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Integrating Non-Euclidean Geometry into High School

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Integrating Non-Euclidean Geometry into High School

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Abstract:

The purpose of this project is to provide the framework for integrating the study of non-Euclidean geometry into a high school math class in ~~such~~ a way that both aligns with the Common Core State Standards and makes use of research-based practices to enhance the learning of traditional geometry. Traditionally, Euclidean geometry has been the only strand of geometry taught in high schools, even though mathematicians have developed several other strands. The non-Euclidean geometry that I focus on in this project is what is known as taxicab geometry. With the Common Core Standards for Math Practice pushing students to "model with mathematics" and "look for and make use of structure", modeling a different geometry with the structure of traditional geometry as a guide can be both highly applicable and highly analytical. This kind of critical thinking is sprinkled throughout the standards. Furthermore, preliminary studies have shown that studying non-Euclidean geometry helps teachers themselves understand notions such as undefined terms more clearly within Euclidean geometry as well. Using resources such as *Taxicab Geometry: An Adventure in Non-Euclidean Geometry* by Eugene Krause, I have constructed new materials for teachers to employ in a high school classroom.

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1. Introduction

1.1 Background:

More than 2,000 years ago, a Greek mathematician named Euclid, in his work known as *The Elements* [1], laid out the foundation for what would become the most commonly studied geometry in the world. The many volumes of that work explained several postulates, definitions, and theorems that define the objects and their properties in two-dimensional and three-dimensional space [1]. Historically, Euclidean geometry has been the primary type of geometry taught in schools. For example, in its introduction to high school geometry standards, the Common Core State Standards Initiative states that "although there are many types of geometry, school mathematics is devoted primarily to plane Euclidean geometry, studied both synthetically (without coordinates) and analytically (with coordinates). Euclidean geometry is characterized most importantly by the Parallel Postulate, that through a point not on a given line there is exactly one parallel line" [2]. The Parallel Postulate is also referred to as Euclid's fifth postulate.

Euclid's fifth postulate became a point of contention with many mathematicians, such as Carl Friedrich Gauss, and led to the creation of geometries that did not follow the same axioms and theorems of Euclidean geometry [3], primarily the parallel or fifth postulate. Since Euclidean geometry is considered the norm, we call such geometries non-Euclidean geometry. These geometries include spherical geometry, hyperbolic geometry, elliptic geometry, and taxicab geometry. In this project, I focus on taxicab geometry because it is the most accessible to high school students and most applicable to the modern world.

Developed by H. Minkowski in the late nineteenth century, taxicab geometry aims to provide a "better model of the artificial world urban world man has built" than Euclidean geometry [4]. Rather than the Euclidean method of measuring distance "as the crow flies", taxicab geometry measures distance using only vertical and horizontal units that emulate street blocks in a city. Hence, this is how the name "taxicab geometry" came to be. In Figure 1, Euclidean geometry would use the length of the blue line as the distance from A to B. In taxicab geometry, the red line would be one way to measure the distance from A to B. This key difference, that distance between points can only be measured along horizontal and vertical segments, forms the basis of taxicab geometry.

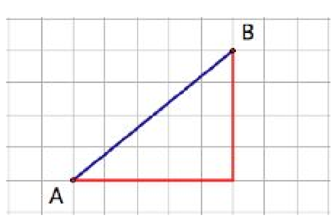

Figure 1

1.2 Completed Work:

Very little work has been done in regards to integrating non-Euclidean geometry into high school classrooms, especially if one looks at taxicab geometry specifically. This could be because the Common Core state standards do not specifically call for curricula to include non-Euclidean geometry. Additionally, many math teachers are unfamiliar with non-Euclidean geometry because of the fact that Euclidean geometry is the mainstream geometry taught. Furthermore, very little has been developed for taxicab geometry because it is a relatively new strand of geometry [3].

