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STOCHASTIC MAXIMUM PRINCIPLE FOR PARTIAL INFORMATION OPTIMAL INVESTMENT AND DIVIDEND PROBLEM OF AN INSURER

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ABSTRACT. We study an optimal investment and dividend problem of an insurer, where the aggregate insurance claims process is modeled by a pure jump Lévy process. We allow the management of the dividend payment policy and the investment of surplus in a continuous-time financial market, which is composed of a risk free asset and a risky asset. The information available to the insurer is partial information. We generalize this problem as a partial information regular-singular stochastic control problem, where the control variable consists of regular control and singular control. Then maximum principles are established to give sufficient and necessary optimality conditions for the solutions of the regular-singular control problem. Finally we apply the maximum principles to solve the investment and dividend problem of an insurer.

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Key words and phrases. Optimal investment, dividend, insurance claim process, maximum principle, regular-singular stochastic control, partial information.

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1. Introduction. Investing the surpluses into financial markets is an effective tool for insurance companies to manage their exposure to risk. The investment problem of an insurer is much more complex than the classic optimal investment problem in financial economics (see, e.g., [20, 21]), for the insurance company is exposed to both financial risk and insurance risk, instead of only financial risk. Financial risk is present because of the fluctuations of financial markets and insurance risk is caused by the liabilities related to insurance claims. Therefore, the problem of optimal investment for an insurer has been one of the widely studied topics in actuarial science, see for example [6, 9, 10, 16, 17] and the references therein.

The problem of optimizing the dividend policy, which is initiated by the work of De Finetti [8], is also a classical problem in actuarial mathematics. With the development of diffusion model, it can be formulated as a singular stochastic control problem. There are extensive literatures on this topic under more general and more realistic model assumptions, such as [2, 5, 7, 18, 22].

Recently there has been an upsurge of interest for optimal investment and dividend problem of an insurer. For example, Azcue and Muler [3] developed a continuous-time model for the optimal investment and dividend problem of an insurance company, where the uncontrolled surplus process evolves as a classical Cramér-Lundberg process. The problem studied in [14] is analogous to the one in [3], where the surplus process is modeled by a regime-switching jump diffusion process. Højgaard and Taksar [12] considered the dividend, reinsurance and portfolio optimization problem in Itô diffusion setting. Jin et al. [13] studied optimal investment, dividend payment and capital policies problem, which can be regarded as a regular-singular-impulse stochastic control problem. In these papers, the associated control problems are mainly solved by integro-differential quasi-variational inequalities (IDQVI), which is possible by assuming that the surplus process is Markovian.

In this paper, we consider the optimal investment and dividend problem of an insurer with partial information, where the aggregate insurance claims is modeled by a pure jump Lévy process. We allow the management of the dividend payment policy and the investment of surplus in a continuous-time financial market consisting of a risk free asset and a risky asset. We consider a general and realistic situation where the information available to the controller is partial information. That is, the insurer decides the investment strategy and dividend payment policy based on partial information, which is less than the full information generated by the market events (see, e.g. [4, 19, 23]). From the view of control theory, such a problem can be generalized as a novel regular-singular stochastic control problem with partial information. Because of the non-Markovian nature of the partial information, this control problem cannot be solved by the well-established IDQVI technique, which motivates us to derive the corresponding maximum principle to handle the partial information case.

There are some results in the maximum principles for regular-singular stochastic control problem. For example, Zhang [24] applied the relaxed control approach to establish a maximum principle for regular-singular control problem, where the control system evolves by forward-backward stochastic differential equation (SDE) driven by Brownian motion. In the partial information case, a similar problem was considered in [11], where the system is governed by mean-field controlled SDE driven by Teugels martingales associated with some Lévy processes and an independent Brownian motion. However, in our situation, since an additional controllable jump

diffusion process is added into the system to model the insurance risk, the existing maximum principles are no longer valid.

We aim to establish sufficient and necessary maximum principles for the regular-singular stochastic control problem, which arises from the optimal investment and dividend problem of an insurer. The maximum principles enable us to give the sufficient and necessary optimality conditions for its solutions. The approaches of the derivations of these maximum principles are similar to the one adopted by Bagheri and Øksendal [4], who derived a maximum principle for regular stochastic control problem under partial information. Since the control variable consists of two components: the regular control and the singular control in our control problem, our results can be regarded as the generalization of [4] to regular-singular stochastic control problem.

The rest of the paper is structured as follows: in the next section we formulate the partial information optimal investment and dividend problem of an insurer. In Section 3 we generalize the optimal investment and dividend problem of an insurer as a regular-singular stochastic control problem with partial information. Then sufficient and necessary maximum principles are established to give the optimality conditions for this general control problem. In Section 4 we apply the maximum principles obtained in Section 3 to solve the optimal investment and dividend problem of an insurer. Finally we conclude the whole paper in Section 5.

2. The partial information optimal investment and dividend problem of an insurer. As always, we start with a complete probability space (Ω, \mathcal{F}, P) to introduce the optimal investment and dividend problem of an insurer. Suppose there is a continuous-time financial market with two investment possibilities:

- A risk free asset (e.g. a bond), with unit price $S_0(t)$ at time t given by

$$dS_0(t) = \rho(t)S_0(t)dt, \quad S_0(0) = 1, \quad \text{for all } t \in [0, T], \quad T \in (0, \infty).$$

- A risky asset (e.g. a stock), with unit price $S_1(t)$ at time t given by

$$dS_1(t) = S_1(t) [\zeta(t)dt + \pi(t)dB(t)], \quad S_1(0) > 0, \quad \text{for all } t \in [0, T],$$

where $\rho(t)$ is the interest rate of the risk free asset at time t , $\zeta(t)$ and $\pi(t)$ are the appreciation rate and the volatility of the risky asset at time t , and $\{B(t) | t \in [0, T]\}$ is a standard Brownian motion on (Ω, \mathcal{F}, P) with respect to its right-continuous P -completed filtration $\{\mathcal{F}_t^B | t \in [0, T]\}$. We assume that $\rho(t)$, $\zeta(t)$ and $\pi(t)$ are \mathcal{F}_t^B -predictable processes such that

$$\int_0^T \{|\rho(t)| + |\zeta(t)| + \pi^2(t)\} dt < \infty, \quad \text{a.s..}$$

Moreover, we impose the following assumptions on the financial market:

- (1) The risk free asset and the risky asset can be traded continuously over time on $[0, T]$.
- (2) The market is frictionless, that is, there are no transaction costs and taxes involved in trading.
- (3) The assets are divisible so that any fractional units of the assets can be traded.

We define the aggregate insurance claims process $\eta(t)$ by a pure jump Lévy process on (Ω, \mathcal{F}, P) as follows:

$$\eta(t) = \int_0^t \int_{\mathbb{R}_0} z N(ds, dz), \quad t \in [0, T],$$

where $N(\cdot, \cdot)$ is Poisson random measure of $\eta(\cdot)$ and $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. Assume that $\eta(t)$ is bounded on $[0, T]$. Then $\eta(t)$ can be written as

$$\eta(t) = \int_0^t \int_{\mathbb{R}_0} z \tilde{N}(ds, dz) + \int_0^t \int_{\mathbb{R}_0} z \nu(dz) ds, \quad t \in [0, T],$$

where

$$\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$$

is the compensation of Poisson random measure $N(\cdot, \cdot)$ and ν is Lévy measure of $\eta(\cdot)$. We refer to [1] for background on Lévy process and Poisson random measures.

For $t \in [0, T]$, let $\kappa(t)$ be the premium rate and let $R(t)$ denote the surplus process of the insurer in absence of investment and dividend. Then we have

$$\begin{aligned} R(t) &= R_0 + \int_0^t \kappa(s) ds - \eta(t) + a\tilde{B}(t) \\ &= R_0 + \int_0^t \left(\kappa(s) - \int_{\mathbb{R}_0} z \nu(dz) \right) ds - \int_0^t \int_{\mathbb{R}_0} z \tilde{N}(ds, dz) + a\tilde{B}(t), \end{aligned}$$

where $R_0 \in \mathbb{R}$ is the initial surplus and $a \in \mathbb{R}$ is the diffusion coefficient. Here $\{\tilde{B}(t) | t \in [0, T]\}$ is another Brownian motion defined on (Ω, \mathcal{F}, P) with respect to its right-continuous P -completed filtration $\{\mathcal{F}_t^{\tilde{B}} | t \in [0, T]\}$. It describes an additional source of the insurance uncertainty (see, e.g., [15]). We assume that $B(t)$, $\tilde{B}(t)$ and $\tilde{N}(dt, dz)$ are mutually independent under P , for $t \in [0, T]$.

