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Minimal Convex Extensions and Intersections of Primary *l*-Ideals in *f*-rings

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INTRODUCTION

Let n be a positive integer. An f-ring A is said to satisfy the left nth-convexity property if for any $u, v \in A$ such that $v \geqslant 0$ and $0 \leqslant u \leqslant v^n$, there exists a $w \in A$ such that $u = wv$. The right nth-convexity property is defined similarly and an f-ring is said to satisfy the nth-convexity property if it satisfies both the left and the right nth-convexity property. In this paper we study embedding a commutative semiprime f-ring into a commutative semiprime f-ring with a convexity property and apply these results to study intersections of primary ideals in commutative semiprime f-rings. Except where explicitly stated, all rings will be assumed to be commutative and semiprime.

Those f-rings which satisfy one or more of these convexity properties have been studied by several authors. In [GJ, 1D], L. Gillman and M. Jerison note that any $C(X)$, the f-ring of all real-valued continuous functions defined on a topological space X, satisfies the nth-convexity property for all $n \geqslant 2$, and in [GJ, 14.25], they give several properties that in $C(X)$ are equivalent to the 1st-convexity property. M. Henriksen proves some results about the ideal theory of an f-ring satisfying the 2nd-convexity property in [H], and S. Steinberg studies left quotient rings of f-rings satisfying the left 1st-convexity property in [S]. In [HP, Sects. 3, 4] C. Huijsmans and B. de Pagter use the 2nd-convexity property to prove some results about the ideal theory of uniformly complete archimedean f-algebras, and in [HP, Sect. 6; HP 1; HP 2; P] they give several properties that in archimedean f-algebras with identity element are equivalent to the 1st-convexity property. The author has looked at f-rings satisfying a convexity property in [L], giving several results concerning ideal theory and unitability of an f-ring satisfying a convexity property.

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Since f-rings which satisfy one of the convexity conditions have some nice properties, we consider how to start with an arbitrary f-ring and "get to" an f-ring satisfying a convexity property. Section II studies embedding an f-ring in an f-ring satisfying a convexity property, and finding a minimal such embedding for a commutative semiprime f-ring.

Section III gives an application showing how embedding an f-ring in a minimal f-ring satisfying a convexity property can be used in problems that do not originally mention a convexity property. There it is shown that in a commutative semiprime f-ring with identity element, an l-ideal I satisfying $I = \langle I\sqrt{I}\rangle$ or $I = I{:}\sqrt{I}$ is an intersection of primary l-ideals and a pseudoprime l-ideal I satisfying $I = \langle I\sqrt{I}\rangle$ or $I = I{:}\sqrt{I}$ is primary.

The problem of identifying l-ideals which are intersections of primary ideals in $C(X)$ has been studied by R. D. Williams in [W], and our results generalize some of that work.

I. Preliminaries

By an ideal we will always mean a ring ideal. Suppose A is a ring and I an ideal of A. We will use the notation $I(a)$ for cosets of I. The ideal I is called *semiprime* (*prime*) if whenever J (J_1, J_2) is an ideal such that $J_2 \subseteq I (J_1 J_2 \subseteq I)$, $J \subseteq I (J_1 \subseteq I$ or $J_2 \subseteq I)$. The ring A is called semiprime (prime) if $\{0\}$ is a semiprime (prime) ideal.

An f-ring is a subdirect product of totally ordered rings. For background material on f-rings see [BKW]. A prime f-ring is a totally ordered domain and a semiprime f-ring is a subdirect product of totally ordered domains.

An ideal I of an f-ring A is said to be an l-ideal if $|x| \leqslant |y|$, $y \in I$ implies $x \in I$. Given a subset $S \subseteq A$ there is a smallest l-ideal containing S, and we will denote this by $\langle S\rangle$. It is well known that the sum of two l-ideals is again an l-ideal. It is also well known that the l-ideals containing a given prime l-ideal form a chain.

Recall that if n is a positive integer, then an f-ring A is said to satisfy the left nth-convexity property if for any $u, v \in A$ such that $v \geqslant 0$ and $0 \leqslant u \leqslant v^n$, there exists a $w \in A$ such that $u = wv$. The right nth-convexity property is defined similarly and an f-ring satisfies the nth-convexity property if it satisfies both the right and the left nth-convexity property. If $n \geqslant 2$ and A satisfies the (left) nth-convexity property, then we may assume that the element w satisfies $0 \leqslant w \leqslant v^{n-1}$ (by replacing the element w, if necessary, by $(w \wedge v^{n-1}) \vee 0$). It is easily seen that

(1.1) An f-ring satisfying the 1st-convexity property also satisfies the nth-convexity property for all $n \geqslant 2$.

