomorphisms map to a semiprime f-ring. Then:*

(1) *A satisfies the 2nd-convexity property.*
(2) *The sum of any two semiprime ideals of A is a semiprime ideal.*
(3) *The sum of any two prime ideals of A is a prime ideal.*
(4) *The sum of any two primary ℓ-ideals of A is a primary ℓ-ideal.*

Proof. First we show that if A_1, A_2 are 1-convex f-rings, B is a semiprime f-ring and $A = A_1 \times_B A_2$, then A satisfies the 2nd-convexity property. Suppose that $0 \leqslant (a_1, a_2) \leqslant (b_1, b_2)^2$ in A. Then for $i = 1, 2$, $0 \leqslant a_i \leqslant b_i^2$ in A_i and since A_i is 1-convex, there is a $w_i \in A_i$ such that $a_i = w_i b_i^2$ and $0 \leqslant w_i \leqslant 1$. Now in B, $0 = \phi_1(a_1) - \phi_2(a_2) = \phi_1(w_1 b_1^2) - \phi_2(w_2 b_2^2) = \phi_1(w_1)\phi_1(b_1^2) - \phi_2(w_2)\phi_2(b_2^2) = \phi_1(w_1)\phi_1(b_1^2) - \phi_2(w_2)\phi_1(b_1^2) = [\phi_1(w_1) - \phi_2(w_2)][\phi_1(b_1)]^2$. Since B is semiprime, $[\phi_1(w_1) - \phi_2(w_2)]\phi_1(b_1) = 0$. It follows that $0 = [\phi_1(w_1) - \phi_2(w_2)]\phi_1(b_1) = \phi_1(w_1)\phi_1(b_1) - \phi_2(w_2)\phi_1(b_1) = \phi_1(w_1)\phi_1(b_1) - \phi_2(w_2)\phi_2(b_2) = \phi_1(w_1 b_1) - \phi_2(w_2 b_2)$. Hence $(w_1 b_1, w_2 b_2) \in A$ and $(a_1, a_2) = (w_1 b_1, w_2 b_2)(b_1, b_2)$.

Next we show that if A_1, A_2 are finitely 1-convex f-rings, each satisfying the 2nd-convexity property, B is a semiprime f-ring and $A = A_1 \times_B A_2$, then A satisfies the 2nd-convexity property. Suppose that $0 \leqslant (a_1, a_2) \leqslant (b_1, b_2)^2$ in A. Then for $i = 1, 2$, $0 \leqslant a_i \leqslant b_i^2$ in A_i and since A_i satisfies the 2nd-convexity property, $a_i = w_i b_i$ for some $w_i \in A_i$. We may assume that $0 \leqslant w_i \leqslant b_i$ for each i. Then $0 = \phi_1(a_1) - \phi_2(a_2) = \phi_1(w_1 b_1) - \phi_2(w_2 b_2) = \phi_1(w_1)\phi_1(b_1) - \phi_2(w_2)\phi_2(b_2) = \phi_1(w_1)\phi_2(b_2) - \phi_2(w_2)\phi_2(b_2) = [\phi_1(w_1) - \phi_2(w_2)]\phi_2(b_2)$. Now B is semiprime, and so by 9.3.1 of [1], $[\phi_1(w_1) - \phi_2(w_2)]\phi_2(b_2) = 0$ implies $|\phi_1(w_1) - \phi_2(w_2)| \wedge |\phi_2(b_2)| = 0$. However, since each $0 \leqslant w_i \leqslant b_i$, we have $|\phi_1(w_1) - \phi_2(w_2)| \leqslant |\phi(b_2)|$ in B. So $0 = |\phi_1(w_1) - \phi_2(w_2)| \wedge |\phi_2(b_2)| = |\phi_1(w_1) - \phi_2(w_2)|$. Thus, $\phi_1(w_1) - \phi_2(w_2) = 0$. So $(w_1, w_2) \in A$ and $(a_1, a_2) = (w_1, w_2)(b_1, b_2)$. Hence A satisfies the 2nd-convexity property.

