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Finite N-Quandles of Twisted Double Handcuff and Complete Graph

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Abstract

The Double Handcuff and K^4 graphs can be generalized to a single family of spatial graphs by adding a variable number of twists between two edges. We can identify spatial graphs by calculating a quotient of the fundamental quandle, known as an N-quandle, which is a spatial graph invariant. In this paper, we prove that the N-quandle associated with this family of spatial graphs is finite when all but two edges are given a label of 2, and the remaining two edges are assigned labels from the natural numbers. To prove that the N-quandle is finite, we produce Cayley graphs for each of the N-quandle components, providing corresponding proofs and analysis.

Contents

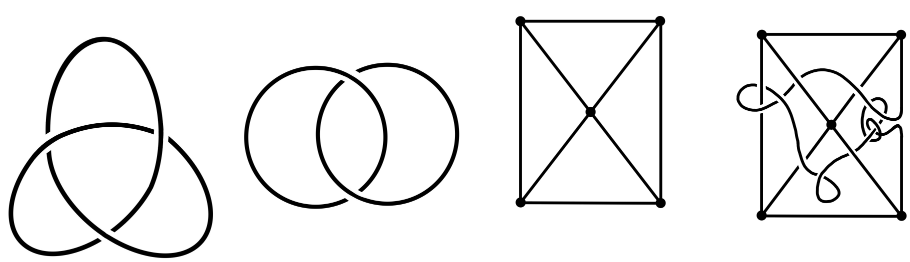
CHAPTER 1

Introduction

1.1. Spatial Graphs

A graph G consists of a set V of vertices and a set E of edges which connect distinct vertices. These graphs can be directed or undirected, but for our purposes we will focus on the former. A spatial graph is a finite graph embedded in $\mathbb{R}^3$, and as such can be considered a generalization of knots, which are simple closed curves in $\mathbb{R}^3$, and links, which are collections of one or more knots which may overlap but may not intersect. One important distinction between spatial graphs and these other topological objects is that spatial graphs allow for the existence of intersections, which result in vertices.

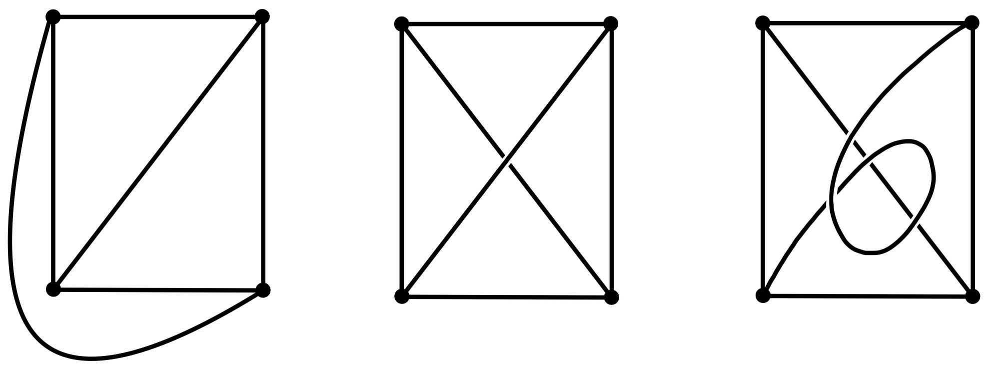
We can represent three dimensional spatial graphs in two dimensions using diagrams, which project the image of a graph from $\mathbb{R}^3$ to $\mathbb{R}^2$. To differentiate between *intersections*, where two edges of a graph physically meet at a vertex, and *crossings* in which two non-intersecting edges overlap in the projection, we represent the lower curve of a crossing as broken. An example of a diagram of a knot, link, graph and spatial graph is presented in Figure 1.



(A) Knot (B) Link (C) Graph (D) Spatial graph

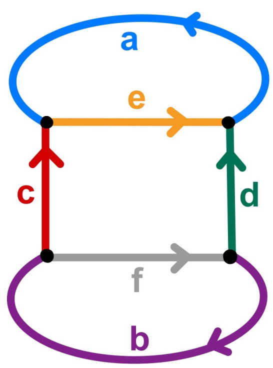
FIGURE 1. Knot, link and graph diagrams

One important question in topology and knot theory is how to distinguish spatial graphs. As Figure 2 demonstrates, diagrams do not uniquely identify spatial graphs, since the edges of the graph can be transformed and twisted so as to change the number of crossings without changing the fundamental structure of a spatial graph. In this paper, we will explore alternative approaches to identifying a topological object.

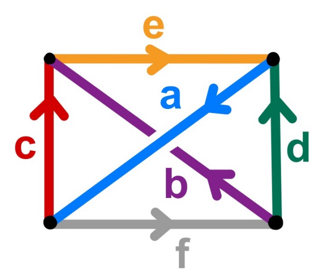


(A) No crossings (B) One crossing (C) Four crossings

Figure 2. Various representations of K_4.



(A) Double Handcuff



(B) Complete K4

Figure 3

In particular, we will consider two well known graphs: the Double Handcuff Graph (Figure 3a) and the Complete (K4) Graph (Figure 3b). We can add k twists between the a and b edges in each graph to generate a corresponding spatial graph, which now is embedded in $\mathbb{R}^3$. The direction of the twists is depicted in Figure 4. Note that these two graphs can be generalized to have the same form as depicted in Figure 5, where G is a tangled double handcuff graph when k is even and it is a tangled K4 graph when k is odd.

1.2. Quandles, n-Quandles and N-Quandles

To distinguish spatial graphs, we must rely on a spatial graph invariant. One such invariant is the *fundamental quandle* of a spatial graph.

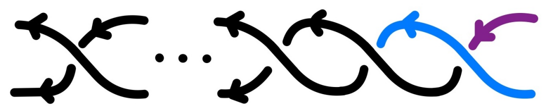


FIGURE 4. k twists between the a and b edges.

A *quandle* is defined as a set Q with two binary operators, $\triangleright$ and $\triangleright^{-1}$ which satisfy the following three axioms:

(1) $\forall x \in Q$, $x \triangleright x = x$
(2) $\forall x, y \in Q$, $(x \triangleright y) \triangleright^{-1} y = x$
(3) $\forall x, y, z \in Q$, $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

Alternatively, quandle operators can be denoted with exponential notation such that $x^y = x \triangleright y$ and $x^{\overline{y}} = x \triangleright^{-1} y$. For the remainder of the paper, we will utilize exponential notation.