In order to transfer the risk, the insurer invests its surplus in the financial market. Let $u(t) := u(t, \omega)$ denote the amount invested in the risky asset which we call *portfolio strategy*. We denote by $X(t)$ the corresponding surplus process with investment. Then, for $t \in [0, T]$, the dynamics of $X(t)$ is given by

$$\begin{cases} dX(t) = \left\{ \kappa(t) + \rho(t)X(t) + u(t) [\zeta(t) - \rho(t)] - \int_{\mathbb{R}_0} z \nu(dz) \right\} dt \\ \quad + \pi(t)u(t)dB(t) + a d\tilde{B}(t) - \int_{\mathbb{R}_0} z \tilde{N}(dt, dz), \\ X(0) = x_0, \end{cases}$$

where $x_0 \in \mathbb{R}$ is the initial surplus.

In addition to investment, the insurer pays dividends to its shareholders by a dividend strategy. Let $\xi(t)$ represent the cumulative amount of dividends paid up to time t . Then the surplus process $X(t)$ in presence of investment and dividend is given by

$$\begin{cases} dX(t) = \left\{ \kappa(t) + \rho(t)X(t) + u(t) [\zeta(t) - \rho(t)] - \int_{\mathbb{R}_0} z \nu(dz) \right\} dt \\ \quad + \pi(t)u(t)dB(t) + a d\tilde{B}(t) - \int_{\mathbb{R}_0} z \tilde{N}(dt, dz) - d\xi(t), \\ X(0) = x_0, \end{cases} \quad (1)$$

where $\xi(t) = \xi(t, \omega)$ is a càdlàg non-decreasing process satisfying $\xi(0) = 0$. Since $d\xi(t)$ may be singular with respect to Lebesgue measure dt , the process $\{\xi(t) | t \geq 0\}$ is called a singular control. In this model, the portfolio strategy $u(t)$ and the dividend policy $\xi(t)$ are controlled by the insurer. Then we define $\{(u(t), \xi(t)) | t \in [0, T]\}$ as a *regular-singular control*.

Now we specify the information structure of the model. As we have defined above, the filtrations $\{\mathcal{F}_t^B | t \in [0, T]\}$ and $\{\mathcal{F}_t^{\tilde{B}} | t \in [0, T]\}$ are the right-continuous, P -completed, natural filtrations generated by $\{B(t) | t \in [0, T]\}$ and $\{\tilde{B}(t) | t \in [0, T]\}$,

respectively. Let $\{\mathcal{F}_t^\eta | t \in [0, T]\}$ denote the P -augmentation of the σ -field generated by the insurance claims process $\{\eta(t) | t \in [0, T]\}$. For each $t \in [0, T]$, we define the enlarged σ -algebra

$$\mathcal{F}_t := \mathcal{F}_t^\eta \vee \mathcal{F}_t^B \vee \mathcal{F}_t^{\tilde{B}},$$

which is the minimal σ -field generated by \mathcal{F}_t^η , \mathcal{F}_t^B and $\mathcal{F}_t^{\tilde{B}}$. Then \mathcal{F}_t is the information generated by the surplus process of the insurer and the price process of the risky asset up to and including time t . So $\{\mathcal{F}_t | t \in [0, T]\}$ represents the full information involved in the model.

In real world, however, the insurer can only get partial information instead of full information. That is, we have a subfiltration

$$\mathcal{G}_t \subseteq \mathcal{F}_t \quad \text{for all } t \in [0, T]$$

such that the control $(u(t), \xi(t))$ is required to be \mathcal{G}_t -adapted. For example, the insurer could have a delayed information compared to \mathcal{F}_t :

$$\mathcal{G}_t := \mathcal{F}_{(t-\delta)+}, \quad \text{for } t \in [0, T], \quad \delta > 0 \text{ is a given constant.}$$

In this case, the insurer decides the portfolio strategy $u(t)$ and the dividend policy $\xi(t)$ at time t based on the information $\mathcal{F}_{(t-\delta)+}$, namely, there is a delay $\delta > 0$.

The *utility function* of the insurer is defined as follows:

$$\mathcal{J}(u, \xi) = E \left[- \int_0^T Qu^2(t)dt - \frac{1}{2} (X(T) - D)^2 + \int_0^T e^{-\int_0^t \rho(s)ds} d\xi(t) \right],$$

where E denotes the expectation with respect to P , $Q \geq 0$ and D are given constants and $\int_0^T Qu^2(t)dt$ is the accumulated cost on $[0, T]$.

Let $\mathcal{A}_\mathcal{G}$ be a family of admissible control (u, ξ) , contained in the set of \mathcal{G}_t -adapted (u, ξ) such that (1) has a unique strong solution and

$$E \left[\int_0^T Qu^2(t)dt + \frac{1}{2} (X(T) - D)^2 + \int_0^T e^{-\int_0^t \rho(s)ds} d\xi(t) \right] < \infty.$$

The objective of the insurer is to find the value function $\Phi_\mathcal{G} \in \mathbb{R}$ and an optimal admissible control $(\hat{u}(\cdot), \hat{\xi}(\cdot)) \in \mathcal{A}_\mathcal{G}$ such that

$$\Phi_\mathcal{G} = \sup_{(u, \xi) \in \mathcal{A}_\mathcal{G}} \mathcal{J}(u, \xi) = \mathcal{J}(\hat{u}, \hat{\xi}). \quad (2)$$

3. Maximum principles for regular-singular stochastic control problem with partial information. In this section, we generalize the investment and dividend problem of an insurer (2) to a regular-singular stochastic control problem with partial information. Then maximum principles are established to give sufficient and necessary optimality conditions for its solutions. The idea of the derivations of these maximum principles is similar to the one presented in [4], where there is only one regular control variable in the control problem. Since the control variable consists of regular control and singular control in our control problem, our results can be regarded as the generalization of [4] to regular-singular stochastic control problem.

3.1. A general formulation of regular-singular stochastic control problem with partial information. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete filtered probability space. Suppose that the state process $\tilde{X}(t) = X(t, \omega)$; $t \in [0, T]$, $\omega \in \Omega$, is described by the following controlled singular jump diffusion:

$$\begin{cases} dX(t) = b(t, X(t), u(t), \omega)dt + \sigma(t, X(t), u(t), \omega)dB(t) + \tilde{\sigma}(t, X(t), u(t), \omega)d\tilde{B}(t) \\ \quad + \int_{\mathbb{R}_0} \gamma(t, X(t), u(t), z, \omega)\tilde{N}(dt, dz) + \lambda(t, X(t), \omega)d\xi(t) \\ X(0) = x \in \mathbb{R}, \end{cases} \quad (3)$$

where the coefficients

$$\begin{aligned} b(t, x, u, \omega) &: [0, T] \times \mathbb{R} \times U \times \Omega \rightarrow \mathbb{R}, \\ \sigma(t, x, u, \omega) &: [0, T] \times \mathbb{R} \times U \times \Omega \rightarrow \mathbb{R}, \\ \tilde{\sigma}(t, x, u, \omega) &: [0, T] \times \mathbb{R} \times U \times \Omega \rightarrow \mathbb{R}, \\ \gamma(t, x, u, z, \omega) &: [0, T] \times \mathbb{R} \times U \times \mathbb{R}_0 \times \Omega \rightarrow \mathbb{R}, \\ \lambda(t, x, \omega) &: [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R} \end{aligned}$$

are given \mathcal{F}_t -predictable processes, and U is a given nonempty open convex subset of \mathbb{R} . We assume that $b, \sigma, \tilde{\sigma}, \gamma, \lambda$ are continuously differentiable with respect to x , there exists $\epsilon > 0$ such that

$$\frac{\partial \gamma}{\partial x}(t, x, z) \geq \epsilon - 1 \quad \text{a.s. for all } (t, x, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}_0,$$

and $B(t)$, $\tilde{B}(t)$ and $\tilde{N}(dt, dz)$ are mutually independent under P , $t \in [0, T]$. The process $u(t) = u(t, \omega) \in U$ is a regular stochastic control and $\xi(t) = \xi(t, \omega)$ is a singular control with $\xi(0) = 0$. Then we call $(u(t), \xi(t))$ a *regular-singular control*, $t \in [0, T]$.