Part (2) now follows from the fact that in a finitely 1-convex f-ring, every semiprime ideal is an ℓ-ideal and from Corollary 2.3 of [10], which shows that an f-ring with the 2nd convexity property also has the property that the sum of any two semiprime ℓ-ideals is a semiprime ℓ-ideal. Part (3) follows from part (2) and the fact that in a commutative f-ring, a semiprime ℓ-ideal that contains a prime ideal is prime. Part (4) now follows from Theorem 3.3 of [9], which states that in a commutative semiprime f-ring with identity element that satisfies the 2nd-convexity property, the sum of any two primary ℓ-ideals is primary. □

We note that Theorem 4.4 of [8] asserts that in an f-ring satisfying the 2nd-convexity condition, the product of two ℓ-ideals is an ℓ-ideal. So, in a finitely 1-convex f-ring satisfying the hypothesis of Theorem 21, the product of two ℓ-ideals is an ℓ-ideal.

The hypothesis that the surjective ℓ-homomorphisms map to a semiprime f-ring cannot be omitted from the previous theorem as is shown by our next example and in fact, the following theorem will show that for many finitely 1-convex f-rings, if the surjective ℓ-homomorphisms used in constructing a finitely 1-convex f-ring do not map to a semiprime f-ring then there must be two semiprime (prime) ideals whose sum is not semiprime (prime).

Example 22. Let $\mathbf{R}[x]$ denote the ring of polynomials over the reals in one indeterminate. Totally order $\mathbf{R}[x]$ lexicographically, so that $1 \gg x \gg x^2 \gg \cdots$. Now let $B = \{\frac{p}{q}\colon\ p, q \in \mathbf{R}[x],\ q(0) \neq 0 \text{ and } q > 0\}$ under the usual addition and multiplication of quotients of polynomials and under the order induced by the order on $\mathbf{R}[x]$. That is, $\frac{p_1}{q_1} \leqslant \frac{p_2}{q_2}$ if and only if $p_1 q_2 \leqslant p_2 q_1$. Then B is a totally ordered 1-convex f-ring. Let $Q = \{\frac{p}{q} \in B\colon\ p \leqslant n x^3$ for some natural number $n\}$. Let $A = \{(f, g) \in B \times B\colon\ f - g \in Q\}$. Then in A, $0 \leqslant (x^3, x^4) \leqslant (x, x)^2$, but it is impossible to write $(x^3, x^4) = (\frac{p}{q}, \frac{r}{s})(x, x)$ for some $(\frac{p}{q}, \frac{r}{s}) \in A$. So A does not satisfy the 2nd-convexity property. The ideals $\pi_1^{-1}(\{0\})$ and $\pi_2^{-1}(\{0\})$ are prime ideals of A and hence are also semiprime ideals, but their sum is not prime or semiprime.

The following theorem goes a step further than the previous example by showing that for many finitely 1-convex f-rings, the sum of any two semiprime (prime) ideals of A is semiprime (prime), only if the construction of the fibre product employs surjective ℓ-homomorphisms with a semiprime kernel.

Theorem 23. *Let A be a commutative semiprime f-ring with identity element. Suppose $A = A_1 \times_B A_2$, where A_1, A_2 are finitely 1-convex f-rings and $\phi_1 : A_1 \to B$, $\phi_2 : A_2 \to B$ are two ℓ-homomorphisms mapping onto an f-ring B. If in A, the sum of any two semiprime (prime) ideals of A is semiprime (prime), then B is a semiprime f-ring.*

Proof. Suppose that B is not a semiprime f-ring and $\phi_1 : A_1 \to B$ and $\phi_2 : A_2 \to B$ are surjective ℓ-homomorphisms and the sum of any two semiprime ideals of A is a semiprime ideal. Then the ideal $\ker(\phi_1)$, is not semiprime and there exists $x_1 \in A_1$ such that $x_1 \notin \ker(\phi_1)$, but $x_1^2 \in \ker(\phi_1)$. Let $x \in A$ such that $\pi_1(x) = x_1$. Assume that $x = (x_1, x_2)$. Since $x_1^2 \in \ker(\phi_1)$ and $\phi_1(x_1) = \phi_2(x_2)$, then $x_2^2 \in \ker(\phi_2)$. Hence $(x_1^2, 0), (0, x_2^2) \in A$. Now in A, let $I_1 = \pi_1^{-1}(\{0\})$ and $I_2 = \pi_2^{-1}(\{0\})$. Then I_1, I_2 are semiprime ideals of A and $(x_1, x_2)^2 = (x_1^2, x_2^2) = (0, x_2^2) + (x_1^2, 0) \in I_1 + I_2$. Since by hypothesis, $I_1 + I_2$ is semiprime and $(x_1, x_2)^2 \in I_1 + I_2$, then $(x_1, x_2) \in I_1 + I_2$. But this means $(x_1, x_2) = (0, y_2) + (y_1, 0)$ for some $(0, y_2) \in I_1$, $(y_1, 0) \in I_2$. Then $x_1 = y_1, x_2 = y_2$, and yet this is impossible since $\phi_1(0) \neq \phi_2(x_2)$ and $\phi_1(x_1) \neq \phi_2(0)$ implies $(0, x_2), (x_1, 0) \notin A$. This contradiction shows that $\ker(\phi_1), \ker(\phi_2)$, and B must be semiprime.