We assign quandle structure to spatial graphs by defining relationships at the crossings and vertices of the graph. Consider a spatial graph G with vertices V and edges E, some of which may be knotted or linked. The *generators* of its fundamental quandle, Q, are the arcs (i.e. the portions of edges between crossings) of G. We define relationships between these elements at the points where they cross and intersect. First we will describe the relationship at crossings. Consider an arc x_j that crosses over a component, splitting it into arcs, x_i on the left and x_k on the right. We then say that $x_i = x_k \triangleright x_j$. Next, we will describe the relationship at vertices. Consider n arcs, $a_1, a_2, \cdots a_n$ which are ordered counter-clockwise and meet at a vertex v. Then,

$$((x \triangleright^{\epsilon_1} a_1) \triangleright^{\epsilon_2} a_2) \cdots \triangleright^{e_n} a_n = x \quad \forall x \in Q$$

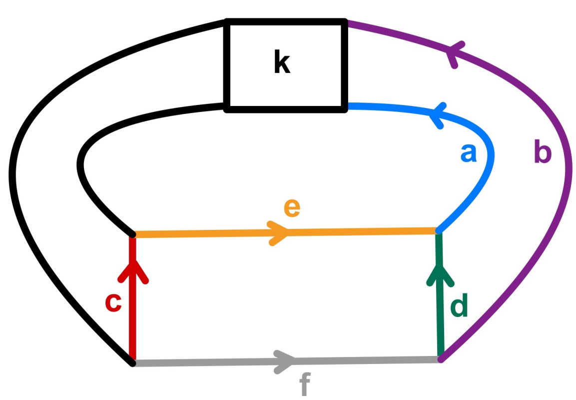


FIGURE 5. A complete/double-handcuff graph with a block of k twists.

where $\epsilon_i = 1$ when a_i points out of v and $\epsilon_i = -1$ otherwise. These relationships are illustrated in Figure 6.

A subset $C \subset Q$ is called a *component* of Q if it is closed under $\triangleright$ and $\triangleright^{-1}$ such that for all $x, y \in C$, $x \triangleright y \in C$ and $x \triangleright^{-1} y \in C$. Note that multiple generators may be in the same component.

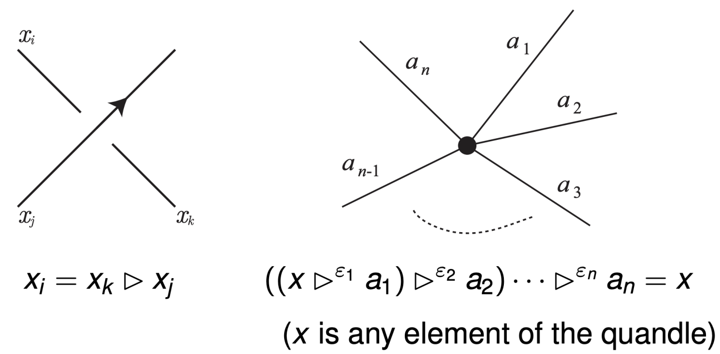


FIGURE 6. Relations of the quandle at crossings and vertices.

Using these relationships, we can prove three lemmas that we will use throughout the remainder of the paper.

LEMMA 1.1. *For all $x, y, z \in Q$,*

$$x^{\overline{yz}} = x^{\overline{z}\,\overline{y}}$$

PROOF. By the second axiom of quandles,

$$\begin{aligned} x^{(\overline{yz})(yz)} &= x \\ x^{(\overline{yz})yz\overline{z}} &= x^{\overline{z}} \\ x^{(\overline{yz})y} &= x^{\overline{z}} \\ x^{(\overline{yz})y\overline{y}} &= x^{\overline{z}\,\overline{y}} \\ x^{\overline{yz}} &= x^{\overline{z}\,\overline{y}} \end{aligned}$$

□

LEMMA 1.2. *For all $x, y, z \in Q$,*

$$x^{(y^z)} = x^{\overline{z}yz}$$

PROOF. The second axiom of quandles gives us that $(x^{\overline{z}})^z = x$, so $x^{(y^z)} = ((x^{\overline{z}})^z)^{(y^z)}$. By the third axiom of quandles,

$$((x^{\overline{z}})^z)^{(y^z)} = ((x^{\overline{z}})^y)^z = x^{\overline{z}yz}$$

Therefore, $x^{(y^z)} = x^{\overline{z}yz}$ □

LEMMA 1.3. *For all* $x, y, z \in Q$,

$$x^{(\overline{y^z})} = x^{\overline{z}\,\overline{y}\,z}$$

PROOF. By Lemma 1.2, $x^{(\overline{y^z})} = x^{\overline{\overline{z}\,y\,z}}$. Moreover, by Lemma 1.1, $x^{\overline{\overline{z}\,y\,z}} = x^{\overline{z}\,\overline{y\overline{z}}} = x^{\overline{z}\,\overline{y}\,z}$. Thus, $x^{(\overline{y^z})} = x^{\overline{z}\,\overline{y}\,z}$. □

Although the fundamental quandle is a spatial graph invariant, it is infinitely large for all spatial graphs except the *unknot* and the *Hopf Link*, and therefore fundamental quandles are almost as difficult to identify as spatial graphs themselves.

To partially address this, we can define a quotient of the fundamental quandle, called an n-quandle, which appends an additional axiom to the three quandle axioms:

4. $\forall x, y \in Q$, $x^{(y^n)} = x$ for some $n \in \mathbb{N}$

A more generalized form of the n-quandle is called an N-quandle. Suppose that a quandle, Q, has k distinct components, $C_1, C_2, \ldots, C_k$. Then, the N-quandle of G has the property

4. $\forall x \in Q$, $x^{(s_i)^{n_i}} = x$ for all generators $s_i \in C_i$, where $1 \leq i \leq k$ and $N = \{n_1, n_2, \ldots n_k\}$ is a set of natural numbers.

Depending on the choice of N, the N-quandle of a given spatial graph may be finite, which allows for graphs to be distinguished from each other.