One program at Oregon State University used non-Euclidean geometries as a means of teaching K-12 teachers about Euclidean geometry. Among other types of non-Euclidean geometry, the program taught in-service teachers taxicab geometry [5]. The program found that "teachers completing the geometry course reported an improved understanding of the role of definitions and undefined terms in geometry" [5]. If a teacher were to supplement mainstream geometry instruction with taxicab geometry, there would be a purpose and not be used simply as "busy work".

2. Project Objective

Following the current set of standards and status quo, students will only learn about one geometry in their academic careers when several exist in our world. To practice effective critical pedagogy, teachers must be able to teach students different perspectives in mathematics rather than just the dominant narrative [6]. Furthermore, learning about taxicab geometry can help students understand Euclidean geometry even more. As the other studies have shown, knowledge of taxicab geometry can help students have a better grasp on undefined terms in Euclidean geometry [4] [5].

With the new Common Core State Standards, through the Mathematical Practices, students are being called to "model with mathematics", "attend to precision", and to "make use of structure" when learning mathematics [2]. Taxicab geometry provides students with a way to model real life urban settings with mathematics and can give students a glimpse of how the structure imposed by distance is something that can change.

The result of this project is an easily implemented lesson module that teachers can use in their classrooms to teach students more about non-Euclidean geometry while supplementing and reinforcing Euclidean geometry standards. With engaging activities, applications to the real world, and class discussions, this lesson caters to all learners, especially audiovisual and kinetic learners.

3. Unit Module

To best implement taxicab geometry into a Euclidean geometry class, we suggest that the teacher uses the following module for one standard school week after the typical first chapter of most geometry textbooks. This topic is best understood after students have learned about undefined terms, the Euclidean distance formula, and the Euclidean definitions of planar shapes such as circles and squares. The unit first lays down a foundation of defining terms that form the basis of taxicab geometry and then, using familiar concepts and terms from Euclidean geometry, build preliminary theorems and structures in taxicab geometry to help understand the structures in Euclidean geometry. In this way, learning taxicab geometry can be a slight detour that helps students explore the meaning of several common terms in Euclidean geometry to show a different way of looking at similar ideas. Thus, the introduction of taxicab geometry is purposeful, and, if done at the beginning of the course, will not distract from other chapters that students will learn later on. Rather, the taxicab unit will reinforce and deepen understanding and intuition for Euclidean geometry. Note that homework for the unit is not discussed. It is up to the teacher to decide what to assign.

3.1 Day One

The teacher should open the unit with a day revolving around defining what taxicab geometry is, its undefined terms, and the history of how it was developed. The goal of this lesson is to have students possess the necessarily terminology and basic introductory concepts of taxicab geometry. For this lesson, Etch-A-Sketches are suggested.

Begin class with a hook that shows the *Sesame Street* character Grover driving a taxi [8]. Have students write down all the places to which Grover suggests the other character Fat Blue should go. ~~to.~~ This forces students to have a task while watching the video and focuses them on paying attention. This video allows students to decompress before getting into the more rigorous parts of the lesson, laugh, and provide the teacher with a reference to be used in future examples and activities. After the video, have students share with their partners which places they wrote down during the video and then confirm the correct answers with the class. This sharing adds to students being held accountable.

Next, explain to the students that the geometry they have learned all their lives is actually just one of many geometries in existence and that this week they are going to explore a new type of geometry known as taxicab geometry. It should also be emphasized to the students that learning taxicab geometry will help them better understand Euclidean geometry. Have students participate in a think-pair-share activity where they discuss with a partner how they might measure the distance Grover would travel in the middle of a city after first thinking about the answer to themselves. This discussion could get students to start thinking about the differences between Euclidean and taxicab distance. After some time discussing with a partner, have the students share their thoughts from discussing with their partners with the class. At this point, show the class the short video clip that explains the basic definition of what taxicab geometry is [9]. Explain further to the class that for the purposes of this course, taxicab points will be the same as any Euclidean point they have dealt with before because the dilemma of whether or not to include all Euclidean points or just lattice points in a model of taxicab geometry inhibited past programs [5]. To remedy this for the purpose of this project, all Euclidean points will be used. In other words, taxicab points will not be limited simply to lattice points. If available, students can use an Etch-A-Sketch to model how taxicab distance is calculated. For visual learners, this is a great manipulative to engage students in the technical definition of taxicab distance and further add to the feeling that the students are constructing taxicab geometry themselves.