Under the stated hypotheses, every semiprime ideal and every prime ideal is an ℓ-ideal. By Theorem 2.2 of [10], if in an f-ring the sum of two prime ℓ-ideals is prime then the sum of two semiprime ℓ-ideals is a semiprime ℓ-ideal. It follows from our work above and these facts that if in A, the sum of any two prime ideals of A is prime, then B is a semiprime f-ring. □

As shown in [10], an f-ring with the 2nd-convexity property does not necessarily have the property that the sum of two z-ideals is a z-ideal. However, we will be able to show that many homogeneously finitely 1-convex f-rings do. We start by showing this for a commutative semiprime 1-convex f-ring with identity element.

Lemma 24. *Let A be a commutative semiprime 1-convex f-ring with identity element. Then in A the sum of any two z-ideals is a z-ideal.*

Before proving this lemma and the following Theorem 25, we note that Corollary 2.5 of [10] states that in a commutative f-ring with identity element, the sum of any two minimal prime ℓ-ideals is a prime z-ideal if and only if the sum of any two z-ideals which are ℓ-ideals is a z-ideal. In light of Theorem 21 and the fact that under the hypotheses of this lemma and theorem, prime ideals and z-ideals are ℓ-ideals, it will be sufficient to prove that the sum of two minimal prime ideals is a z-ideal.

Proof. Suppose A is a commutative semiprime 1-convex f-ring with identity element. Then every maximal ideal of A contains a unique minimal prime ideal (since every maximal ideal of a 1-convex f-ring has rank 1 by Theorem 5.6 of [12]). Suppose that Q, Q' are distinct minimal prime ideals of A. Then Q, Q' are contained in distinct maximal ideals of A. If $Q + Q'$ were a proper ideal, then it would be contained in a maximal ideal M, which would imply that $Q, Q' \subseteq M$. But this would say that M does not contain a unique minimal prime ideal, a contradiction. So $Q + Q' = A$, which is a z-ideal. □

We are now ready to show that in many homogeneously finitely 1-convex f-rings, the sum of two z-ideals is a z-ideal.

Theorem 25. *Let A be a commutative semiprime f-ring with identity element. Suppose A is a homogeneously finitely 1-convex f-ring constructed from n copies of the 1-convex f-ring B and the z-ideal Q of B. Then the sum of any two z-ideals of A is a z-ideal.*

Proof. Let Q be a z-ideal of B and suppose $A = \{(a_1, a_2, \dots, a_n) \in \prod_{i=1}^n B\colon a_i - a_j \in Q$ for all $i, j\}$. Again it will be sufficient to show that the sum of two minimal prime ideals of A is a z-ideal. So suppose R, R' are minimal prime ideals of A. We may assume that either $R = \pi_1^{-1}(P)$, $R' = \pi_1^{-1}(P')$ or $R = \pi_1^{-1}(P)$ and $R' = \pi_2^{-1}(P')$ for minimal prime ideals P, P' of B. Suppose first that $R = \pi_1^{-1}(P)$, $R' = \pi_1^{-1}(P')$ for minimal prime ideals P, P' of B. Then $R + R' = \pi_1^{-1}(P) + \pi_1^{-1}(P')$. It is straightforward to show that $\pi_1^{-1}(P) + \pi_1^{-1}(P') = \pi_1^{-1}(P + P')$. In B, the ideal $P + P'$ is the sum of two minimal prime ideals which are z-ideals, and hence $P + P'$ is a z-ideal by the previous lemma. Then by Theorem 11 (6), $R + R' = \pi_1^{-1}(P) + \pi_1^{-1}(P') = \pi_1^{-1}(P + P')$ is a z-ideal.