CHAPTER 2

Proof of Finite N-Quandle of Twisted Double Handcuff and Complete Graph

2.1. Introduction to Twisted Double Handcuff and Complete Graph

In this section, we will prove that the Twisted Double Handcuff Graph and Twisted Complete Graph, which we previously showed can be generalized to a spatial graph G of the same form in Figure 5, have a finite N-Quandle, denoted $Q_N(G)$. Observe that $Q_N(G)$ has six generators, a, b, c, d, e, f. We will show that the $(2, 2, n_1, n_2, 2, 2)$-quandle of G is finite for all $n_1, n_2 \in \mathbb{N}$. In particular,

$$|Q_N(G)| = 2kn_1 + 2kn_2 + 4kn_1n_2$$

First, we will give a presentation for $Q_N(G)$. Our choice of $N = (2, 2, n_1, n_2, 2, 2)$ gives us six primary relations, namely,

- $x^{a^2} = x \ \forall x \in Q_N(G)$
- $x^{b^2} = x \ \forall x \in Q_N(G)$
- $x^{c^{n_1}} = x \ \forall x \in Q_N(G)$
- $x^{d^{n_2}} = x \ \forall x \in Q_N(G)$
- $x^{e^2} = x \ \forall x \in Q_N(G)$
- $x^{f^2} = x \ \forall x \in Q_N(G)$

Notice that this implies that $x^w = x^{\overline{w}}$ for all $w \in \{a, b, e, f\}$. Next, we will utilize the relations given by Figure 6 to identify any secondary relations present in the graph. We can use the crossing relations to determine the N-quandle elements assigned to the two left-most arcs of G in terms of a, b and k.

Lemma 2.1. *Let x denote the outer arc on the left side of G and let y denote the inner arc. When k is even,*

$$x = b^{a(ba)^{\frac{k-2}{2}}}$$
$$y = a^{(ba)^{\frac{k}{2}}}$$

When k is odd,

$$x = a^{(ba)^{\frac{k-1}{2}}}$$
$$y = b^{a(ba)^{\frac{k-1}{2}}}$$

Proof. We will prove the result using induction. The process described below is demonstrated graphically in Figure 1.

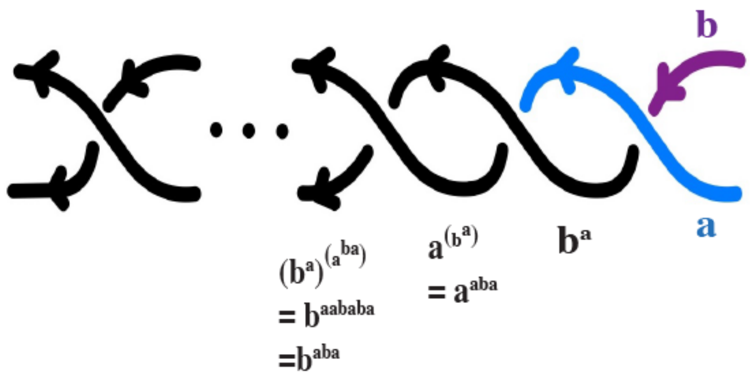


FIGURE 1. Quandle relations on twists.

Base case: Using the crossings relations, when $k = 1$, $x = a$ and $y = b^a$, and when $k = 2$, $x = b^a$ and $y = a^{(b^a)} = a^{aba} = a^{ba}$ by Lemma 1.2. Thus, the base cases hold.

Inductive step: Now suppose that for an odd number of twists k,

$$x_k = a^{(ba)^{\frac{k-1}{2}}}$$

$$y_k = b^{a(ba)^{\frac{k-1}{2}}}$$

Then for $k+1$ twists,

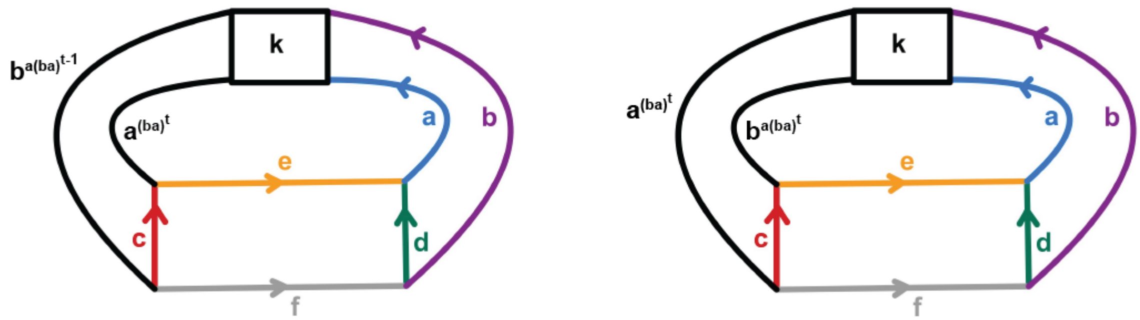
$$x_{k+1} = y_k = b^{a(ba)^{\frac{k-1}{2}}} = b^{a(ba)^{\frac{(k+1)-2}{2}}}$$

$$\begin{aligned} y_{k+1} &= x_k^{(y_k)} \\ &= (a^{(ba)^{\frac{k-1}{2}}})^{b^{a(ba)^{\frac{k-1}{2}}}} \\ &= a^{(ba)^{\frac{k-1}{2}}\overline{a(ba)^{\frac{k-1}{2}}}ba(ba)^{\frac{k-1}{2}}} \\ &= a^{(ba)^{\frac{k-1}{2}}(ab)^{\frac{k-1}{2}}aba(ba)^{\frac{k-1}{2}}} \\ &= a^{(ba)^{\frac{(k+1)}{2}}} \end{aligned}$$

Therefore, the result holds by induction for the odd to even case. Using the same logic, we get that result also holds for the even to odd case. □

The newly labelled graphs are given in Figure 2a when k is even and Figure 2b when k is odd.

Now, we can apply the vertex relations at the four vertices to get four secondary relationships that must hold for all $x \in Q_N(G)$:



(A) Labelling on G when k is even, where $t = \frac{k}{2}$.

(B) Labelling on G when k is odd, where $t = \frac{k-1}{2}$.