Now that the class has defined what a point is in taxicab geometry, move the class conversation into what we might define a line to be in taxicab geometry, keeping in mind and reminding the class of the definition of a line in Euclidean geometry. If the class did not reach this conclusion themselves, establish that lines in taxicab geometry simply are defined as all the Euclidean points that connect two points such that only horizontal and vertical distances are used. Finally, have the class compare and contrast the differences between undefined terms in taxicab geometry versus Euclidean geometry.

3.2 Day Two

The lesson for this day of the unit will explore the notion of distance in taxicab geometry and compare it to Euclidean distances. One activity in this segment of the module will require the desks to be in a grid shape with easy access in between desks. Prepare the classroom accordingly.

At the beginning of class, randomly pair students and have them discuss and review the taxicab geometry definitions of undefined terms with their partners. For the first activity have all students get up out of their seats with their new partners. Model the following activity before allowing students to complete it with their partners: Two students each choose a desk to be their position and walk towards each other, calculating and recording how many desks they need to walk to get to the other student. Students could alternate between who records the distance and walks to whom. Assigning these roles helps structure this activity and provide students with a specific, concrete way to contribute to the work and be held accountable [7]. This activity is specifically tailored for the kinesthetic learners and provide a hands-on way for all students to first grapple with taxicab distance. Depending on how students are handling the activity and how much time is permitted, the teacher could ask the students to try to find the shortest path between their positions.

After students have run through this activity, bring the students back to their desks for a discussion with them about the difference between calculating such a distance the way they just did and calculating Euclidean distance. Then transition into explaining that horizontal and vertical distances form the basis of what taxicab geometry is. The next discussion should cover what the shortest distance in Euclidean geometry is versus the shortest distance of taxicab geometry. After a think-pair share activity, have the class discuss this distinction. A common axiom in Euclidean geometry is that two points have one shortest distance between each other. In Figures 2, 3, and 4 below, one can see that unlike lines in Euclidean geometry, taxicab points have many different shortest paths between them where the red lines represent the various shortest taxicab distances [10]. Lead student discussion in this direction using whatever scaffolds are necessary.