Suppose that the information available to the controller is partial information. That is, let

$$\mathcal{G}_t \subseteq \mathcal{F}_t; \quad t \in [0, T]$$

be a subfiltration of \mathcal{F}_t . Then the regular-singular control $(u(t), \xi(t))$ are \mathcal{G}_t -adapted. Assume in addition that the process $t \rightarrow \lambda(t, x)$ is \mathcal{G}_t -adapted.

Let

$$\begin{aligned} f(t, x, u, \omega) &: [0, T] \times \mathbb{R} \times U \times \Omega \rightarrow \mathbb{R}, \\ h(t, x, \omega) &: [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R} \end{aligned}$$

be given \mathcal{F}_t -predictable processes and let $g(x, \omega)$ an \mathcal{F}_T -measurable random variable for each x . Assume that f, g and h are continuously differentiable with respect to x . Then we define the *performance functional* as follows:

$$\mathcal{J}(u, \xi) = E \left[\int_0^T f(t, X(t), u(t), \omega)dt + g(X(T), \omega) + \int_0^T h(t, X(t), \omega)d\xi(t) \right], \quad (4)$$

where E denotes expectation with respect to P .

Let $\mathcal{A}_{\mathcal{G}}$ denote a given family of controls (u, ξ) , contained in the set of \mathcal{G}_t -adapted (u, ξ) such that the system (3) has a unique strong solution and

$$E \left[\int_0^T |f(t, X(t), u(t), \omega)| dt + |g(X(T), \omega)| + \int_0^T |h(t, X(t), \omega)| d\xi(t) \right] < \infty.$$

Then $\mathcal{A}_{\mathcal{G}}$ is called the admissible control set.

The partial information regular-singular stochastic control problem is to find the value function $\Phi_{\mathcal{G}} \in \mathbb{R}$ and optimal regular-singular control $(\hat{u}, \hat{\xi}) \in \mathcal{A}_{\mathcal{G}}$ such that

$$\Phi_{\mathcal{G}} = \sup_{(u, \xi) \in \mathcal{A}_{\mathcal{G}}} \mathcal{J}(u, \xi) = \mathcal{J}(\hat{u}, \hat{\xi}). \quad (5)$$

3.2. Necessary maximum principle for regular-singular stochastic control problem with partial information. In this subsection, we establish maximum principle to give necessary optimality conditions for the regular-singular stochastic control problem (5). Then, we prove that these necessary optimality conditions are sufficient for a directional sub-stationary point of the performance functional.

To give a Hamiltonian-based maximum principle, we firstly define the Hamiltonian.

Definition 3.1. Let $\xi^C(t)$ be the continuous part of $\xi(t)$ and let $\Delta\xi(t) = \xi(t) - \xi(t-)$ be the purely discontinuous part of $\xi(\cdot)$ at time t . The Hamiltonian

$$H : [0, T] \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \rightarrow \mathcal{D}$$

is defined by

$$\begin{aligned} & H(t, x, u, p, q, \tilde{q}, r(\cdot))(dt, d\xi) \\ &= \left[f(t, x, u) + pb(t, x, u) + q\sigma(t, x, u) + \tilde{q}\tilde{\sigma}(t, x, u) + \int_{\mathbb{R}} r(t, z)\gamma(t, x, u, z)\nu(dz) \right] dt \\ &+ [p\lambda(t, x) + h(t, x)] d\xi^C(t) + \lambda(t, x) \int_{\mathbb{R}_0} r(\{t\}, z)N(\{t\}, dz)\Delta\xi(t). \end{aligned} \quad (6)$$

Here \mathcal{R} is the set of functions $r(\cdot, \cdot) : [0, T] \times \mathbb{R}_0 \rightarrow \mathbb{R}$ such that (6) is well defined and \mathcal{D} is the set of all sums of stochastic dt - and $d\xi$ -differentials. For a given process $F(t, z)$, we denote

$$\int_{\mathbb{R}_0} F(\{t\}, z)N(\{t\}, dz) := \begin{cases} F(t, z), & \text{if Lévy process } \eta \text{ has a jump of size } z \text{ at } t, \\ 0, & \text{else.} \end{cases}$$

The adjoint processes $(p(t), q(t), \tilde{q}(t), r(\cdot)(t, z))$ associated to (u, ξ) are given by the following backward SDE:

$$\begin{cases} dp(t) = -\frac{\partial H}{\partial x}(t, X(t), u(t), p(t), q(t), \tilde{q}(t), r(\cdot))(dt, d\xi) + q(t)dB(t) \\ \quad + \tilde{q}(t)d\tilde{B}(t) + \int_{\mathbb{R}_0} r(t, z)\tilde{N}(dt, dz), \\ p(T) = g'(X(T)). \end{cases} \quad (7)$$

Assumption 1. We make the following assumptions:

- (I) For all t, h satisfying $0 \leq t < t+h \leq T$ and all bounded \mathcal{G}_t -measurable random variables $\theta(\omega)$, the control $(\beta(s), 0)$ with

$$\beta(s) = \theta(\omega)\chi_{[t, t+h]}(s); \quad s \in [0, T]$$

belongs to $\mathcal{A}_{\mathcal{G}}$, where $\chi_{[t, t+h]}$ is the indicator function of $[t, t+h]$.

- (II) For all $(u, \xi) \in \mathcal{A}_{\mathcal{G}}$ and all bounded $(\beta, \varsigma) \in \mathcal{A}_{\mathcal{G}}$, there exists $\delta > 0$ such that

$$(u(t) + y\beta(t), \xi(t) + y\varsigma(t)) \in \mathcal{A}_{\mathcal{G}}, \text{ for all } y \in (-\delta, \delta), t \in [0, T].$$

For the sake of simplicity and clarification, we introduce the following short hand notations:

$$\begin{aligned}\frac{\partial b}{\partial x}(t) &= \frac{\partial b}{\partial x}(t, X(t), u(t), \omega), \quad \frac{\partial b}{\partial u}(t) = \frac{\partial b}{\partial u}(t, X(t), u(t), \omega), \\ \frac{\partial H}{\partial x}(dt, d\xi) &= \frac{\partial H}{\partial x}(t, X(t), u(t), p(t), q(t), \tilde{q}(t), r(t, \cdot))(dt, d\xi), \\ \frac{\partial H}{\partial u}(dt, d\xi) &= \frac{\partial H}{\partial u}(t, X(t), u(t), p(t), q(t), \tilde{q}(t), r(t, \cdot))(dt, d\xi)\end{aligned}$$

and similarly for other derivatives.

For a bounded (β, ς) , we define the derivative process $\alpha(t, \beta, \varsigma)$ by

$$\alpha(t, \beta, \varsigma) := \lim_{y \rightarrow 0^+} \frac{1}{y} [X^{u+y\beta, \xi+y\varsigma}(t) - X^{u, \xi}(t)].$$

Then, we obtain by (3) that

$$\begin{aligned}d\alpha(t, \beta, \varsigma) &= \alpha(t, \beta, \varsigma) \left[\frac{\partial b}{\partial x}(t)dt + \frac{\partial \sigma}{\partial x}(t)dB(t) + \frac{\partial \tilde{\sigma}}{\partial x}(t)d\tilde{B}(t) + \frac{\partial \lambda}{\partial x}(t)d\xi(t) \right. \\ &\quad \left. + \int_{\mathbb{R}_0} \frac{\partial \gamma}{\partial x}(t, z)\tilde{N}(dt, dz) \right] + \beta(t) \left[\frac{\partial b}{\partial u}(t)dt + \frac{\partial \sigma}{\partial u}(t)dB(t) \right. \\ &\quad \left. + \frac{\partial \tilde{\sigma}}{\partial u}(t)d\tilde{B}(t) + \int_{\mathbb{R}_0} \frac{\partial \gamma}{\partial u}(t, z)\tilde{N}(dt, dz) \right] + \lambda(t, x)d\varsigma(t)\end{aligned}\quad (8)$$

with

$$\alpha(0, \beta, \varsigma) = 0.$$

We are now ready to state and prove maximum principle to give necessary optimality conditions for the solutions of the control problem (5).