FIGURE 2

- $x^{ae\bar{d}} = x \ \forall x \in Q_N(G)$
- $x^{bd\bar{f}} = x \ \forall x \in Q_N(G)$
- $x^{e(ab)^k a\bar{c}} = x \ \forall x \in Q_N(G)$
- $x^{c(ab)^{k-1}af} = x \ \forall x \in Q_N(G)$

The last two relations hold for both the even and odd case since

$$x^{e\overline{a^{(ba)^{\frac{k}{2}}}}\,\bar{c}} = x^{e(ab)^{\frac{k}{2}}a(ba)^{\frac{k}{2}}\bar{c}} = x^{e(ab)^k a\bar{c}} = x^{e(ab)^{\frac{k-1}{2}}(ab)a(ba)^{\frac{k-1}{2}}\bar{c}} = x^{e\overline{b^{a(ba)^{\frac{k-1}{2}}}}\,\bar{c}}$$

and

$$x^{c\overline{b^{a(ba)^{\frac{k}{2}-1}}}\,f} = x^{c\overline{a(ba)^{\frac{k}{2}-1}}\,ba(ba)^{\frac{k}{2}-1}f} = x^{c(ab)^{\frac{k}{2}}a(ba)^{\frac{k}{2}-1}f} = x^{c(ab)^{k-1}af}$$
$$= x^{c(ab)^{\frac{k-1}{2}}a(ba)^{\frac{k-1}{2}}f} = x^{c\overline{a^{(ba)^{\frac{k-1}{2}}}}\,f}$$

These relations yield the following quandle presentation for $Q_N(G)$:

$$Q_N(G) = \langle a, b, c, d, e, f \mid x^{a^2} = x, x^{b^2} = x, x^{c^{n_1}} = x, x^{d^{n_2}} = x, x^{e^2} = x, x^{f^2} = x$$
$$x^{ae\bar{d}} = x, x^{bd\bar{f}} = x, x^{e(ab)^k a\bar{c}} = x, x^{c(ab)^{k-1}af} = x\rangle \quad (1)$$

To show that $Q_N(G)$ is finite, we will utilize a method developed by Winker, which relies on the construction of a Cayley graph of $Q_N(G)$ [**1**] and is described in detail in [**2**]. The Cayley graph associated with an N-quandle is finite if and only if the N-quandle is finite, where the vertices of the graph represent the unique elements of the quandle, and the edges represent relations between these elements. In other words, if two vertices, x and y of the Cayley graph are connected by an edge labelled as a, it implies that $x^a = y$.

In the following sections, we will prove that the Cayley graph associated with $Q_N(G)$ is finite, with $2kn_1+2kn_2+4kn_1n_2$ vertices. More specifically, $Q_N(G)$ consists of six components, each generated by one of the six generators of $Q_N(G)$. The four components associated with a, b, e and f have size kn_1n_2. The component associated with c has $2kn_2$ vertices and the component associated with d has $2kn_1$ vertices.

2.2. Important Relationships

Before proving the size of the N-quandle, we define several important lemmas that will prove useful in later sections.

LEMMA 2.2. *For any element $x \in Q$,*

- $x^{bd} = x^{\bar{d}\,b} = x^f$
- $x^{db} = x^{b\bar{d}}$
- $x^{ad} = x^{\bar{d}\,a} = x^e$
- $x^{da} = x^{a\bar{d}}$

PROOF. Given that $x^{ae\bar{d}} = x$,

$$x^{ae} = x^d$$
$$(x^a)^{(ae)} = (x^a)^d$$
$$x^e = x^{ad}$$

Thus, we obtain that, $x^{\bar{e}} = x^{\overline{ad}}$ so $x^e = x^{\bar{d}\,a}$. Therefore,

$$x^{ad} = x^{\bar{d}\,a}$$
$$x^{adad} = x$$
$$x^{dadad} = x^d$$
$$x^{dadad\bar{d}} = x^{d\bar{d}}$$
$$x^{dada} = x$$
$$x^{da} = x^{a\bar{d}}$$

Similarly, given that $x^{bdf} = x$, $x^{bd} = f$ and $x^{\overline{bd}} = x^{\bar{d}\,b} = x^{\bar{f}} = x^f$. Then, $x^{bdbd} = x \Rightarrow x^{dbdb} = x \Rightarrow x^{db} = x^{b\bar{d}}$. $\square$

Notice that this lemma implies that $x^a = x^{dad} = x^{\bar{d}\,a\bar{d}}$ and $x^b = x^{dbd} = x^{\bar{d}\,b\bar{d}}$. Therefore,

LEMMA 2.3. *For any element $x \in Q$,*

$$x^{ab} = x^{dab\bar{d}} = x^{\bar{d}\,abd} = x^{ef}$$

PROOF. By Lemma 2.2,

$$x^{ab} = x^{(dad)(\bar{d}\,b\bar{d})} = x^{dab\bar{d}} = x^{ef}$$

Similarly,

$$x^{ab} = x^{(\bar{d}\,a\bar{d})(dbd)} = x^{\bar{d}\,abd} = x^{ef}$$

□

Lemma 2.4. *For any element $x \in Q$,*

$$x^{(ab)^k} = x^{\bar{c}\bar{d}} = x^{(ef)^k}$$

Proof. Given the initial relationship $x^{c(ab)^{k-1}af} = x$, we see

$$\begin{aligned} x &= x^{c(ab)^{k-1}af} \\ &= x^{c(ab)^k(ba)af} \\ &= x^{c(ab)^k bf} \\ &= x^{c(ab)^k b(bd)} \quad \text{(Lemma 2.2)} \\ &= x^{c(ab)^k d} \end{aligned}$$

Therefore, $x^{(ab)^k} = x^{\bar{c}\bar{d}} = x^{(ef)^k}$ by Lemma 2.3. □

Lemma 2.5. *For all $x \in Q$,*

$$x^{cd\bar{c}\bar{d}} = x$$

Proof. By Lemma 2.4,

$$\begin{aligned} x &= x^{c(ab)^k d} \\ &= x^{c(dab\bar{d})^k d} \quad \text{(Lemma 2.3)} \\ &= x^{cd(ab)^k\bar{d}d} \\ &= x^{cd(ab)^k} \\ &= x^{cd\bar{c}\bar{d}} \quad \text{(Lemma 2.4)} \end{aligned}$$

□

Lemma 2.6. *For $w \in \{a, b, e, f\}$ and for all $x \in \mathbb{Q}$,*

$$x^{c^i d^j w} = x^{w\bar{c}^i\bar{d}^j}$$

Proof. We will prove the conclusion for a, b, e and f separately.