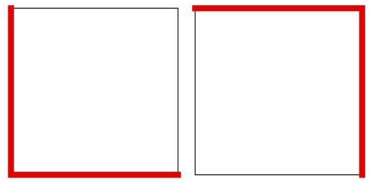

Figure 2

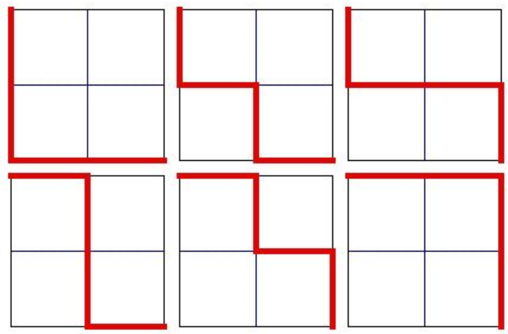

Figure 3

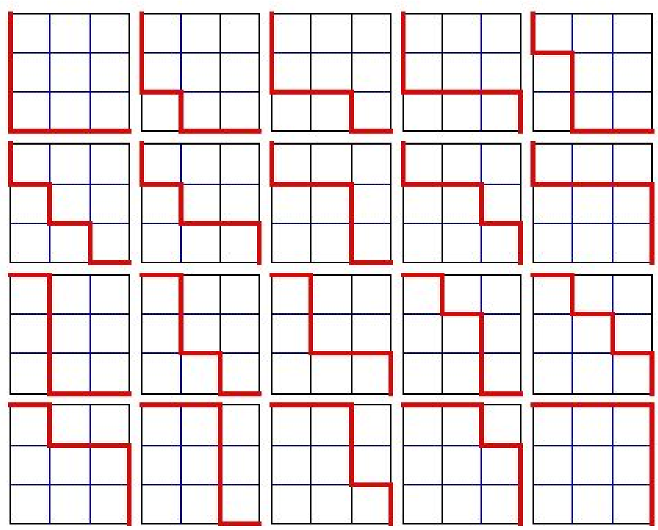

Figure 4

After realizing that there are many ways to find the shortest distance between taxicab points, students will be ready to work~~ing~~ with distance formulas. First, review the Euclidean distance formula students learned previously. After doing so, have students work individually with several taxicab line segments such as those in Figure 5. Direct students to calculate the Euclidean distance using the distance formula and the taxicab distance by counting. While doing this, students are ~~to~~ trying to derive ~~a~~ the distance formula for taxicab geometry. Some of the above-grade-level students may get the formula, but most students will struggle. This struggle is welcome~~d~~ and in line with the Common Core Math Practices of trying out all possible entry points of problem [2].

After this activity, have students write on a Post-It or other piece of paper a possible conjecture for the taxicab distance formula as the lesson's exit ticket. If students have not gone over this word before in a different class, explain precisely what a "conjecture" is in mathematics. If students do not have a conjecture, students may write down a prediction of whether taxicab distances are generally going to be longer than, shorter than, or the same as Euclidean distances and why that might be the case. Finally, if students have a question they may ask it on this exit ticket and the teacher can answer directly on the exit ticket to give immediate feedback. Students could turn in the exit ticket on their way out of class.

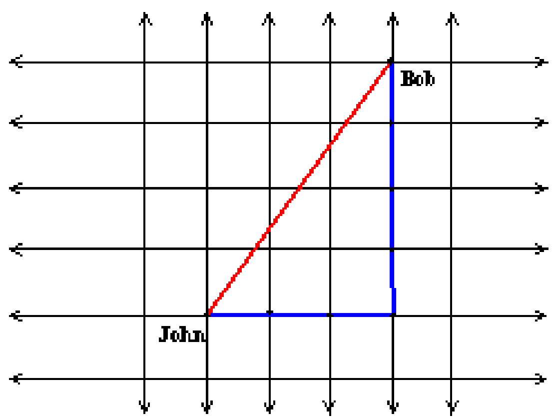

Figure 5

3.3 Day Three

Today's lesson fine tunes the taxicab distance formula and introduces students to the structure of circles in taxicab geometry. Aside from practice with the computational and algebraic aspect of the formulas, the main goal of this lesson is to explore what truly makes a circle a circle across all geometries. Hence, the Common Core Math Practice emphasized in this lesson will be Math Practice 7 which has students "look for and make use of structure" [2]. The goal for this lesson is for students to master or at least feel comfortable with the taxicab distance formula and begin to explore taxicab circles.

Begin the class by having students complete warm-up problems that review the Euclidean distance formula and taxicab distance from the previous lesson. As students are working individually, the teacher should walk around the classroom and assist students as necessary. After the students have completed this activity have students share with their partners the answers they got and how they got those answers before the class convenes to discuss the warm-up questions together. Next, review with the class any conjectures that students came up with on their exit tickets from the previous lesson. The teacher will write on the board which formulas are the closest to the actual formula, if any are present.