Let A be a semiprime f-ring. The following appears in [L, 2.1]:

(1.2) If $n \geqslant 2$ and if whenever $u, v \in A$ with $v \geqslant 0$ and $0 \leqslant u \leqslant v^n$, there is a $w \in A$ such that $0 \leqslant w \leqslant v^{n-1}$ and $u = wv$, then the element w is unique.

The following is proved in [L, 2.3, 3.9] when $n \geqslant 1$ and A satisfies the nth-convexity property.

(1.3) Any l-homomorphic image of A satisfies the nth-convexity property.

(1.4) If A has an identity element and if $0 \leqslant u \leqslant v$ and $u^{-1} \in A$, then $v^{-1} \in A$.

In [L, 4.4] the following is shown.

(1.5) Let A be an f-ring satisfying the 2nd-convexity property. If I, J are l-ideals of A, then IJ is also an l-ideal in A.

II

In this section, we discuss embedding an f-ring into an f-ring satisfying a convexity property.

2.1. DEFINITION. Let A be an f-ring. An f-ring B is an n-convexity cover of A if A is embedded in B and B satisfies the nth-convexity property.

In the next theorem necessary and sufficient conditions for the existence of an n-convexity cover of a semiprime, but not necessarily commutative, f-ring are given. Recall that a (noncommutative) domain R is a left Ore domain if for $a, b \in R$, there exist $a_1, b_1 \in R\backslash\{0\}$ such that $b_1 a = a_1 b$.

THEOREM 2.2. *Let $n \geqslant 1$. If A is a semiprime f-ring, then A has a (semiprime) n-convexity cover if and only if A can be embedded in a direct product of totally ordered division rings.*

Proof. Suppose A has a semiprime n-convexity cover B. By (1.3), B is a subdirect product of totally ordered domains which satisfy the left nth-convexity property. It follows that each of these totally ordered domains is a left Ore domain and hence is embeddable in a totally ordered division ring. ∎

In [J, II 6.1], D. Johnson gives an example of a totally ordered l-simple domain that cannot be embedded in a totally ordered division ring. So not every totally ordered domain has an n-convexity cover.

However, the last theorem does imply that every semiprime commutative f-ring has an n-convexity cover. Next we ask, for a semiprime commutative f-ring is there a minimal such cover, and if there is, does it enjoy a universal mapping property? To facilitate this discussion we make the following definitions.

2.3. DEFINITIONS. Let $n \geqslant 1$ and A be an f-ring.

(1) An n-convexity cover $K_n A$ of A with $e\colon A \to K_n A$ is a minimal n-convexity cover of A if whenever $\phi\colon A \to B$ is an embedding into a semiprime f-ring B which satisfies the nth-convexity property, there is an embedding $\bar{\phi}\colon K_n A \to B$ such that $\phi = \bar{\phi} \circ e$.

(2) Suppose $K_n A$ is a minimal n-convexity cover of A with $e\colon A \to K_n A$. Then $K_n A$ satisfies the universal mapping property if whenever $\phi\colon A \to B$ is a homomorphism of A into a semiprime f-ring B satisfying the nth-convexity property, there is a homomorphism $\bar{\phi}\colon K_n A \to B$ such that $\bar{\phi} \circ e = \phi$.

For a commutative semiprime f-ring, we will always be able to find a minimal n-convexity cover if $n \geqslant 2$. If $n = 1$, the problem is not as easy.

THEOREM 2.4. *Let $n \geqslant 2$ and A be a commutative semiprime f-ring. Then there is a unique (up to isomorphism) commutative semiprime f-ring $K_n A$ which is a minimal n-convexity cover of A, and which satisfies the universal mapping property. If A is a subdirect product of the totally ordered domains A_i and $\Pi Q(A_i)$ denotes the direct product of the quotient fields $Q(A_i)$, then $K_n A$ is isomorphic to a unique sub-f-ring of $\Pi Q(A_i)$. Moreover, if A is a direct sum (direct product) of the A_i, then $K_n A$ is a direct sum (direct product) of the $K_n(A_i)$, the minimal convexity covers of the A_i.*

Portions of the proof will be separated out and stated in the following lemmas.