First consider the case when $w = a$. Observe that

$$x^{c^i d^j a} = x^{c^i d^{j-1}(da)} = x^{c^i d^{j-1}a\bar{d}}$$

by Lemma 2.2. We can repeat this process j times to obtain $x^{c^i d^j a} = x^{c^i a\bar{d}^j}$. Now observe that the relationship $x^{e(ab)^k a\bar{c}} = x$ yields the following equations:

$$x^c = x^{e(ab)^k a}$$

$$x^{\bar{c}} = x^{a(ba)^k e}$$

Therefore,

$$\begin{aligned}
x^{c^i d^j a} &= x^{c^i a \bar{d}^j} \\
&= x^{[e(ab)^k a]^i a \bar{d}^j} \\
&= x^{[ea(ba)^k]^i a \bar{d}^j} \\
&= x^{[ead(ba)^k \bar{d}]^i a \bar{d}^j} \quad \text{(Lemma 2.3)} \\
&= x^{[ee(ba)^k \bar{d}]^i a \bar{d}^j} \quad \text{(Lemma 2.2)} \\
&= x^{[(ba)^k \bar{d}]^i a \bar{d}^j} \\
&= x^{[aa(ba)^k \bar{d}]^i a \bar{d}^j} \\
&= x^{a[a(ba)^k \bar{d} a]^i \bar{d}^j} \\
&= x^{a[a(ba)^k e]^i \bar{d}^j} \quad \text{(Lemma 2.2)} \\
&= x^{a \bar{c}^i \bar{d}^j}
\end{aligned}$$

Next, consider the case when $w = b$. Notice that

$$x^{c^i d^j b} = x^{c^i d^{j-1}(db)} = x^{c^i d^{j-1} b \bar{d}}$$

by Lemma 2.2. We can repeat this process j times to obtain $x^{c^i d^j b} = x^{c^i b \bar{d}^j}$. Also, observe that the relationship $x^{c(ab)^{k-1} af} = x$ yields

$$x^c = x^{fb(ba)^k}$$

$$x^{\bar{c}} = x^{(ab)^k bf}$$

Then,

$$\begin{aligned}
x^{c^i d^j b} &= x^{c^i b \bar{d}^j} \\
&= x^{[fb(ba)^k]^i b \bar{d}^j} \\
&= x^{[bdb(ba)^k]^i b \bar{d}^j} \\
&= x^{b[db(ba)^k b]^i \bar{d}^j} \\
&= x^{b[d(ab)^k]^i \bar{d}^j} \\
&= x^{b[d\bar{d}(ab)^k d]^i \bar{d}^j} \\
&= x^{b[(ab)^k bf]^i \bar{d}^j} \quad \text{(Lemma 2.2)} \\
&= x^{b \bar{c}^i \bar{d}^j}
\end{aligned}$$

Next, consider the case when $w = e$. Notice that

$$x^{c^i d^j e} = x^{c^i d^{j-1}(de)} = x^{c^i d^{j-1} e \bar{d}}$$

by Lemma 2.2. We can repeat this process j times to obtain $x^{c^i d^j e} = x^{c^i e \bar{d}^j}$. Then,

$$\begin{aligned} x^{c^i d^j e} &= x^{c^i e \bar{d}^j} \\ &= x^{[e(ab)^k a]^i e \bar{d}^j} \\ &= x^{[ea(ba)^k]^i e \bar{d}^j} \\ &= x^{e[a(ba)^k e]^i \bar{d}^j} \\ &= x^{e \bar{c}^i \bar{d}^j} \end{aligned}$$

Finally, consider the case when $w = f$. Observe that

$$x^{c^i d^j f} = x^{c^i d^{j-1}(df)} = x^{c^i d^{j-1} f \bar{d}}$$

by Lemma 2.2. We can repeat this process j times to obtain $x^{c^i d^j f} = x^{c^i f \bar{d}^j}$. Then,

$$\begin{aligned} x^{c^i d^j f} &= x^{c^i f \bar{d}^j} \\ &= x^{[fb(ba)^k]^i f \bar{d}^j} \\ &= x^{f[b(ba)^k f]^i \bar{d}^j} \\ &= x^{f[(ab)^k bf]^i \bar{d}^j} \\ &= x^{f \bar{c}^i \bar{d}^j} \end{aligned}$$

Therefore in all cases, $x^{c^i d^j w} = x^{w \bar{c}^i \bar{d}^j}$.

□

2.3. The a and b Components

In this section we will first show that the size of the a-component in the quandle Q has size kn_1n_2. To illustrate the structure of the component, Figure 3 presents a Cayley graph representation of the a-component of the quandle when $n_1 = 3$, $n_2 = 2$ and $k = 4$. The Cayley graph is distinctly divided into k layers, each with $n_1 n_k$ points, such that there are a total of kn_1n_2 points, as desired. Each of these layers has a torus-like structure, a diagram of which is presented in Figure 4. There are n_1 vertical cross-sections of the torus, representing the c-edges, and n_2 horizontal cross sections representing the d-edges. Each torus can also be represented by a square using its identification space, as displayed in Figure 4. The set of tori are then connected by alternating pairs of a/e and b/f edges.

The torus structure of each level is given by Lemma 2.5, which tells us that $x^{cd\bar{c}\bar{d}} = x$. This relationship gives us the c and d square patterns in Figure 4. The alternating pairs of a/e and b/f edges between the tori can be explained by the initial relationships $x^{ae\bar{d}} = x$ and $x^{bdf} = x$. Since $x^{bdf} = x$, we can start at any point in a torus, move up to the following torus along a b-edge, travel horizontally along a d-edge of the torus, and return to the original point via an f-edge. The same occurs with a/e edges. Thus, the behavior of the Cayley graph at and between tori is accounted for.