Theorem 3.2 (Necessary maximum principle). *Suppose $(\hat{u}, \hat{\xi}) \in \mathcal{A}_G$ is the solution of the control problem (5). Let $\hat{X}(t)$, $\hat{p}(t)$, $\hat{q}(t)$, $\hat{\tilde{q}}(t)$, $\hat{r}(\cdot)(t, z)$, $\hat{\alpha}(t, \beta, \varsigma)$ be the solutions of the equations (3), (7) and (8) corresponding to $(\hat{u}, \hat{\xi})$. Set*

$$\hat{U}(t) := \hat{p}(t)\lambda(t, \hat{X}(t)) + h(t, \hat{X}(t)) \quad (9)$$

and

$$\hat{V}(t) := \lambda(t, \hat{X}(t)) \int_{\mathbb{R}_0} \hat{r}(\{t\}, z)\tilde{N}(\{t\}, dz). \quad (10)$$

Moreover, we assume that

$$\begin{aligned}E \left[\int_0^T \hat{\alpha}^2(t, \beta, \varsigma) \left\{ \hat{q}^2(t) + \hat{\tilde{q}}^2(t) + \int_{\mathbb{R}^n} \hat{r}^2(t, z)\nu(dz) \right\} dt \right] &< \infty, \\ E \left[\int_0^T \hat{p}^2(t) \left\{ \left(\hat{\alpha}(t, \beta, \varsigma) \frac{\partial \hat{\sigma}}{\partial x}(t) + \frac{\partial \hat{\sigma}}{\partial u}(t)\beta(t) \right)^2 \right. \right. \\ &\quad \left. \left. + \left(\hat{\alpha}(t, \beta, \varsigma) \frac{\partial \hat{\tilde{\sigma}}}{\partial x}(t) + \frac{\partial \hat{\tilde{\sigma}}}{\partial u}(t)\beta(t) \right)^2 + \int_{\mathbb{R}_0} \gamma^2(t, \hat{X}(t), \hat{u}(t), z)\nu(dz) \right\} dt \right] < \infty.\end{aligned}$$

Then the following holds for almost all $t \in [0, T]$

$$E \left[\frac{\partial H}{\partial u} \left(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{\tilde{q}}(t), \hat{r}(t, \cdot) \right) (dt, d\hat{\xi}) \middle| \mathcal{G}_t \right] = 0. \quad (11)$$

$$E \left[\hat{U}(t) \middle| \mathcal{G}_t \right] \leq 0 \quad \text{and} \quad E \left[\hat{U}(t) \middle| \mathcal{G}_t \right] d\hat{\xi}^C(t) = 0. \quad (12)$$

$$E \left[\hat{V}(t) \middle| \mathcal{G}_t \right] \leq 0 \quad \text{and} \quad E \left[\hat{V}(t) \middle| \mathcal{G}_t \right] \triangle \hat{\xi}(t) = 0. \quad (13)$$

Proof of Theorem 3.2. Suppose that $(\hat{u}, \hat{\xi}) \in \mathcal{A}_{\mathcal{G}}$ is the solution of the control problem (5). Then

$$\lim_{y \rightarrow 0^+} \frac{1}{y} \left[\mathcal{J}(\hat{u} + y\beta, \hat{\xi} + y\varsigma) - \mathcal{J}(\hat{u}, \hat{\xi}) \right] \leq 0 \quad (14)$$

holds for all bounded $(\beta, \varsigma) \in \mathcal{A}_{\mathcal{G}}$. By the definition of $\mathcal{J}(u, \xi)$, (14) leads to

$$\begin{aligned} E \left[\int_0^T \left\{ \frac{\partial \hat{f}}{\partial x}(t) \hat{\alpha}(t, \beta, \varsigma) + \frac{\partial \hat{f}}{\partial u}(t) \beta(t) \right\} dt + g'(\hat{X}(T)) \hat{\alpha}(T, \beta, \varsigma) \right. \\ \left. + \int_0^T \frac{\partial h}{\partial x}(t, \hat{X}(T)) \hat{\alpha}(t, \beta, \varsigma) d\hat{\xi}(t) + \int_0^T h(t, \hat{X}(T)) d\varsigma(t) \right] \leq 0, \end{aligned} \quad (15)$$

where $\frac{\partial \hat{f}}{\partial x}(t) = \frac{\partial f}{\partial x}(t, \hat{X}(t), \hat{u}(t))$ and similarly for $\frac{\partial \hat{f}}{\partial u}(t)$, $\frac{\partial \hat{b}}{\partial x}(t)$, $\frac{\partial \hat{b}}{\partial u}(t)$, $\frac{\partial \hat{\sigma}}{\partial x}(t)$, $\frac{\partial \hat{\sigma}}{\partial u}(t)$, $\frac{\partial \hat{\gamma}}{\partial x}(t)$, $\frac{\partial \hat{\gamma}}{\partial u}(t)$, $\frac{\partial \hat{H}}{\partial x}(dt, d\hat{\xi})$, $\frac{\partial \hat{H}}{\partial u}(dt, d\hat{\xi})$.

We firstly consider $E \left[g'(\hat{X}(T)) \hat{\alpha}(T, \beta, \varsigma) \right]$. By Itô formula, we see that

$$\begin{aligned} E \left[g'(\hat{X}(T)) \hat{\alpha}(T, \beta, \varsigma) \right] &= E \left[\hat{p}(T) \hat{\alpha}(T, \beta, \varsigma) \right] \\ &= E \left[\int_0^T \left\{ \hat{p}(t) \left(\hat{\alpha}(t, \beta, \varsigma) \frac{\partial \hat{b}}{\partial x}(t) + \frac{\partial \hat{b}}{\partial u}(t) \beta(t) \right) + \hat{q}(t) \left(\hat{\alpha}(t, \beta, \varsigma) \frac{\partial \hat{\sigma}}{\partial x}(t) \right. \right. \right. \\ &\quad \left. \left. + \frac{\partial \hat{\sigma}}{\partial u}(t) \beta(t) \right) + \hat{q}(t) \left(\hat{\alpha}(t, \beta, \varsigma) \frac{\partial \hat{\sigma}}{\partial x}(t) + \frac{\partial \hat{\sigma}}{\partial u}(t) \beta(t) \right) \right. \\ &\quad \left. + \int_{\mathbb{R}_0} \hat{r}(t, z) \left(\hat{\alpha}(t, \beta, \varsigma) \frac{\partial \hat{\gamma}}{\partial x}(t, z) + \frac{\partial \hat{\gamma}}{\partial u}(t, z) \beta(t) \right) \nu(dz) \right\} dt \\ &\quad + \hat{p}(t) \hat{\alpha}(t, \beta, \varsigma) \frac{\partial \lambda}{\partial x} \left(t, \hat{X}(t) \right) d\hat{\xi}^C(t) + \hat{p}(t) \lambda \left(t, \hat{X}(t) \right) d\varsigma^C(t) \\ &\quad - \hat{\alpha}(t, \beta, \varsigma) \frac{\partial \hat{H}}{\partial x} \left(dt, d\hat{\xi} \right) + \sum_{0 \leq t \leq T} \left\{ \hat{\alpha}(t, \beta, \varsigma) \frac{\partial \lambda}{\partial x} \left(t, \hat{X}(t) \right) \triangle \hat{\xi}(t) \right. \\ &\quad \left. + \lambda \left(t, \hat{X}(t) \right) \triangle \varsigma(t) \right\} \int_{\mathbb{R}_0} r(\{t\}, z) \tilde{N}(\{t\}, dz) \right], \end{aligned} \quad (16)$$

where $\triangle \varsigma(t) = \varsigma(t) - \varsigma(t-)$ and $\varsigma^C(t)$ are pure discontinuous part and continuous part of $\varsigma(t)$, respectively. By the definition of Hamiltonian (6), we have

$$\begin{aligned} \frac{\partial H}{\partial x}(dt, d\hat{\xi}) &= \left(\frac{\partial f}{\partial x}(t) + p \frac{\partial b}{\partial x}(t) + q \frac{\partial \sigma}{\partial x}(t) + \tilde{q} \frac{\partial \tilde{\sigma}}{\partial x}(t) + \int_{\mathbb{R}} r(t, z) \frac{\partial \gamma}{\partial x}(t, z) \nu(dz) \right) dt \\ &\quad + \left(p \frac{\partial \lambda}{\partial x}(t) + \frac{\partial h}{\partial x}(t) \right) d\hat{\xi}^C(t) + \frac{\partial \lambda}{\partial x}(t) \int_{\mathbb{R}_0} r(\{t\}, z) N(\{t\}, dz) \triangle \xi(t) \end{aligned} \quad (17)$$

and

$$\frac{\partial H}{\partial u}(dt, d\xi) = \left(\frac{\partial f}{\partial u}(t) + p \frac{\partial b}{\partial u}(t) + q \frac{\partial \sigma}{\partial u}(t) + \tilde{q} \frac{\partial \tilde{\sigma}}{\partial u}(t) + \int_{\mathbb{R}} r(t, z) \frac{\partial \gamma}{\partial u}(t, z) \nu(dz) \right) dt. \quad (18)$$

Then, substituting (16), (17), (18) into (15), we get

$$E \left[\int_0^T \frac{\partial \hat{H}}{\partial u}(dt, d\hat{\xi}) \beta(t) dt + \int_0^T \hat{U}(t) d\varsigma^C(t) + \sum_{0 \leq t \leq T} \hat{V}(t) \Delta \varsigma(t) \right] \leq 0, \quad (19)$$

where $\hat{U}(t)$ and $\hat{V}(t)$ are defined by (9) and (10), respectively.