LEMMA 2.5. *Let $n \geqslant 2$ and $\{A_i : i \in I\}$ be a collection of f-rings contained in the semiprime f-ring A. If each A_i satisfies the* nth-*convexity property, then $\bigcap \{A_i : i \in I\}$ satisfies the* nth-*convexity property.*

Proof. Suppose $0 \leqslant u \leqslant v^n$ and $v \geqslant 0$ in $\bigcap \{A_i : i \in I\}$. Then $0 \leqslant u \leqslant v^n$ and $v \geqslant 0$ in A and in each A_i. By (1.2), there is a unique element $w \in A$ such that $0 \leqslant w \leqslant v^{n-1}$ and $u = wv$. Since each A_i satisfies the nth-convexity property, $w \in \bigcap \{A_i : i \in I\}$. ∎

LEMMA 2.6. *Let $n \geqslant 2$ and let B be an n-convexity cover of the f-ring A with embedding $e\colon A \to B$. If B is the convex sub-l-ring of B generated by $e(A)$ then the following hold.*

(1) *For every l-ideal I of A, $\langle e(I) \rangle \cap e(A) = e(I)$.*

(2) *For every semiprime l-ideal I of A, $\sqrt{\langle e(I) \rangle} \cap e(A) = e(I)$, where $\sqrt{\langle e(I) \rangle}$ denotes the smallest semiprime l-ideal of $K_n A$ containing $e(I)$.*

Moreover, if B is a minimal n-convexity cover of A, or if C is a semiprime n-convexity cover of A and B is the intersection of all the sub-f-rings of C

which satisfy the nth-*convexity property and which contain* $e(A)$, *then* B *satisfies the* nth-*convexity property and* B *is the convex sub-l-ring of* B *generated by* $e(A)$.

Proof. (1) The fact that $\langle e(I)\rangle \cap e(A)=e(I)$ follows easily from the hypothesis.

(2) Suppose that $a\in\sqrt{\langle e(I)\rangle}\cap e(A)$. Then $a^m\in\langle e(I)\rangle\cap e(A)$ for some m. But by (1), $\langle e(I)\rangle\cap e(A)=e(I)$, so $a^m\in e(I)$. Hence $a\in e(I)$.

Now suppose either that B is a minimal n-convexity cover of A, or that C is a semiprime n-convexity cover of A and B is the intersection of all the sub-f-rings of C which satisfy the nth-convexity property and which contain $e(A)$. Let $B'=\{b\in B: |b|\leqslant e(a)$ for some $a\in A^+\}$. Then B' is a sub-f-ring of B. Suppose $v\geqslant 0$ and $0\leqslant u\leqslant v^n$ in B'. Then $v\leqslant e(a)$ for some $a\in A^+$. Also, there is a $w\in B$ such that $u=wv$ and $0\leqslant w\leqslant v^{n-1}$. So $0\leqslant w\leqslant v^{n-1}\leqslant e(a^{n-1})$, which implies $w\in B'$. Thus B' satisfies the nth-convexity property. By hypothesis, B either is embedded in or is contained in B'. ∎

LEMMA 2.7. *Let* $n\geqslant 2$ *and suppose* A *is a sub-*f*-ring of an* f*-ring* B *which satisfies the* nth-*convexity property. Suppose* $I\subseteq A$ *is a semiprime l-ideal in* B. *Then if* A/I *satisfies the* nth-*convexity property,* A *also satisfies the* nth-*convexity property.*

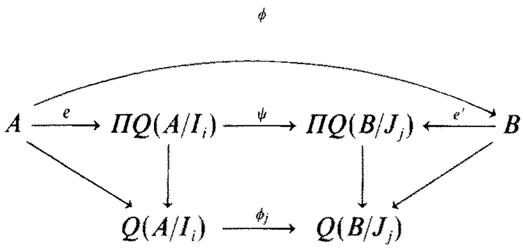
Proof. Suppose $0\leqslant u\leqslant v^n$ and $v\geqslant 0$ in A. Then there is a $w\in B$ such that $u=wv$ and $0\leqslant w\leqslant v^{n-1}$. Now $0\leqslant I(u)\leqslant I(v^n)$ in A/I. So there is an element $w'\in A$ such that $I(u)=I(w'v)$ and $0\leqslant I(w')\leqslant I(v^{n-1})$. Since I is semiprime, B/I is semiprime. In B/I, $I(u)=I(wv)$ with $0\leqslant I(w)\leqslant I(v^{n-1})$ and at the same time, $I(u)=I(w'v)$ with $0\leqslant I(w')\leqslant I(v^{n-1})$. By (1.2), $I(w)=I(w')$. That is, $w=w'+b$ for some $b\in I\subseteq A$. Therefore $w\in A$. ∎

We now give the proof of Theorem 2.4.