If less than half of all students have written the correct formula, reveal to them a hint that the taxicab formula is similar to the Euclidean distance formula but simpler. Allow students in groups of three or four to continue grappling with deriving the taxicab distance formula. As usual, the teacher will walk around as a facilitator to work with and assist groups when necessary. After some time, the class will reconvene and discuss the formula yet again. If a group of students derived the formula, they can present it to the class on the board and explain how they got the formula. If no group was able to derive the formula, the teacher should explain how to derive the formula and explicitly give it to the class. Class discussion of its similarities to the Euclidean formula can follow and the teacher can have groups test out the formula on past examples. Note: the Euclidean distance formula for the distance *d* between points (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_1 - x_2)^2 + (y_2 - y_1)^2}$ and the taxicab distance formula for the distance *d* between points (x_1, y_1) and (x_2, y_2) is $d = |x_1 - x_2| + |y_2 - y_1|$.

Following the reveal of the taxicab distance formula, direct students to do a think-pair-share activity where students first think to themselves, discuss with a partner, and then finally share out to the class what a circle is. It is important to note that you should help students "attend to precision" as laid out in the Common Core Math Practice 6 and try to define a circle rigorously [2]. Many students will define a circle as a closed, curved line. Tell the students to keep such a definition in mind, but to now discuss in the form of another think-pair-share activity with their partner what a circle would look like in taxicab geometry. When the class discusses this part of the lesson, point out the rigorous definition of a circle is "the set of all points in a plane at a fixed distance from a fixed point in the plane" [11]. Nowhere in this definition is there any mention of a curved line being required to be a circle. Breaking down the mathematical jargon for the students, explain that the "fixed distance" the definition mentions is what many of them already know as the radius of a circle and the teacher will draw the Euclidean circle they know well. Then, send the students off again to discuss again what the taxicab circle may look like, given the same fixed-distance definition. Once some time has passed, ask students to sketch what they think the taxicab looks like and explain why as their exit ticket. Questions can also be asked here as always and students can receive immediate feedback. Again, this could be done on a Post-It note that students would turn in on their way out of class.

3.4 Day Four

For today's lesson, students continue and finish discussing the structure of circles in taxicab geometry while introducing the concept of taxicab triangles. Like the previous lesson, the "look for and make use of structure" Math Practice of the Common Core State Standards will be the main focus [2]. The main objective of this lesson is to have students understand that makes a circle a circle are the points that are equidistant to its center, **not** the curved line around the center and to start thinking about the notion of a triangle in taxicab geometry.

Very similarly to the before, begin the class by having students complete warm-up problems that review the Euclidean distance formula and taxicab distance formulas from the previous lesson. Again, as students are working individually, walk around the classroom and assist students as necessary. After the students have completed this activity, have students share with their partners the answers they got and how they got those answers before the class convenes to discuss the warm-up questions together. Next, review with the class any of the sketches that students came up with on their exit tickets from the previous lesson and write them on the board. Circle on the board which of their sketches are the closest to the actual shape of a taxicab circle.

If less than half of all students have gotten the shape of the taxicab circle, divulge a hint that the taxicab circle looks like a familiar Euclidean shape they are very familiar with before letting students in groups of three or four work on comparing their sketches with one another as well as trying to draw the graph of a taxicab circle. After the students have worked with their groups for a sufficient amount of time, the class can come together to discuss their findings. The taxicab circle looks like a Euclidean square, as shown in Figure 6, and will most likely surprise and intrigue students [11]. If the students come up with the correct drawing themselves, have students go up to the board and present how they found their drawing. If no group could come up with the correct drawing, then reveal it to them. Class discussion about why the drawing is indeed a circle should ensue with major emphasis on the definition of a circle.

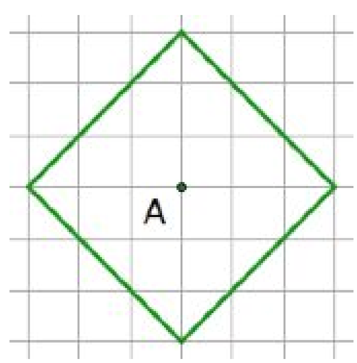

Figure 6

Afterwards, move the lesson along to have students think to themselves what a triangle in taxicab geometry might look like. Again, have students recall individually the rigorous definition of a triangle. Then students will participate in a think-pair-share activity where they share what they think a triangle is defined to be with their partners before sharing to the class. After a small class discussion, write on the board the formal definition which is a closed figure with three line segments as sides. Then hand out a worksheet with several mini-coordinate planes to draw on. The students will then individually try to draw triangles in taxicab geometry, keeping in mind the definition of a triangle. As always, walk around and facilitate the activity.