Since the inequality (19) holds for all bounded $(\beta, \varsigma) \in \mathcal{A}_{\mathcal{G}}$, one can choose $\varsigma \equiv 0$ and has

$$E \left[\int_0^T \frac{\partial \hat{H}}{\partial u}(dt, d\hat{\xi}) \beta(t) \right] \leq 0. \quad (20)$$

In particular, for a fixed $s \in [0, T)$, the inequality (20) holds for all bounded $(\beta(s), 0) \in \mathcal{A}_{\mathcal{G}}$ with

$$\beta(s) = \beta(s, \omega) = \theta(\omega) \chi_{[t, t+h]}(s), \quad s \in [0, T],$$

where $\theta(\omega)$ is a bounded \mathcal{G}_t -measurable random variable. Then (20) can be written as

$$E \left[\int_t^{t+h} \frac{\partial \hat{H}}{\partial u}(dt, d\hat{\xi}) \theta \right] \leq 0. \quad (21)$$

Since (21) holds for both θ and $-\theta$, it follows that

$$E \left[\int_t^{t+h} \frac{\partial \hat{H}}{\partial u}(dt, d\hat{\xi}) \theta \right] = 0. \quad (22)$$

Differentiating (22) with respect to h at $h = 0$, we have

$$E \left[\frac{\partial \hat{H}}{\partial u}(dt, d\hat{\xi}) \theta \right] = 0$$

holds for all bounded \mathcal{G}_t -measurable random variable θ . Then we conclude that

$$E \left[\frac{\partial \hat{H}}{\partial u}(dt, d\hat{\xi}) \middle| \mathcal{G}_t \right] = 0$$

proving (11).

Next we prove (12) and (13). Note that the inequality (19) holds for all bounded $(\beta, \varsigma) \in \mathcal{A}_{\mathcal{G}}$. Thus, by letting $\beta = 0$, we obtain

$$E \left[\int_0^T \hat{U}(t) d\varsigma^C(t) + \sum_{0 \leq t \leq T} \hat{V}(t) \Delta \varsigma(t) \right] \leq 0, \quad \text{for all } (0, \varsigma) \in \mathcal{A}_{\mathcal{G}}. \quad (23)$$

In order to prove (12), we choose ς in the following way:

$$d\varsigma(t) = a(t) dt, \quad t \in [0, T],$$

where $a(t) \geq 0$ is \mathcal{G}_t -adapted continuous stochastic process. Then it follows from (23) that

$$E \left[\int_0^T \hat{U}(t) a(t) dt \right] \leq 0$$

holds for all \mathcal{G}_t -adapted $a(t) \geq 0$, which implies that

$$E \left[\hat{U}(t) \middle| \mathcal{G}_t \right] \leq 0, \quad \text{for almost all } t \in [0, T].$$

Moreover, by choosing $\varsigma(t) = \hat{\xi}^C(t)$, together with (23), we have

$$E \left[\int_0^T \hat{U}(t) d\hat{\xi}^C(t) \right] \leq 0. \quad (24)$$

Similarly, let $\varsigma(t) = -\hat{\xi}^C(t)$ in (23). Then we get

$$E \left[\int_0^T \hat{U}(t) \left(-d\hat{\xi}^C(t) \right) \right] \leq 0. \quad (25)$$

Combining (24) and (25), we obtain

$$E \left[\int_0^T \hat{U}(t) d\hat{\xi}^C(t) \right] = E \left[\int_0^T E \left[\hat{U}(t) \middle| \mathcal{G}_t \right] d\hat{\xi}^C(t) \right] = 0. \quad (26)$$

Since $\hat{\xi}(\cdot)$ is singular control, we have $d\hat{\xi}^C(t) \geq 0$. Hence, it follows from (3.2) and (26) that

$$E \left[\hat{U}(t) \middle| \mathcal{G}_t \right] d\hat{\xi}^C(t) = 0.$$

In order to prove (13), we fix $t \in [0, T]$ and choose ς such that

$$d\varsigma(s) = a(\omega) \delta_t(s); \quad s \in [0, T],$$

where $a(\omega) \geq 0$ is \mathcal{G}_t -measurable and bounded, and $\delta_t(s)$ is the unit point mass at t . In this case, we obtain by (23) that

$$E \left[\hat{V}(t) a(\omega) \right] \leq 0$$

holds for all bounded \mathcal{G}_t -measurable $a(\omega) \geq 0$. This gives

$$E \left[\hat{V}(t) \middle| \mathcal{G}_t \right] \leq 0.$$

Let $\hat{\xi}^d(t)$ denote the purely discontinuous part of $\hat{\xi}(t)$. Choosing $\varsigma(t) = \hat{\xi}^d(t)$, we get by (23) that

$$E \left[\sum_{0 \leq t \leq T} \hat{V}(t) \Delta \hat{\xi}(t) \right] \leq 0. \quad (27)$$

Similarly, by letting $\varsigma(t) = -\hat{\xi}^d(t)$, we have

$$E \left[\sum_{0 \leq t \leq T} \hat{V}(t) \left(-\Delta \hat{\xi}(t) \right) \right] \leq 0. \quad (28)$$

Combining (27) and (28), we see that

$$E \left[\sum_{0 < t \leq T} \hat{V}(t) \Delta \hat{\xi}(t) \right] = E \left[\sum_{0 < t \leq T} E \left[\hat{V}(t) \middle| \mathcal{G}_t \right] \Delta \hat{\xi}(t) \right] = 0. \quad (29)$$

It is obvious that $\Delta \hat{\xi}(t) \geq 0$, for $\hat{\xi}(\cdot)$ is singular control. Thus we conclude from (3.2) and (29) that

$$E \left[\hat{V}(t) \middle| \mathcal{G}_t \right] \Delta \hat{\xi}(t) = 0 \quad \text{for all } t \in [0, T],$$

which completes the whole proof. \square

In Theorem 3.2, we have derived a maximum principle to give the necessary optimality conditions ((11), (12) and (13)) for the solutions of the control problem (5). Next we show that these necessary optimality conditions are sufficient for a directional sub-stationary point of $\mathcal{J}(u, \xi)$.

Theorem 3.3. *Let $(\hat{u}, \hat{\xi}) \in \mathcal{A}_{\mathcal{G}}$ satisfy (11), (12) and (13). Then $(\hat{u}, \hat{\xi})$ is a directional sub-stationary point for $\mathcal{J}(u, \xi)$, in the sense that*

$$\lim_{y \rightarrow 0^+} \frac{1}{y} \left[\mathcal{J}(\hat{u} + y\beta, \hat{\xi} + y\varsigma) - \mathcal{J}(\hat{u}, \hat{\xi}) \right] \leq 0, \quad \text{for all bounded } (\beta, \varsigma) \in \mathcal{A}_{\mathcal{G}}.$$