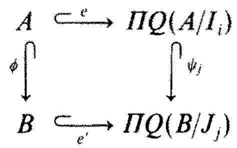
Proof. Let $\{I_i: i\in\Gamma\}$ denote the collection of all proper prime l-ideals in A. Then A is a subdirect product of the totally ordered domains A/I_i. So there is an embedding $e: A\to \Pi Q(A/I_i)$ given by $[e(a)]_i=I_i(a)$. Note that $\Pi Q(A/I_i)$ is a semiprime f-ring satisfying the nth-convexity property. Let K_nA be the intersection of all sub-f-rings of $\Pi Q(A/I_i)$ which contain $e(A)$ and which satisfy the nth-convexity property. By Lemma 2.5, K_nA satisfies the nth-convexity property.

Now suppose that $\phi: A\to B$ embeds A into a semiprime f-ring B satisfying the nth-convexity property. Let C be the intersection of all sub-f-rings of B which contain $\phi(A)$ and which satisfy the nth-convexity property. By 2.6 C satisfies the nth-convexity property, and C is the convex sub-l-ring of C generated by $\phi(A)$. There is no harm in assuming that $C=B$. Let

$\{J_j: j \in \Sigma\}$ denote the collection of all proper prime l-ideals in B. There is a natural embedding $e': B \to \Pi Q(B/J_j)$ given by $[e'(b)]_j = J_j(b)$. Define a mapping $\psi: \Pi Q(A/I_i) \to \Pi Q(B/J_j)$ by the following. For each $j \in \Sigma$ there exists $k \in \Gamma$ with $J_j \cap \phi(A) = \phi(I_k)$ since B is the convex sub-l-ring generated by $\phi(A)$. Then ϕ induces mappings $\phi_j: Q(A/I_k) \to Q(B/J_j)$, and the ϕ_j induce a mapping $\psi: \Pi Q(A/I_i) \to \Pi Q(B/J_j)$ such that the following diagram commutes.



We now have embeddings defined so that the following diagram commutes.



But $e'(B) \supseteq e' \circ \phi(A) = \psi \circ e(A)$ and satisfies the nth convexity property. Therefore $\psi(K_n A) \subseteq e'(B)$. Thus there is an embedding of $K_n A$ into B. So $K_n A$ is a minimal n-convexity cover of A.

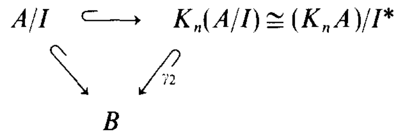
Next, we show that this minimal cover is unique up to isomorphism. Suppose that C also is a minimal n-convexity cover of A and let $e_1: A \to C$ be an embedding. Then there is an embedding $\gamma: C \to K_n A$ such that $\gamma \circ e_1(A) = e(A)$. But $K_n A$ is a sub-f-ring of $\Pi Q(A/I_i)$, so we may consider γ to map C into $\Pi Q(A/I_i)$. Now $\gamma(C)$ satisfies the nth-convexity property and contains $e(A) = \gamma \circ e_1(A)$. So $\gamma(C) \subseteq K_n A \subseteq \gamma(C)$.

Next we show that $K_n A$ satisfies the universal mapping property. Suppose $\phi: A \to B$ is an l-homomorphism into a semiprime f-ring satisfying the nth-convexity property. Let $I = \ker \phi$ and $I^* = \sqrt{\langle e(I) \rangle}$ be the smallest semiprime l-ideal of $K_n A$ containing $e(I)$. By Lemma 2.6, $I^* \cap e(A) = e(I)$.

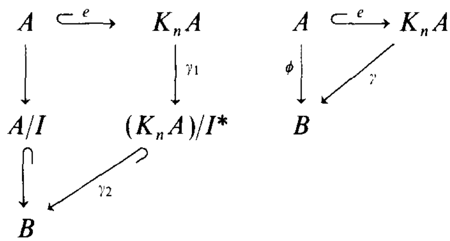
Note that A/I is a semiprime commutative f-ring and so we may consider $K_n(A/I)$. We will show that $K_n(A/I) \cong (K_n A)/I^*$. Since $I^* \cap e(A) = e(I)$, A/I is embedded in $(K_n A)/I^*$. So there are embeddings such that $A/I \to K_n(A/I) \to (K_n A)/I^*$. Thus there is a sub-$f$-ring $C \supseteq I^*$ of $K_n A$ such that $C/I^* \cong K_n(A/I)$ and $e(A) \subseteq C$. By Lemma 2.7, C satisfies the

nth-convexity property. We have $e(A) \subseteq C \subseteq K_n A$ and C satisfies the nth-convexity property. By our choice of $K_n A$, $C = K_n A$, and $K_n(A/I) \cong C/I^* \cong (K_n A)/I^*$.