By the end of the lesson, the class will discuss what they conjectured taxicab triangles looked like and what taxicab triangles look like is revealed. Triangles in taxicab geometry are blocky and must have very defined vertices [4]. Figure 7 reveals what a right triangle may look like. The exit ticket for this lesson would be a challenge question where on the Post-It note students would try to draw an equilateral triangle in taxicab geometry, or ask any clarifying questions they may have. Students would turn this in on their way out of class.

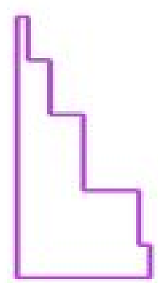

Figure 7

3.5 Day Five

The fifth and final day of the unit will finish the exploration of triangles in taxicab geometry as well as a final wrap-up summarizing taxicab geometry and its close connection to Euclidean geometry. For this lesson, the primary goal is for students to see the connections Euclidean and taxicab geometry have and to reflect on how learning taxicab geometry helped, or in some cases may not have helped, them understand Euclidean geometry better.

To begin class, remind students of what a taxicab triangle looks like and why they look the way they do. Then, segway into talking about the exit ticket from the previous lesson. Guiding questions for this day's think-pair-share activity would be: What is an equilateral triangle defined to be? How would this look in taxicab geometry? After the students have discussed and drawn together what they think an equilateral triangle looks like in taxicab geometry, the class will meet again for discussion. If a pair of students got the correct picture, they would present to the class on the board what the picture is and how they got it. If no student got the correct picture, reveal what an equilateral triangle would look like and why the picture looks the way it does. To give students even more interesting and creative yet informal information, show students Figure 8, which illustrates common Euclidean shapes in their taxicab forms.

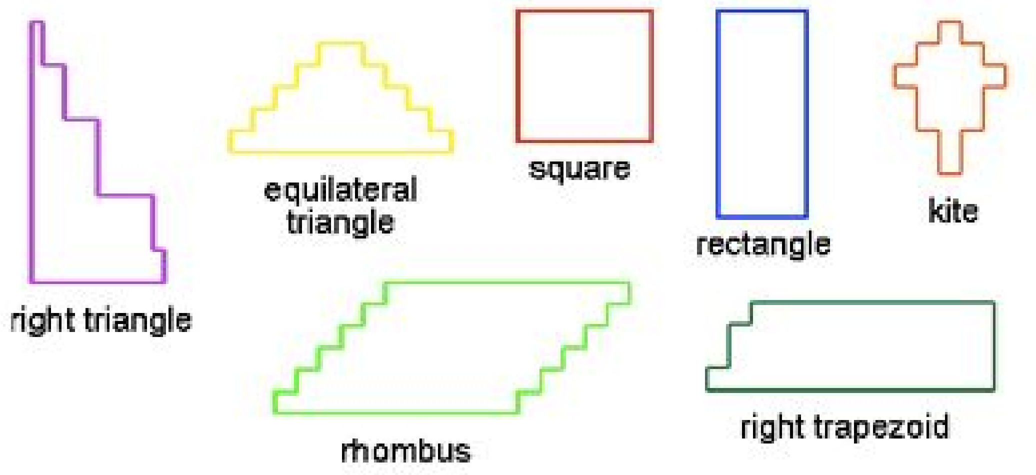

Figure 8

The rest of the lesson will have students tying together their brief excursion into taxicab geometry with Euclidean geometry. In groups of three or four, students will fill out a worksheet, shown below as Figure 9, to define common terms and draw how they look in each geometry.