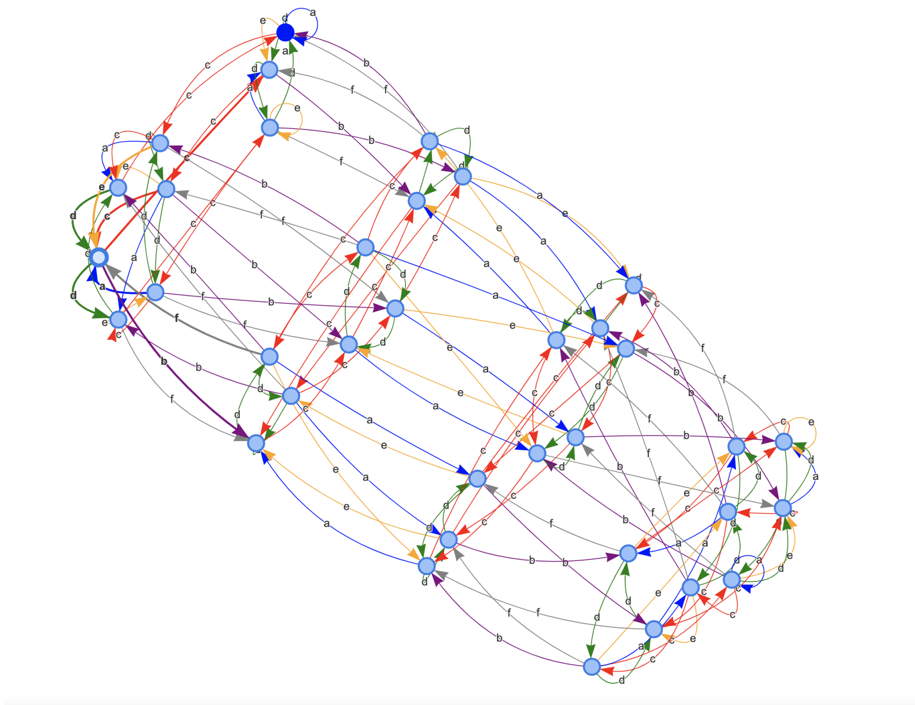


FIGURE 3. Cayley graph of the a-component when $n_1 = n_2 = 3$ and $k = 4$.

Therefore, the remainder of this section will explain the behavior at the bottom and top most tori. At the bottom-most torus, which contains the a-element, b and f-edges serve as connectors to the following torus. By Theorem 2.1 and Theorem 2.2, the a and e-edges on the bottom torus connect to a different point within the same torus.

THEOREM 2.1. *For all $i, j \geq 0$,*

$$a^{c^i d^j a} = a^{\bar{c}^i \bar{d}^j}$$

This theorem follows immediately from Lemma 2.6.

THEOREM 2.2. *For all $i, j \geq 0$,*

$$a^{c^i d^j e} = a^{\bar{c}^i \bar{d}^{j-1}}$$

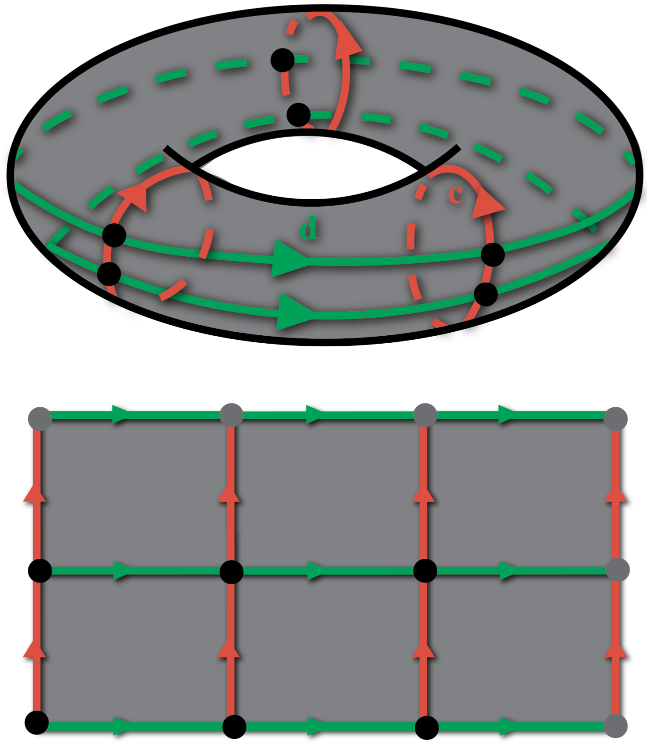


FIGURE 4. Each a-component contains k tori as depicted above. The torus above would be found in a graph where $n_1 = 3$ and $n_2 = 2$. Each torus has n_1 vertical cross sections (represented in red) and n_2 horizontal cross section (represented in green). The intersections of the cross sections are the points of the Cayley graph. The torus can also be represented using its square identification space, where opposite points are identified (e.g. the bottom left point is identified with the top left, top right, and bottom right points.)

PROOF. Lemma 2.6 gives us that $a^{c^i d^j e} = a^{e\bar{c}^i\bar{d}^j}$. Therefore,

$$\begin{aligned} a^{c^i d^j e} &= a^{e\bar{c}^i\bar{d}^j} \\ &= a^{ad\bar{c}^i\bar{d}^j} && \text{(Lemma 2.2)} \\ &= a^{a\bar{c}^i d\bar{d}^j} && \text{(Lemma 2.5)} \\ &= a^{\bar{c}^i\bar{d}^{j-1}} \end{aligned}$$

□

The top-most torus varies depending on whether k is even or odd. When k is even, b and f-edges again serve as connectors between the torus and the one below it. Theorem 2.3 and Theorem 2.4 again indicate that the a and e-edges connect to other points on the torus. When k is odd, a and e-edges now serve as connectors between the torus and the one below it. Theorem 2.5 and Theorem 2.6 demonstrate that the b and f-edges connect to other points on the torus.