Since $K_n(A/I)$ is a minimal n-convexity cover of A/I, there is an embedding $\gamma_2 \colon K_n(A/I) \to B$ such that the diagram commutes.



Let $\gamma_1 \colon K_n A \to (K_n A)/I^*$ be the natural l-homomorphism and $\gamma = \gamma_2 \circ \gamma_1$. Then $\gamma \colon K_n A \to B$ and the following diagrams commute.



The proofs of the remaining assertions of the theorem are routine and omitted. ▮

Remarks. (1) In different terms, Theorem 2.4 states that K_n is a functor which preserves monics in the category of commutative semiprime f-rings and $e \colon I \to K_n$ is a monic natural transformation.

(2) Theorem 2.4 is easily generalized to hold under the hypothesis that A is a semiprime f-ring for which every prime l-homomorphic image is a left Ore domain.

This argument may not be used to obtain a minimal 1-convexity cover for arbitrary commutative semiprime f-rings since in that case, we may not apply Lemma 2.5. That is, we do not have a result stating that in a semiprime f-ring with the 1st-convexity property, the intersection of all sub-f-rings satisfying the 1st-convexity property also satisfies the 1st-convexity property. The reason we do not have such a result is that (1.2) does not hold for the 1st-convexity property. When $0 \leqslant u \leqslant v$ in an f-ring satisfying the 1st-convexity property, there is not necessarily a unique element w such that $u = wv$ or even a unique element w such that $0 \leqslant w \leqslant 1$ and $u = wv$ when an identity element is present.

The following theorem gives a condition under which a minimal 1-convexity cover exists and under which the minimal 1-convexity cover is the same as the minimal n-convexity cover for $n \geqslant 2$. Recall that given a commutative f-ring A and a subset S without zero divisors, there is a f-ring A_s, called the localization of A at S, and an embedding $\lambda: A \to A_s$ such that (i) for every $s \in S$, $\lambda(s)$ is invertible in A_s, and (ii) for any l-homomorphism $\phi: A \to B$ mapping A into an f-ring B such that every $\phi(s)$ is invertible, there exists an l-homomorphism $\bar{\phi}: A_s \to B$ such that $\bar{\phi} \circ \lambda = \phi$.

THEOREM 2.8. *Let A be a commutative f-ring with identity element in which every finitely generated ideal of A is principal. If $S = \{s \in A: s \geqslant 1\}$, then A_s, the localization of A at S, satisfies the* 1st-*convexity property. Thus, for $n \geqslant 1$, A_s is a minimal n-convexity cover of A which satisfies the universal mapping property.*

Proof. Suppose A_s is the localization of A at S, and $\lambda: A \to A_s$ is the embedding. Suppose $0 \leqslant u \leqslant v$ in A_s. There is an $x \in \lambda(S)$ such that xu, $xv \in \lambda(A)$. By hypothesis, $(xu, xv)_{\lambda(A)} = (d)_{\lambda(A)}$ for some $d \in \lambda(A)$. We may assume $xv \neq 0$, $d \neq 0$. So there are $p, q, r, s \in \lambda(A)$ such that $xu = pd$, $xv = qd$, and $rxu + sxv = d$. Let $I = \{a \in \lambda(A): ad = 0\}$. Then I is a semiprime l-ideal of $\lambda(A)$. Now $(rp + sq) - 1 \in I$, so $I(rp + sq) = I(1)$. Also, $|p| - |p| \wedge |q| \in I$, so $I(|p|) \leqslant I(|q|)$. Thus $I(1) = I(rp + sq) \leqslant I((|r| + |s|)|q|)$. This implies there is an $i \in I$ such that $1 \leqslant (|r| + |s|)|q| + i$. Hence $((|r| + |s|)|q| + i)^{-1} \in A_s$. So in A_s, $xu = |p|\,|d| = |p|((|r| + |s|)|q| + i)^{-1}((|r| + |s|)|q| + i)|d| = |p|((|r| + |s|)|q| + i)^{-1}(|r| + |s|)|qd| = |p|((|r| + |s|)|q| + i)^{-i}(|r| + |s|)\,xv$. So there is an element $w \in A_s$ such that $xu = wxv$. But $x^{-1} \in A_s$, so $u = wv$. Therefore A_s satisfies the 1st-convexity property.