Proof of Theorem 3.3. Suppose (11), (12) and (13) hold for $(\hat{u}, \hat{\xi}) \in \mathcal{A}_{\mathcal{G}}$. Then, by Assumption 1, we can choose $(\beta, \varsigma) \in \mathcal{A}_{\mathcal{G}}$ such that

$$(\hat{u}(t) + y\beta(t), \hat{\xi}(t) + y\varsigma(t)) \in \mathcal{A}_{\mathcal{G}}, \quad t \in [0, T],$$

for all $y \in [0, \delta]$ for some $\delta > 0$. In this case, we have

$$d\hat{\xi}^C(t) + y d\varsigma^C(t) \geq 0 \tag{30}$$

and

$$\Delta(\hat{\xi}(t) + y\varsigma(t)) \geq 0. \tag{31}$$

Given $\delta > 0$, for $y \in [0, \delta]$, we consider

$$\begin{aligned} & yE \left\{ \int_0^T \frac{\partial \hat{H}}{\partial u}(dt, d\hat{\xi}) \beta(t) dt + \int_0^T \hat{U}(t) d\varsigma^C(t) + \sum_{0 < t \leq T} \hat{V}(t) \Delta \varsigma(t) \right\} \\ = & yE \left\{ \int_0^T E \left[\frac{\partial \hat{H}}{\partial u}(dt, d\hat{\xi}) \middle| \mathcal{G}_t \right] \beta(t) dt \right. \\ & \left. + \int_0^T E \left[\hat{U}(t) \middle| \mathcal{G}_t \right] d\varsigma^C(t) + \sum_{0 < t \leq T} E \left[\hat{V}(t) \middle| \mathcal{G}_t \right] \Delta \varsigma(t) \right\} \\ = & E \left\{ \int_0^T E \left[\hat{U}(t) \middle| \mathcal{G}_t \right] d\hat{\xi}^C(t) + \sum_{0 < t \leq T} E \left[\hat{V}(t) \middle| \mathcal{G}_t \right] \Delta \hat{\xi}(t) \right\} \\ & + yE \left\{ \int_0^T E \left[\frac{\partial \hat{H}}{\partial u}(dt, d\hat{\xi}) \middle| \mathcal{G}_t \right] \beta(t) dt \right. \\ & \left. + \int_0^T E \left[\hat{U}(t) \middle| \mathcal{G}_t \right] d\varsigma^C(t) + \sum_{0 < t \leq T} E \left[\hat{V}(t) \middle| \mathcal{G}_t \right] \Delta \varsigma(t) \right\} \\ = & E \left\{ \int_0^T E \left[\hat{U}(t) \middle| \mathcal{G}_t \right] d(\hat{\xi}^C(t) + y\varsigma^C(t)) + \sum_{0 < t \leq T} E \left[\hat{V}(t) \middle| \mathcal{G}_t \right] \Delta(\hat{\xi}(t) + y\varsigma(t)) \right\}. \end{aligned}$$

From (12), (13), (30) and (31), we see that

$$\int_0^T E \left[\hat{U}(t) \middle| \mathcal{G}_t \right] d(\hat{\xi}^C(t) + y\varsigma^C(t)) \leq 0$$

and

$$\sum_{0 \leq t \leq T} E \left[\hat{V}(t) \middle| \mathcal{G}_t \right] \Delta \left(\hat{\xi}(t) + y \varsigma(t) \right) \leq 0.$$

Therefore, (19) in the proof of Theorem 3.2 leads to

$$\begin{aligned} & \lim_{y \rightarrow 0^+} \frac{1}{y} \left[\mathcal{J}(\hat{u} + y\beta, \hat{\xi} + y\varsigma) - \mathcal{J}(\hat{u}, \hat{\xi}) \right] \\ &= E \left[\int_0^T \frac{\partial \hat{H}}{\partial u}(dt, d\hat{\xi}) \beta(t) dt + \int_0^T \hat{U}(t) d\varsigma^C(t) + \sum_{0 \leq t \leq T} \hat{V}(t) \Delta \varsigma(t) \right] \leq 0. \end{aligned}$$

□

3.3. Sufficient maximum principle for regular-singular stochastic control problem with partial information. In this subsection, we impose some additional conditions such that the necessary optimality conditions (11), (12) and (13) are also sufficient for the solutions of the control problem (5).

Theorem 3.4 (Sufficient maximum principle). *Suppose that $(\hat{u}, \hat{\xi}) \in \mathcal{A}_{\mathcal{G}}$ satisfies (12), (13) and*

$$\begin{aligned} & \sup_u E \left[H \left(t, \hat{X}(t), u, \hat{p}, \hat{q}, \hat{\hat{q}}, \hat{r}(\cdot) \right) (dt, d\hat{\xi}) \middle| \mathcal{G}_t \right] \\ &= E \left[H \left(t, \hat{X}(t), \hat{u}, \hat{p}, \hat{q}, \hat{\hat{q}}, \hat{r}(\cdot) \right) (dt, d\hat{\xi}) \middle| \mathcal{G}_t \right]. \end{aligned} \quad (32)$$

Let $\hat{X}(t)$, $\hat{p}(t)$, $\hat{q}(t)$, $\hat{\hat{q}}(t)$, $\hat{r}(\cdot)(t, z)$ be the solutions of the equations (3) and (7) corresponding to $(\hat{u}, \hat{\xi})$. We assume that

$$\begin{aligned} & E \left[\int_0^T \left(\hat{X}(t) - X(t) \right)^2 \left\{ \hat{q}^2(t) + \hat{\hat{q}}^2(t) + \int_{\mathbb{R}_0} \hat{r}^2(t, z) \nu(dz) \right\} dt \right] < \infty, \\ & E \left[\int_0^T \hat{p}^2(t) \left\{ \sigma^2(t, X(t), u(t)) + \tilde{\sigma}^2(t, X(t), u(t)) \right. \right. \\ & \quad \left. \left. + \int_{\mathbb{R}_0} \gamma^2(t, X(t), u(t), z) \nu(dz) \right\} dt \right] < \infty, \quad \text{for all } u. \end{aligned}$$

Moreover, suppose that

$$x \rightarrow g(x) \quad \text{is concave}$$

and

$$(x, u, \xi) \rightarrow H(t, x, u, \hat{p}(t), \hat{q}(t), \hat{\hat{q}}(t), \hat{r}(t, \cdot))(dt, d\xi) \text{ is concave, for all } t \in [0, T].$$

Then $(\hat{u}, \hat{\xi})$ is the optimal regular-singular control for the control problem (5).

Proof of Theorem 3.4. In the following we use the notation

$$\begin{aligned} \hat{H}(t)(dt, d\hat{\xi}) &= H(t, \hat{x}, \hat{u}, \hat{p}, \hat{q}, \hat{\hat{q}}, \hat{r}(\cdot))(dt, d\hat{\xi}), \\ H(t)(dt, d\xi) &= H(t, x, u, \hat{p}, \hat{q}, \hat{\hat{q}}, \hat{r}(\cdot))(dt, d\xi) \end{aligned}$$

and similarly with $\hat{f}(t)$, $f(t)$, $\hat{b}(t)$, $b(t)$, $\hat{\sigma}(t)$, $\sigma(t)$, $\hat{\tilde{\sigma}}(t)$, $\tilde{\sigma}(t)$, $\hat{\gamma}(t, z)$, $\gamma(t, z)$. Then, by the definition of $\mathcal{J}(u, \xi)$, we see that

$$\mathcal{J}(u, \xi) - \mathcal{J}(\hat{u}, \hat{\xi}) = I_1 + I_2 + I_3, \quad (33)$$

where

$$\begin{aligned} I_1 &= E \left[\int_0^T (f(t) - \hat{f}(t)) dt \right], \\ I_2 &= E \left[g(X(T)) - g(\hat{X}(T)) \right], \\ I_3 &= E \left[\int_0^T h(t, X(t)) d\xi(t) - \int_0^T h(t, \hat{X}(t)) d\hat{\xi}(t) \right]. \end{aligned}$$

We firstly consider I_1 . It follows from the definition of Hamiltonian (6) that

$$\begin{aligned} I_1 &= E \left[\int_0^T H(t)(dt, d\xi) - \hat{H}(t)(dt, d\hat{\xi}) - \int_0^T \left\{ \hat{p}(t) (b(t) - \hat{b}(t)) \right. \right. \\ &\quad + \hat{q}(t) (\sigma(t) - \hat{\sigma}(t)) + \hat{\hat{q}}(t) (\tilde{\sigma}(t) - \hat{\tilde{\sigma}}(t)) \\ &\quad + \int_{\mathbb{R}_0} \hat{r}(t, z) (\gamma(t, z) - \hat{\gamma}(t, z)) \nu(dz) \Big\} dt \\ &\quad - \int_0^T (\hat{p}(t) \lambda(t, X(t)) + h(t, X(t))) d\xi^C(t) \\ &\quad + \int_0^T (\hat{p}(t) \lambda(t, \hat{X}(t)) + h(t, \hat{X}(t))) d\hat{\xi}^C(t) \\ &\quad \left. - \sum_{0 < t \leq T} \left(\lambda(t, X(t)) \Delta \xi(t) - \lambda(t, \hat{X}(t)) \Delta \hat{\xi}(t) \right) \int_{\mathbb{R}_0} \hat{r}(\{t\}, z) N(\{t\}, dz) \right]. \end{aligned} \quad (34)$$