THEOREM 2.3. *When k is even,*

$$a^{(ba)^{\frac{k}{2}-1}bc^id^ja} = a^{(ba)^{\frac{k}{2}-1}b\bar{c}^{i-1}\bar{d}^{j-1}}$$

PROOF. By Lemma 2.6,

$$a^{(ba)^{\frac{k}{2}-1}bc^id^ja} = a^{(ba)^{\frac{k}{2}-1}ba\bar{c}^i\bar{d}^j} = a^{(ba)^{\frac{k}{2}}\bar{c}^i\bar{d}^j}$$

Therefore,

$$\begin{aligned}
a^{(ba)^{\frac{k}{2}-1}bc^id^ja} &= a^{(ba)^{\frac{k}{2}}\bar{c}^i\bar{d}^j} \\
&= a^{(ba)^{\frac{k}{2}}(\bar{c}\bar{d})\bar{c}^{i-1}\bar{d}^{j-1}} && \text{(Lemma 2.5)} \\
&= a^{(ba)^{\frac{k}{2}}(ab)^k\bar{c}^{i-1}\bar{d}^{j-1}} && \text{(Lemma 2.4)} \\
&= a^{(ab)^{\frac{k}{2}}\bar{c}^{i-1}\bar{d}^{j-1}} \\
&= a^{(ba)^{\frac{k}{2}-1}b\bar{c}^{i-1}\bar{d}^{j-1}}
\end{aligned}$$

□

THEOREM 2.4. *When k is even,*

$$a^{(ba)^{\frac{k}{2}-1}bc^id^je} = a^{(ba)^{\frac{k}{2}-1}b\bar{c}^{i-1}\bar{d}^{j-2}}$$

PROOF. By Lemma 2.6,

$$a^{(ba)^{\frac{k}{2}-1}bc^id^je} = a^{(ba)^{\frac{k}{2}-1}ba\bar{c}^i\bar{d}^{j-1}} = a^{(ba)^{\frac{k}{2}}\bar{c}^i\bar{d}^{j-1}}$$

Also using the same argument as in Theorem 2.3, we can see that

$$a^{(ba)^{\frac{k}{2}}\bar{c}^i\bar{d}^{j-1}} = a^{(ba)^{\frac{k}{2}-1}b\bar{c}^{i-1}\bar{d}^{j-2}}$$

Therefore,

$$a^{(ba)^{\frac{k}{2}-1}bc^id^je} = a^{(ba)^{\frac{k}{2}-1}b\bar{c}^{i-1}\bar{d}^{j-2}}$$

□

THEOREM 2.5. *When k is odd,*

$$a^{(ba)^{\frac{k-1}{2}}c^id^jb} = a^{(ba)^{\frac{k-1}{2}}\bar{c}^{i+1}\bar{d}^{j+1}}$$

PROOF. By Lemma 2.6,

$$a^{(ba)^{\frac{k-1}{2}} c^i d^j b} = a^{(ba)^{\frac{k-1}{2}} b\overline{c}^{\,i}\overline{d}^{\,j}}$$

Therefore,

$$\begin{aligned}
a^{(ba)^{\frac{k-1}{2}} c^i d^j b} &= a^{(ba)^{\frac{k-1}{2}} b\overline{c}^{\,i}\overline{d}^{\,j}} \\
&= a^{(ab)^{\frac{k+1}{2}} \overline{c}^{\,i}\overline{d}^{\,j}} \\
&= a^{(ab)^{\frac{k+1}{2}} (dc)\overline{c}^{\,i+1}\overline{d}^{\,j+1}} \\
&= a^{(ab)^{\frac{k+1}{2}} (ba)^k\overline{c}^{\,i+1}\overline{d}^{\,j+1}} \\
&= a^{(ba)^{k-\frac{k+1}{2}} \overline{c}^{\,i+1}\overline{d}^{\,j+1}} \\
&= a^{(ba)^{\frac{k-1}{2}} \overline{c}^{\,i+1}\overline{d}^{\,j+1}}
\end{aligned}$$

□

THEOREM 2.6. *When k is odd,*

$$a^{(ba)^{\frac{k-1}{2}} c^i d^j f} = a^{(ba)^{\frac{k-1}{2}} \overline{c}^{\,i+1}\overline{d}^{\,j}}$$

PROOF. By Lemma 2.6,

$$a^{(ba)^{\frac{k-1}{2}} c^i d^j f} = a^{(ba)^{\frac{k-1}{2}} f\overline{c}^{\,i}\overline{d}^{\,j}}$$

Therefore,

$$\begin{aligned}
a^{(ba)^{\frac{k-1}{2}} c^i d^j f} &= a^{(ba)^{\frac{k-1}{2}} f\overline{c}^{\,i}\overline{d}^{\,j}} \\
&= a^{(ba)^{\frac{k-1}{2}} bd\overline{c}^{\,i}\overline{d}^{\,j}} \\
&= a^{(ba)^{\frac{k-1}{2}} b\overline{c}^{\,i}\overline{d}^{\,j-1}}
\end{aligned}$$

Therefore, using the same logic as in Theorem 2.5,

$$a^{(ba)^{\frac{k-1}{2}} c^i d^j f} = a^{(ba)^{\frac{k-1}{2}} b\overline{c}^{\,i}\overline{d}^{\,j-1}} = a^{(ba)^{\frac{k-1}{2}} \overline{c}^{\,i+1}\overline{d}^{\,j}}$$

□

Therefore, every element in the a-component belongs to an $n_1 \times n_2$ sized torus and there are k such tori. Furthermore, it is easy to verify that all secondary relations are satisfied. Therefore, the a-component has kn_1n_2 elements.

To verify that the b-component also has kn_1n_2 elements, we can easily check that there is an automorphism between the Graph A (left) and Graph B (right) in Figure 5, where a maps to b, d maps to $\overline{c}$ and e maps to f. Therefore, the b-component must have the same size as the a-component.

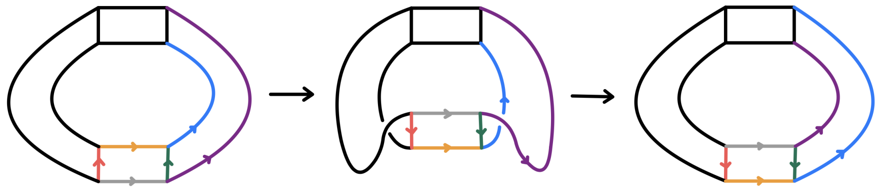


Figure 5. On the basis of isotopic moves, we can transform Graph A (left) into Graph B (right).