If $n \geqslant 1$ and $\phi: A \to B$ is a homomorphism into an f-ring B satisfying the nth-convexity property, then for every $s \in S$, $\phi(s)$ is invertible in B. Hence there exists a homomorphism $\bar{\phi}: A_s \to B$ such that $\bar{\phi} \circ \lambda = \phi$. ∎

III

In this section, we give two results whose proofs use Lemma 2.6 but whose statements do not involve any of the convexity properties. This application will show how the nth-convexity property can be used in problems that do not originally mention it. In this section, we assume that A has a identity element (in addition to the assumption that A is commutative and semiprime).

An ideal I in a ring A is *primary* if $ab \in I$, and $a \notin I$ implies $b^n \in I$ for some positive integer n. Primary ideals in $C(X)$ have been studied by L. Gillman and C. Kohls in [GK] and by C. Kohls in [K]. The problem of identifying l-ideals which are intersections of primary ideals has been studied by

R. D. Williams in [W]. There he investigates necessary and sufficient conditions for an l-ideal of $C(X)$ to be an intersection of primary ideals. Recall that if I, J are ideals of a ring then $I: J = \{a \in A: aJ \subseteq I\}$. We will generalize some of his results to show that in a commutative semiprime f-ring with identity element, if an l-ideal I satisfies $I = \langle I \sqrt{I} \rangle$ or $I = I: \sqrt{I}$, then I is an intersection of primary l-ideals. As a corollary, we show that if I is a pseudoprime l-ideal satisfying $I = \langle I \sqrt{I} \rangle$ or $I = I: \sqrt{I}$, then I is primary.

First, we need some facts concerning primary l-ideals in semiprime commutative f-rings satisfying a convexity property. An f-ring A with identity is said to satisfy the *bounded inversion property* if $a \geqslant 1$ in A implies $a^{-1} \in A$. By (1.4), an f-ring with identity element satisfying the nth-convexity property also satisfies the bounded inversion property. For $C(X)$, the result of the next lemma appears in [GK, 4.6]. The result holds in the more general context given next, and we omit the proof.

LEMMA 3.1. *Let A be a commutative f-ring with identity element which satisfies the bounded inversion property. Let P be a prime l-ideal of A. If a is a positive nonunit of A/P, then*

$$mP|^{a} = \{b \in A/P: |b|^{m} < a^{m-1} \text{ for all } m \in \mathbf{N}\}$$

and

$$mP|_{a} = \{b \in A/P: |b|^{m} \leqslant a^{m+1} \text{ for some } m \in \mathbf{N}\}$$

are primary l-ideals of A/P, and $a \in mP|^{a}$, $a \notin mP|_{a}$.

Suppose A is an f-ring satisfying the hypotheses of Lemma 3.1 and P is a prime l-ideal of A. For each primary l-ideal $mP|^{a}$ (respectively $mP|_{a}$) of A/P, we may associate a primary l-ideal of A, namely $\{b \in A: P(b) \in mP|^{a}\}$ (respectively $\{b \in A: P(b) \in mP|_{a}\}$). We will denote these by $P|^{f}$ and $P|_{f}$, respectively, where $f \in A$ is an element such that $P(f) = a$.

Recall that a *pseudoprime* ideal I is an ideal with the property that $xy = 0$ implies $x \in I$ or $y \in I$. Part (1) of the next lemma has been shown by H. Subramanian in [Su]. The result of Part (2) is shown to hold in a $C(X)$ by L. Gillman and C. Kohls in [GK]. However, their proof is valid for any semiprime f-ring with identity element.

LEMMA 3.2. *Let A be a commutative semiprime f-ring with identity element.*

(1) *An l-ideal I is pseudoprime if and only if it contains a prime l-ideal.*

(2) *An l-ideal I is an intersection of pseudoprime l-ideals.*

We are now ready to give two results concerning $I\sqrt{I}$ and $I:\sqrt{I}$ in a commutative semiprime f-ring A with identity element which satisfies the 2nd-convexity property. R. D. Williams has shown that $I\sqrt{I}$ is an intersection of primary l-ideals in [W, 2.8], and our first proof will mimic his.