Next we consider I_2 . Let $\tilde{X}(T) := X(T) - \hat{X}(T)$. Since $g(x)$ is concave in x , we have

$$I_2 = E \left[g(X(T)) - g(\hat{X}(T)) \right] \leq E \left[g'(\hat{X}(T)) (X(T) - \hat{X}(T)) \right] = E \left[\hat{p}(T) \tilde{X}(T) \right].$$

Applying Itô formula to $E \left[\hat{p}(T) \tilde{X}(T) \right]$, we get

$$\begin{aligned} I_2 &\leq E \left[\int_0^T \left\{ \hat{p}(t) (b(t) - \hat{b}(t)) + \hat{q}(t) (\sigma(t) - \hat{\sigma}(t)) + \hat{\hat{q}}(t) (\tilde{\sigma}(t) - \hat{\tilde{\sigma}}(t)) \right. \right. \\ &\quad + \int_{\mathbb{R}_0} \hat{r}(t, z) (\gamma(t, z) - \hat{\gamma}(t, z)) \nu(dz) \Big\} dt \Big] - \int_0^T \tilde{X}(t) \frac{\partial \hat{H}}{\partial x}(t)(dt, d\hat{\xi}) \\ &\quad + \int_0^T \hat{p}(t) \lambda(t, X(t)) d\xi^C(t) - \int_0^T \hat{p}(t) \lambda(t, \hat{X}(t)) d\hat{\xi}^C(t) \\ &\quad + \sum_{0 < t \leq T} \left(\lambda(t, X(t)) \Delta \xi(t) - \lambda(t, \hat{X}(t)) \Delta \hat{\xi}(t) \right) \int_{\mathbb{R}_0} \hat{r}(\{t\}, z) N(\{t\}, dz) \Big]. \end{aligned} \quad (35)$$

Substituting (34) and (35) into (33), we obtain

$$\begin{aligned} &\mathcal{J}(u, \xi) - \mathcal{J}(\hat{u}, \hat{\xi}) \\ &\leq E \left[\int_0^T H(t)(dt, d\xi) - \int_0^T \hat{H}(t)(dt, d\hat{\xi}) - \int_0^T \tilde{X}(t) \frac{\partial \hat{H}}{\partial x}(t)(dt, d\hat{\xi}) \right]. \end{aligned} \quad (36)$$

Since $(x, u, \xi) \rightarrow H(t, x, u, \hat{p}(t), \hat{q}(t), \hat{\bar{q}}(t), \hat{r}(t, \cdot))(dt, d\xi)$ is concave, we have

$$\begin{aligned} H(t)(dt, d\xi) - \hat{H}(t)(dt, d\hat{\xi}) &\leq \frac{\partial \hat{H}}{\partial x}(t)(dt, d\hat{\xi}) (X(t) - \hat{X}(t)) \\ &\quad + \frac{\partial \hat{H}}{\partial u}(t)(dt, d\hat{\xi}) (u(t) - \hat{u}(t)) \\ &\quad + \nabla_{\xi} \hat{H}(t) (d\xi(t) - d\hat{\xi}(t)), \end{aligned}$$

where $\nabla_{\xi} \hat{H}$ is the Fréchet derivative of \hat{H} at ξ . Then (36) leads to

$$\begin{aligned} &\mathcal{J}(u, \xi) - \mathcal{J}(\hat{u}, \hat{\xi}) \\ &\leq E \left\{ \int_0^T (u(t) - \hat{u}(t)) \frac{\partial \hat{H}}{\partial u}(t)(dt, d\hat{\xi}) + \int_0^T \nabla_{\xi} \hat{H}(t) (d\xi(t) - d\hat{\xi}(t)) \right\} \\ &= E \left\{ \int_0^T (u(t) - \hat{u}(t)) E \left[\frac{\partial \hat{H}}{\partial u}(t)(dt, d\hat{\xi}) \middle| \mathcal{G}_t \right] \right. \\ &\quad + \int_0^T E [\hat{U}(t) | \mathcal{G}_t] (d\xi^C(t) - d\hat{\xi}^C(t)) \\ &\quad \left. + \sum_{0 < t \leq T} E [\hat{V}(t) | \mathcal{G}_t] (\Delta \xi(t) - \Delta \hat{\xi}(t)) \right\}. \end{aligned}$$

Since the control $(\hat{u}, \hat{\xi}) \in \mathcal{A}_{\mathcal{G}}$ satisfies (11), (12) and (13), we conclude that

$$\mathcal{J}(u, \xi) - \mathcal{J}(\hat{u}, \hat{\xi}) \leq E \left\{ \int_0^T E [\hat{U}(t) | \mathcal{G}_t] d\xi^C(t) + \sum_{0 < t \leq T} E [\hat{V}(t) | \mathcal{G}_t] \Delta \xi(t) \right\} \leq 0.$$

holds for all $(u, \xi) \in \mathcal{A}_{\mathcal{G}}$. Therefore, $(\hat{u}, \hat{\xi})$ is optimal for the control problem (5). \square

4. Solutions to the investment and dividend problem of an insurer. In this section, we come back to the investment and dividend problem of an insurer (2) and solve it by Theorem 3.2 and Theorem 3.4.

With the notation of Section 3 we see that in the control problem (2) we have

$$\begin{aligned} b(t, x, u) &= \kappa(t) + \rho(t)x(t) + u(t) [\zeta(t) - \rho(t)] - \int_{\mathbb{R}_0} z\nu(dz); \\ \sigma(t, x, u) &= \pi(t)u(t); \quad \tilde{\sigma}(t, x, u) = a; \quad \gamma(t, x, u, z) = -z; \quad \lambda(t, x) = -1; \\ f(t, x, u) &= -Qu^2(t); \quad g(x) = -\frac{1}{2}(x - D)^2; \quad h(t, x) = e^{-\int_0^t \rho(s)ds}. \end{aligned}$$

Then the corresponding Hamiltonian is

$$\begin{aligned} H &= \left\{ -Qu^2(t) + \left[\kappa(t) + \rho(t)x(t) + u(t) [\zeta(t) - \rho(t)] - \int_{\mathbb{R}_0} z\nu(dz) \right] p(t) \right. \\ &\quad \left. + \pi(t)u(t)q(t) + a\tilde{q}(t) - \int_{\mathbb{R}} r(t, z)z\nu(dz) \right\} dt \\ &\quad + \left[-p(t) + e^{-\int_0^t \rho(s)ds} \right] d\xi^C(t) - \int_{\mathbb{R}_0} r(\{t\}, z)N(\{t\}, dz)\Delta \xi(t), \end{aligned}$$

where the adjoint processes $(p(t), q(t), \tilde{q}(t), r(\cdot)(t, z))$ are given by the following backward SDE:

$$\begin{cases} dp(t) = -\rho(t)p(t)dt + q(t)dB(t) + \tilde{q}(t)d\tilde{B}(t) + \int_{\mathbb{R}_0} r(t, z)\tilde{N}(dt, dz), \\ p(T) = X(T) - D. \end{cases} \quad (37)$$

It is obvious that $-\frac{1}{2}(x - D)^2$ is concave with respect to x and

$$(x, u, \xi) \rightarrow H(t, x, u, \hat{p}(t), \hat{q}(t), \hat{\tilde{q}}(t), \hat{r}(t, \cdot))(dt, d\xi) \text{ is concave.}$$

Therefore, the sufficient and necessary conditions for the solutions of the control problem (2) can be given by Theorem 3.2 and Theorem 3.4.