Term	**Definition/Formula**	**Euclidean**	**Taxicab**
Point			
Line			
Distance			
Circle			
Triangle			
Equilateral Triangle			

Figure 9

After having sufficient time to complete the worksheet, the class will discuss the correct answers and thus allow for students to have a correct and complete worksheet. It is up to you whether or not they will grade this worksheet, but it is highly recommended that students are allowed to keep it to reference throughout the course. The final activity the lesson will end with is a reflection for students to write individually about how their experience with taxicab geometry was. This reflection will serve to gauge how successful the module was and allow students to think about how the lessons impacted them. In the reflection students can say how difficult they found taxicab geometry in general and how difficult it is compared to Euclidean geometry. Additionally, students can discuss whether or not learning taxicab geometry helped them increase their understanding Euclidean geometry and geometry in general. Finally, students can offer feedback on what could make the unit more effective. You can also add whatever they like to the prompt.

4. Possible Complications

Integrating non-Euclidean geometry in high school could be problematic if parents object to adding more to an already rigorous curriculum. Although this paper goes through how exploring taxicab geometry can actually help students understand Euclidean geometry even better, parents may still have issues with implementing such a unit. Thus, we suggest that a teacher who wishes to integrate taxicab geometry works with their administrator and district to clear teaching a unit such as this. Sending a letter home to parents explaining that taxicab geometry will be integrated in the course, that it will only help students understand Euclidean geometry more, and that it will be a relatively quick unit could also remedy the situation. Perhaps first trial runs should also be used with honors geometry classes before trying this module with other classes that have students of different levels of understanding. Once the purpose of the unit is explained and cleared for use in the classroom, we believe a significant amount of parents will understand the importance of trying this module to help students.

5. Further Research

Implementing this lesson with a real class and documenting the results can be very beneficial for seeing whether or not what we expected in theory would be the case in practice. Furthermore, after mastering this week-long module successfully in several classrooms, developing a larger unit could be interesting to see implemented in practice as well. Finally, we created a plan to integrate taxicab geometry into a high school classroom because it is very applicable to the real world and easily compared to Euclidean geometry. However, taxicab geometry is certainly not the only type of non-Euclidean geometry in existence. Hence it would be interesting to look into creating similar modules for other non-Euclidean geometries such as spherical, hyperbolic, or elliptic geometries that could be employed in K-12 classrooms as well.

6. References

[1] Cuomo, Serafina. *Ancient mathematics*. Routledge, 2005.

[2] "Preparing America's Students for Success." *Home | Common Core State Standards Initiative*

[3] Bonola, Roberto. *Non-Euclidean geometry: A critical and historical study of its development*. Courier Corporation, 1955.

[4] Krause, Eugene F. *Taxicab Geometry: An Adventure in Non-euclidean Geometry*. Menlo Park: Dover Publications, 1986.

[5] Damcke, David, Tevian Dray, Maria Fung, Dianne Hart, and Lyn Riverstone. "Using Non-Euclidean Geometry to Teach Euclidean Geometry to K–12 Teachers." Oregon State University

[6] Giroux, Henry A. "Pedagogy and the Politics of Hope Theory, Culture, and Schooling: A Critical Reader." (1997).

[7] Cohen, Elisabeth G., and Rachel A. Lotan. *Designing Groupwork: Strategies for the Heterogeneous Classroom Third Edition*. Teachers College Press, 2014.

[8] *Sesame Street*. PBS. 20 Feb. 1995. <https://www.youtube.com/watch?v=T4p_rX74TgI>.

[9] Taxicab Geometry Public. <https://www.youtube.com/watch?v=csGN_-aF_pI>.

[10] Dickau, Robert M. "Shortest Path Diagrams." *Mathforum.org. 1994*.

[11] Wilson, J. "Taxi Cab Geometry Intro." *jwilson.coe.uga.edu*. University of Georgia. 2009

7. Acknowledgements

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