THEOREM 3.3. *Suppose A is a semiprime commutative f-ring with identity element satisfying the* 2nd-*convexity property and I is an l-ideal of A. Then if $I = I\sqrt{I}$, it is an intersection of primary l-ideals.*

Proof. Let $f \in A \backslash I\sqrt{I}$. We will show there is a primary l-ideal that contains $I\sqrt{I}$ but not f. By 3.2, there is a pseudoprime l-ideal Q containing $I = I\sqrt{I}$ but not f. Now let P be a prime l-ideal contained in Q (by 3.2), and let M be the maximal l-ideal in which P is contained. If $f \notin M$, then M is a prime l-ideal containing Q, and hence $I\sqrt{I}$, but not f. Suppose now that $f \in M$. Then $P(|f|)$ is a nonunit of A/P. Now the l-ideals containing P form a chain, and $f \in P|^{|f|}$ while $f \notin Q$. So $Q \subseteq P|^{|f|}$. Thus $I \subseteq P|^{|f|}$. We now show that $I\sqrt{I} \subseteq P|_{|f|}$. Suppose that $g \in I$, $h \in \sqrt{I}$. Then there is some $k \in \mathbf{N}$ such that $h^k \in I$. Also, since $I \subseteq P|^{|f|}$, $P(|g|^m) < P(|f|^{m-1})$ and $P(|h|^{km}) < P(|f|^{m-1})$ for all $m \in \mathbf{N}$. Thus $P(|gh|^{k(k+2)}) = P(|g|^{k(k+2)})$ $P(|h|^{k(k+2)}) \leqslant P(|f|^{k(k+1)})P(|f|^{k+1}) = P(|f|^{k(k+2)+1})$. So $gh \in P|_{|f|}$. ∎

THEOREM 3.4. *Let $n \geqslant 1$. Suppose A is a semiprime commutative f-ring with identity element satisfying the n*th-*convexity property, and I is an l-ideal of A. Then for any $x \in A \backslash (I:\sqrt{I}):\sqrt{I}$ there is a primary l-ideal which contains $I:\sqrt{I}$ but not x.*

Proof. Since $x \notin (I:\sqrt{I}):\sqrt{I}$, there is a $g \geqslant 0$ in $\sqrt{I}$ such that $xg \notin I:\sqrt{I}$. This implies that there is an $h \geqslant 0$ in $\sqrt{I}$ such that $xgh \notin I$. Let $f = g \vee h$. Then $f \in \sqrt{I}$ and $xf^2 \notin I$. By 3.2, there is a pseudoprime l-ideal Q containing $I:\sqrt{I}$ but not xf. Now let P be a prime l-ideal contained in Q and let M be the maximal ideal containing P. If $x \notin M$, then M is a prime l-ideal containing Q, and therefore containing $I:\sqrt{I}$, but not containing x. Suppose now that $x \in M$. The l-ideals containing P form a chain, and $xf \in P|^{|x|f}$ while $xf \notin Q$. So $Q \subseteq P|^{|x|f}$. Thus $I:\sqrt{I} \subseteq P|^{|x|f}$.

Let k be the smallest integer such that $f^k \in I$. Since $x \notin Q$, $x \notin P + I$. Since A/P is totally ordered, $P(|x|) > P(f^k)$. So $P(|x|)^{k+1} > P(|x|f)^k$ and therefore, $x \notin P|^{|x|f}$. ∎

An l-ideal I of an f-ring A is *square dominated* if $I = \{a \in A : |a| \leqslant x^2$ for some $x \in A$ such that $x^2 \in I\}$. A slight modification of this proof shows that if A is a semiprime commutative f-ring satisfying the nth-convexity property with identity element, and $\sqrt{I}$ is a square dominated l-ideal of A, then $I:\sqrt{I}$ is an intersection of primary l-ideals.

We are now ready to prove our main result of this section.

THEOREM 3.5. *Let A be a commutative semiprime f-ring with identity element and suppose I is an l-ideal of A. Then if $I = \langle I\sqrt{I}\rangle$ or if $I = I : \sqrt{I}$, I is an intersection of primary l-ideals.*

Proof. Let B be a commutative semiprime minimal 2-convexity cover of A with the identity element 1, and with the embedding $e: A \to B$. By 2.6, B is the convex sub-l-ring of B generated by $e(A)$. In B, let J be the l-ideal generated by $e(I)$. By 2.6(1), $J \cap e(A) = e(I)$.

Suppose first that $I = \langle I\sqrt{I}\rangle$. Then $J = J\sqrt{J}$. By Theorem 3.3, J is an intersection of primary l-ideals Q_i in B. Now $Q_i \cap e(A)$ are primary l-ideals of $e(A)$ and so $e(I) = e(\langle I\sqrt{I}\rangle) = J \cap e(A) = (\bigcap Q_i) \cap e(A) = (Q_i \cap e(A))$. Therefore I is an intersection of primary l-ideals.