Next suppose that $R = \pi_1^{-1}(P)$ and $R' = \pi_2^{-1}(P')$ for minimal prime ideals P, P' of B. Suppose that $\mathcal{M}_A(x) = \mathcal{M}_A(y)$ and $x \in \pi_1^{-1}(P) + \pi_2^{-1}(P')$. Then $x = p + q$ for some $p \in \pi_1^{-1}(P)$ and $q \in \pi_2^{-1}(P')$. Then $\pi_1(p) \in P$, and since $\pi_2(p) - \pi_1(p) \in Q$, $\pi_2(p) \in P + Q$. Similarly, $\pi_1(q) \in P' + Q$ and $\pi_2(q) \in P'$. So $\pi_i(x) \in P + Q + P'$ for $i = 1, 2$. Since P, P', Q are z-ideals, by hypothesis $P + Q + P'$ is a z-ideal of B. Since by Theorem 12, every maximal ideal of A has the form $\pi_i^{-1}(M)$ for a maximal ideal M of B, it follows that $\mathcal{M}_B(\pi_i(x)) = \mathcal{M}_B(\pi_i(y))$ for each i. Hence $\pi_i(y) \in P + Q + P'$ for $i = 1, 2$. Then $\pi_1(y) = p_1 + q_1 + r_1$ for some $p_1 \in P$, $q_1 \in Q$, and $r_1 \in P'$. Now $\pi_2(y) = \pi_1(y) + q_2 = p_1 + q_1 + r_1 + q_2$ for some $q_2 \in Q$. Then let w be the element of A for which $\pi_2(w) = p_1 + q_1 + q_2$ and $\pi_i(w) = p_1$ for all $i \neq 2$; and let v be the element for which $\pi_1(v) = q_1 + r_1$, $\pi_2(v) = r_1$, and $\pi_i(v) = \pi_i(y) - p_1$ for all $i \neq 1, 2$. It is easy to see that $w \in A$. Also, since for $i \neq 1, 2$, there exists $q_i \in Q$ such that $\pi_i(y) = \pi_1(y) + q_i$, we have $\pi_1(v) - \pi_i(v) = q_1 + r_1 - \pi_i(y) + p_1 = q_1 + r_1 - (p_1 + q_1 + r_1 + q_i) + p_1 = -q_i \in Q$ and it follows that $v \in A$. Then $y = w + v$. Since $\pi_1(w) = p_1 \in P$, $w \in \pi_1^{-1}(P)$ and since $\pi_2(v) = r_1 \in P'$, $v \in \pi_2^{-1}(P')$. So $y \in \pi_1^{-1}(P) + \pi_2^{-1}(P') = R + R'$. □

A situation similar to that for sums of semiprime ideals in commutative semiprime f-rings with an identity element that are finitely 1-convex holds. That is, for commutative semiprime f-rings with an identity element that are homogeneously finitely 1-convex, the sum of every two z-ideals can be a z-ideal only if the construction of the fibre product employs surjective ℓ-homomorphisms with a kernel that is a z-ideal.

Theorem 26. *Let A be a commutative semiprime f-ring with identity element. If A is a homogeneously finitely 1-convex f-ring constructed from n $(n \geqslant 2)$ copies of the 1-convex f-ring B and semiprime, but non-z-ideal Q of B then there are two z-ideals of A whose sum is not a z-ideal.*

Proof. Suppose A is a homogeneously finitely 1-convex f-ring constructed from n copies of the 1-convex f-ring B and semiprime, but non-z-ideal Q of B. Then there exists $a', b' \in A$ such that $a' \in Q$ while $b' \notin Q$ and $\mathcal{M}_B(a') = \mathcal{M}_B(b')$. Since Q is a semiprime ideal, it is the intersection of prime ideals. So, there is a prime ideal P' such that $Q \subseteq P'$ and $b' \notin P'$. Let $P \subseteq P'$ be a minimal prime ideal. Then $\pi_1^{-1}(P), \pi_2^{-1}(P)$ are z-ideals of A since P, being a minimal prime ideal of B, is a z-ideal of B and by Theorem 11. We now consider $\pi_1^{-1}(P) + \pi_2^{-1}(P)$. Let $a, b, c, d \in A$ be the elements where $\pi_i(a) = a'$ and $\pi_i(b) = b'$ for $i = 1, 2, \dots, n$; $\pi_2(c) = a'$; $\pi_i(c) = 0$ for $i = 1, 3, 4, \dots, n$, and $\pi_2(d) = 0$, $\pi_i(d) = a'$ for $i = 1, 3, 4, \dots, n$.