We firstly consider the control problem (2) in the case of $Q > 0$. Let $(\hat{u}, \hat{\xi}) \in \mathcal{A}_{\mathcal{G}}$ be an optimal regular-singular control and let $\hat{X}(t), \hat{p}(t), \hat{q}(t), \hat{\tilde{q}}(t), \hat{r}(\cdot)(t, z)$ be the corresponding solutions of the equations (1) and (37). Then, by Theorem 3.2 and Theorem 3.4, the optimality condition (11) leads to

$$\hat{u}(t) = \frac{1}{2Q} \{ [\zeta(t) - \rho(t)] E[\hat{p}(t) | \mathcal{G}_t] + \pi(t) E[\hat{q}(t) | \mathcal{G}_t] \}. \quad (38)$$

By the optimality condition (12), we obtain that $\hat{\xi}^C(t)$, which is the continuous part of $\hat{\xi}(t)$, satisfies:

$$E[\hat{p}(t) | \mathcal{G}_t] \geq e^{-\int_0^t \rho(s)ds} \quad \text{and} \quad \left\{ e^{-\int_0^t \rho(s)ds} - E[\hat{p}(t) | \mathcal{G}_t] \right\} d\hat{\xi}^C(t) = 0. \quad (39)$$

And it follows from the optimality condition (13) that

$$E \left[\int_{\mathbb{R}_0} \hat{r}(\{t\}, z) \tilde{N}(\{t\}, dz) \middle| \mathcal{G}_t \right] \geq 0 \quad (40)$$

and

$$E \left[\int_{\mathbb{R}_0} \hat{r}(\{t\}, z) \tilde{N}(\{t\}, dz) \middle| \mathcal{G}_t \right] \Delta \hat{\xi}(t) = 0 \quad (41)$$

holds for all $t \in [0, T]$, where $\Delta \hat{\xi}(t) = \hat{\xi}(t) - \hat{\xi}(t-)$ is the purely discontinuous part of $\hat{\xi}(t)$.

We summarize the above argument in the following theorem.

Theorem 4.1. Assume that $Q > 0$. $(\hat{u}, \hat{\xi}) \in \mathcal{A}_{\mathcal{G}}$ is an optimal regular-singular control for the investment and dividend problem of an insurer (2) if and only if $(\hat{u}, \hat{\xi})$ satisfies (38), (39), (41) and (41), where $\hat{X}(t), \hat{p}(t), \hat{q}(t), \hat{\tilde{q}}(t), \hat{r}(\cdot)(t, z)$ are the solutions of the corresponding equations (1) and (37).

In theorem 4.1, the adjoint processes $\hat{p}(t), \hat{q}(t), \hat{\tilde{q}}(t)$ and $\hat{r}(\cdot)(t, z)$ are defined in terms of backward SDE (37), which is usually hard to solve. Here we leave the solution methods of this backward SDE for future research. Instead we wrap up this paper by giving the explicit solutions in the special case when $Q = 0$ and $\xi = 0$.

Corollary 1. Assume that $Q = 0$ and $\xi(t) = 0$ for $t \in [0, T]$. Then $(\hat{u}, 0) \in \mathcal{A}_{\mathcal{G}}$ is an optimal control for the problem (2) if and only if $\hat{u}(t)$ is given by

$$\hat{u}(t) = \frac{[2\rho(t)\varphi(t) + \varphi'(t)] E[\hat{X}(t) | \mathcal{G}_t] + \rho(t)\psi(t) + \varphi(t) [\kappa(t) - \int_{\mathbb{R}_0} z\nu(dz)] + \psi'(t)}{-\varphi(t)(\zeta(t) - \rho(t))}. \quad (42)$$

Here $\hat{X}(t)$ is the solution of (1) corresponding to $(\hat{u}(t), 0)$. The deterministic processes $\varphi(t)$ and $\psi(t)$ are defined by

$$\varphi(t) = e^{-\int_t^T \{\Gamma_s^2 - 2\rho(s)\}ds} \quad (43)$$

and

$$\psi(t) = e^{-\int_t^T \{\Gamma_s^2 - \rho(s)\} ds} \left\{ \int_t^T \left[\kappa(s) - \int_{\mathbb{R}_0} z \nu(dz) \right] e^{-\int_s^T \{2\Gamma_r^2 - 3\rho(r)\} dr} ds - D \right\}, \quad (44)$$

where

$$\Gamma_t := \frac{\zeta(t) - \rho(t)}{\pi(t)}. \quad (45)$$

Proof. Under the assumptions of $Q = 0$ and $\xi = 0$, we see by the optimality condition (11) that the optimal control \hat{u} is given by

$$[\zeta(t) - \rho(t)] E[\hat{p}(t) | \mathcal{G}_t] + \pi(t) E[\hat{q}(t) | \mathcal{G}_t] = 0, \quad (46)$$

where $\hat{p}(t)$, $\hat{q}(t)$, $\hat{\tilde{q}}(t)$ and $\hat{r}(\cdot)(t, z)$ satisfy the backward SDE (37). In order to get the solutions of (37), we conjecture that $p(t)$ has the following form:

$$p(t) = \varphi(t)X(t) + \psi(t), \quad (47)$$

where $\varphi(t)$ and $\psi(t)$ are deterministic differential functions. Then, applying Itô formula to (47), we have

$$\begin{aligned} dp(t) = & \{ \varphi'(t)X(t) + \psi'(t) + \varphi(t)[\kappa(t) + \rho(t)X(t) + u(t)(\zeta(t) - \rho(t)) \\ & - \int_{\mathbb{R}_0} z \nu(dz)] \} dt + \varphi(t)\pi(t)u(t)dB(t) \\ & + \varphi(t)ad\tilde{B}(t) - \int_{\mathbb{R}_0} z\varphi(t)\tilde{N}(dt, dz). \end{aligned} \quad (48)$$

Comparing (37) and (48), we see that

$$\begin{aligned} -\rho(t)[\varphi(t)X(t) + \psi(t)] = & \varphi(t)[\kappa(t) + \rho(t)X(t) + u(t)(\zeta(t) - \rho(t)) \\ & - \int_{\mathbb{R}_0} z \nu(dz)] + \varphi'(t)X(t) + \psi'(t), \end{aligned} \quad (49)$$

$$q(t) = \varphi(t)\pi(t)u(t), \quad (50)$$

$$\tilde{q}(t) = \varphi(t)a,$$

$$r(t, z) = z\varphi(t).$$

Substituting (47) and (50) into (46), we obtain

$$\hat{u}(t) = \frac{[\zeta(t) - \rho(t)] \left\{ \varphi(t)E[\hat{X}(t) | \mathcal{G}_t] + \psi(t) \right\}}{-\varphi(t)\pi^2(t)}. \quad (51)$$

Now we determine $\varphi(t)$ and $\psi(t)$. It follows from (49) that $\hat{u}(t)$ is represented by (42). Comparing (42) and (51), together with (37), we have

$$\frac{[\zeta(t) - \rho(t)] \varphi(t)}{\varphi(t)\pi^2(t)} = \frac{[2\rho(t)\varphi(t) + \varphi'(t)]}{\varphi(t)(\zeta(t) - \rho(t))}; \quad \varphi(T) = 1$$

and

$$\frac{[\zeta(t) - \rho(t)] \psi(t)}{\varphi(t)\pi^2(t)} = \frac{\rho(t)\psi(t) + \varphi(t) \left(\kappa(t) - \int_{\mathbb{R}_0} z \nu(dz) \right) + \psi'(t)}{\varphi(t)(\zeta(t) - \rho(t))}; \quad \psi(T) = -D.$$

Let Γ_t be defined by (45). Then the above two equations can be simplified to

$$\varphi'(t) = \{\Gamma_t^2 - 2\rho(t)\} \varphi(t); \quad \varphi(T) = 1 \quad (52)$$

and

$$\psi'(t) = \{\Gamma_t^2 - \rho(t)\} \psi(t) - \varphi(t) \left[\kappa(t) - \int_{\mathbb{R}_0} z \nu(dz) \right]; \quad \psi(T) = -D. \quad (53)$$

It is easy to solve the backward ordinary differential equations (52) and (53) and their explicit solutions are given by (43) and (44), which completes the proof. \square

5. Conclusion. We studied an optimal investment and dividend problem of an insurance company in jump diffusions, where the information available to the insurer is partial information. Our model includes financial risk and insurance risk. From the view of optimal control theory, this problem can be generalized as regular-singular stochastic control problem with partial information. We derived maximum principles to give the sufficient and necessary optimality conditions for its solutions. Then, with help of the established maximum principles, we characterized the solutions of the investment and dividend problem of an insurer and gave its explicit solutions in special cases. On the other hand, in our maximum principle formulation, the adjoint processes are defined in terms of a backward SDE, whose explicit solutions are usually hard to get. Therefore, the numerical methods of this type of backward SDEs will be explored in our subsequent work.

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