Next, suppose that $I = I : \sqrt{I}$. Let $e(a) \in (J : \sqrt{J}) : \sqrt{J} \cap e(A)$. Then for any $b, c \in \sqrt{I}$, $e(a)e(b)e(c) \in J \cap e(A) = e(I)$. Thus, $e(a) \in e((I : \sqrt{I}) : \sqrt{I}) = e(I : \sqrt{I}) = e(I)$. We now have $(J : \sqrt{J}) : \sqrt{J} \cap e(A) \subseteq e(I)$. Clearly, the reverse inclusion also holds, and $(J : \sqrt{J}) : \sqrt{J} \cap e(A) = e(I)$.

For any $a \in B \backslash (J : \sqrt{J}) : \sqrt{J}$, there is a primary l-ideal Q_i of B which contains $J : \sqrt{J}$ but not a by Theorem 3.4. Now $Q_i \cap e(A)$ are primary l-ideals of $e(A)$. So $e(I) = e(I : \sqrt{I}) = J \cap e(A) \subseteq J : \sqrt{J} \cap e(A) \subseteq (\bigcap Q_i) \cap e(A) = \bigcap (Q_i \cap e(A)) \subseteq (J : \sqrt{J}) : \sqrt{J} \cap e(A) = e(I)$. Thus $e(I) = \bigcap (Q_i \cap e(A))$ and I is an intersection of primary l-ideals. ∎

COROLLARY 3.6. *Let A be a commutative semiprime f-ring with identity element. If I is a pseudoprime l-ideal that is an intersection of primary l-ideals, then I is itself primary. Thus, if I is a pseudoprime l-ideal satisfying $I = \langle I\sqrt{I}\rangle$ or $I = I : \sqrt{I}$, I is a primary l-ideal.*

Proof. Suppose I is a pseudoprime l-ideal that is an intersection of the primary l-ideals Q_i. Then I contains a prime l-ideal and hence the set of all l-ideals containing I form a chain. Now if $Q_i \supseteq \sqrt{I}$ for all i, then $\sqrt{I} \subseteq \bigcap Q_i = I$. Hence I is semiprime and pseudoprime and therefore prime. We now may assume there is some α such that $Q_\alpha \subset \sqrt{I}$. Suppose that $ab \in I$ and $a \notin I$. There is some β such that $a \notin Q_\beta \subseteq Q_\alpha \subseteq \sqrt{I}$. Since $ab \in Q_\beta$ and $a \notin Q_\beta$, $b \in \sqrt{Q_\beta} \subseteq \sqrt{I}$. Thus I is primary. ∎

Finally, we give an example showing that an l-ideal I with the property $I = \langle I\sqrt{I}\rangle$ or $I = I : \sqrt{I}$ is not the only type of ideal that is an intersection of primary l-ideals in a commutative semiprime f-ring with identity element. Another such example (which is not as simple) is given in [W, 2.11]. The f-ring described in this example was first given in [HP, 4.16].

EXAMPLE 3.7. In $C([0, 1])$, denote by i the function $i(x) = x$, by e the function $e(x) = 1$, and let $w = \sqrt{i}$. Let $\langle i\rangle$ denote the l-ideal of $C([0, 1])$

generated by i, and let $A=\{f\in C([0,1])\colon f=ae+bw+g;\ g\in\langle i\rangle,\ a,b\in\mathbf{R}\}$. Give A the inherited (componentwise) addition, multiplication, and ordering. Then it can be shown that A is an f-ring. Also, A is commutative, semiprime, and possesses an identity element.

Let $I=\{ae+bw+g\in A\colon a=b=0\}$. Then I is an l-ideal of A. Simple calculations show that I is primary. Note that $\sqrt{I}=\{ae+bw+g\in A\colon a=0\}$. Then $\langle I\sqrt{I}\rangle\subseteq\{ae+bw+g\in A\colon a=b=0,\ g\leqslant ni^{3/2}\}\subset I$. Also, $1w\in A$ and $(1w)\sqrt{I}\subseteq I$. This implies $1w\in I\colon\sqrt{I}$ and yet $1w\notin I$. So $I\subset I\colon\sqrt{I}$. Thus $\langle I\sqrt{I}\rangle\subset I\subset I\colon\sqrt{I}$. Note also that I is pseudoprime and so the converse to the corollary is also false.

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