Then $a = c + d$ and $c \in \pi_1^{-1}(P)$, while $d \in \pi_2^{-1}(P)$. Hence $a \in \pi_1^{-1}(P) + \pi_2^{-1}(P)$. Because $\mathcal{M}_B(a') = \mathcal{M}_B(b')$ and every maximal ideal of A has the form given in Theorem 12, it follows that $\mathcal{M}_A(a) = \mathcal{M}_A(b)$. However, we will show that $b \notin \pi_1^{-1}(P) + \pi_2^{-1}(P)$. Suppose to the contrary that $b = s + t$ where $s \in \pi_1^{-1}(P)$ and $t \in \pi_2^{-1}(P)$. Then $\pi_1(s), \pi_2(t) \in P$. This implies $\pi_1(t) \in P + Q$ and so $\pi_1(b) = \pi_1(s) + \pi_1(t) \in P + P + Q = P + Q \subseteq P'$. This contradicts the fact that $\pi_1(b) = b' \notin P'$. Therefore $b \notin \pi_1^{-1}(P) + \pi_2^{-1}(P)$ and $\pi_1^{-1}(P) + \pi_2^{-1}(P)$ is not a z-ideal. $\square$

We conclude with a concrete example of a homogeneously finitely 1-convex f-ring in which there are two z-ideals whose sum is not a z-ideal.

Example 27. Let $\beta\mathbf{N}$ denote the Stone–Čech compactification of the natural numbers $\mathbf{N}$. Let $B = C(\beta\mathbf{N})$. Then B is a 1-convex f-ring. In B, let $Q = \{f \in C(\beta\mathbf{N})\colon |f|_{\mathbf{N}}(x)|^n \leqslant m \cdot \frac{1}{x}$ for some natural numbers $m, n\}$. Then Q is a semiprime ideal of B. But Q is not a z-ideal since if we define $h : \mathbf{N} \to \mathbf{R}$ by $h(x) = \sum_{i=1}^{\infty} \frac{1}{2^n}(\frac{1}{x})^{1/n}$, $g : \mathbf{N} \to \mathbf{R}$ by $g(x) = \frac{1}{x}$ and let $h^\beta, g^\beta : \beta\mathbf{N} \to \mathbf{R}$ denote the continuous extensions of h, g to $\beta\mathbf{N}$, then $h^\beta \notin Q$, $g^\beta \in Q$. However, $\mathcal{M}(h^\beta) = \mathcal{M}(g^\beta)$ since every maximal ideal of $\beta\mathbf{N}$ is of the form $M_x = \{f \in \beta\mathbf{N}\colon f(x) = 0\}$ for some $x \in \beta\mathbf{N}$ and the functions h^β, g^β have the same zerosets (i.e. $\beta\mathbf{N} - \mathbf{N}$). By the previous theorem, the finitely 1-convex f-ring $A = \{(f_1, f_2) \in B \times B\colon f_1 - f_2 \in Q\}$ has two z-ideals whose sum is not a z-ideal. In fact, if we let $\alpha \in \beta\mathbf{N} - \mathbf{N}$, and P_α denote the minimal prime ideal of B defined by $P_\alpha = \{f \in B\colon Z(f)$ contains a neighborhood of $\alpha\}$, then P_α is a z-ideal of B and hence $\pi_1^{-1}(P_\alpha), \pi_2^{-1}(P_\alpha)$ are z-ideals of A. However, $\pi_1^{-1}(P_\alpha) + \pi_2^{-1}(P_\alpha)$ is not a z-ideal since $(g^\beta, g^\beta) = (0, g^\beta) + (g^\beta, 0) \in \pi_1^{-1}(P_\alpha) + \pi_2^{-1}(P_\alpha)$ and $\mathcal{M}((g^\beta, g^\beta)) = \mathcal{M}((h^\beta, h^\beta))$ while $(h^\beta, h^\beta) \notin \pi_1^{-1}(P_\alpha) + \pi_2^{-1}(P_\alpha)$.

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