2.4. The c and d Components

Next, we will show that the d-component in $Q_N(G)$ has $2kn_1$ elements. Again, we illustrate the component using a Cayley graph in Figure 6. The Cayley graph of the d-component consists of $2k$ n_1-cycles of c, connected by alternating a/b and e/f edges. Every vertex has a d-loop at it.

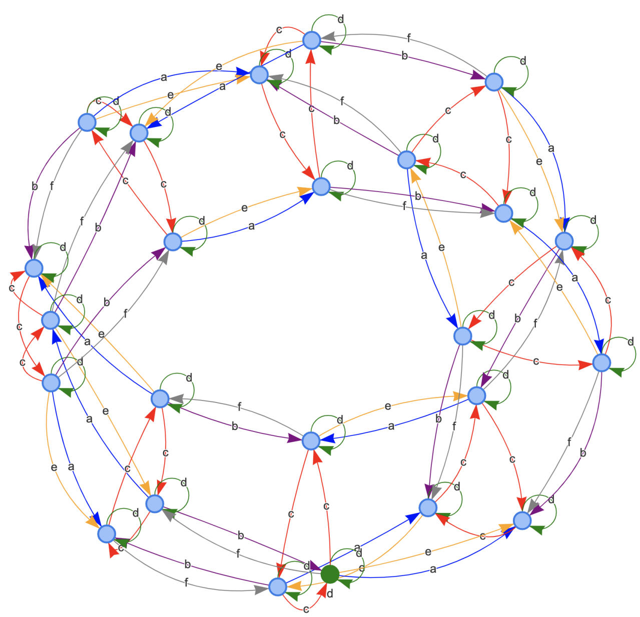


Figure 6. Cayley graph of the d-component of $Q_N(G)$ when $n_1 = 3$, $n_2 = 2$ and $k = 4$.

First, we will prove the existence of the d-loops at every vertex.

Theorem 2.7. *For $i \geq 0$,*

- $d^{(ab)^i d} = d^{(ab)^i}$
- $d^{(ab)^i ad} = d^{(ab)^i a}$

PROOF. We will prove this result using induction. For the base case, note that by Lemma 2.2, $d^a = d^{dad} = d^{ad}$, so

$$d^{abd} = d^{(ad)bd} = d^{ad(\bar{d}b)} = d^{ab} \tag{2}$$

Furthermore, using our result from (2),

$$d^{abad} = d^{(abd)ad} = d^{abd(\bar{d}a)} = d^{aba} \tag{3}$$

Therefore, the base cases hold.

Now, assume that $d^{(ab)^i d} = d^{(ab)^i}$ and $d^{(ab)^i ad} = d^{(ab)^i a}$. Then,

$$\begin{aligned} d^{(ab)^{i+1}d} &= d^{(ab)^i abd} \\ &= d^{(ab)^i a\bar{d}b} \\ &= d^{(ab)^i ab} \quad \text{(Inductive Hypothesis)} \\ &= d^{(ab)^{i+1}} \end{aligned} \tag{4}$$

Moreover,

$$\begin{aligned} d^{(ab)^{i+1}ad} &= d^{(ab)^{i+1}\bar{d}a} \\ &= d^{(ab)^{i+1}a} \end{aligned}$$

using our result from (4). Therefore, $d^{(ab)^i d} = d^{(ab)^i}$ and $d^{(ab)^i ad} = d^{(ab)^i a}$ by mathematical induction.

□

Next, Theorem 2.8 and Theorem 2.9 justify the cyclical a/b and e/f pattern, and show that it is closed.

THEOREM 2.8. *For all $i \geq 0$,*

- $d^{(ab)^i a} = d^{(ab)^i e}$
- $d^{(ab)^i b} = d^{(ab)^i f}$
- $d^{(ab)^i} = d^{(ab)^i ae}$
- $d^{(ab)^{i+1}} = d^{(ab)^i af}$

PROOF. By Theorem 2.7, $d^{(ab)^i a} = d^{(ab)^i ad} = d^{(ab)^i e}$ using the relationship $x^{ae\bar{d}} = x$. Moreover, $d^{(ab)^i} = d^{(ab)^i d} = d^{(ab)^i ae}$.

Similarly, using the relationship $x^{bdf} = x$, we see $d^{(ab)^i b} = d^{(ab)^i \bar{d}b} = d^{(ab)^i f}$. Moreover, $d^{(ab)^{i+1}} = d^{(ab)^i abd} = d^{(ab)^i af}$.

□

THEOREM 2.9.

$$d^{(ab)^{n_1 k}} = d$$

PROOF. First, we will show that for $i \geq 0$, $d^{\bar{c}^i} = d^{(ab)^{ik}}$. The case for $i = 1$ can be inferred directly from Theorem 2.10. Now suppose that for $i = j$, $d^{\bar{c}^j} = d^{(ab)^{jk}}$. Then,

$$d^{\bar{c}^{j+1}} = d^{\bar{c}^j \bar{c}} = d^{(ab)^{jk}\bar{c}}$$

By the base case,

$$d^{(ab)^{jk}\bar{c}} = d^{(ab)^{jk}(ab)^k} = d^{(ab)^{jk+k}} = d^{(ab)^{(j+1)k}}$$

Therefore, the result holds by mathematical induction. From here, it is easy to see that

$$d^{(ab)^{n_1 k}} = d^{\bar{c}^{n_1}} = d$$

□

The n_1-cycle structure for the c edges is justified by the fact that c has order n_1 such that $x^{c^{n_1}} = x$ for all $x \in Q_N(G)$. Theorem 2.10 explains how these c cycles are embeded into the graph's structure. In particular, $\bar{c}$ connects any vertex with the vertex $(ab)^k$ away from it.

THEOREM 2.10. *For all $i \geq 0$,*

$$d^{(ab)^i \bar{c}} = d^{(ab)^{i+k}}$$

PROOF. Given that $x^{(ab)^k} = x^{\bar{c}\bar{d}}$ by Lemma 2.4,

$$d^{(ab)^k} = d^{\bar{c}\bar{d}}$$

$$d^{(ab)^k d} = d^{\bar{c}}$$

$$d^{(ab)^k} = d^{\bar{c}} \tag{5}$$

by Theorem 2.7. Thus,

$$\begin{aligned} d^{(ab)^i \bar{c}} &= d^{(ab)^i (ab)^k} \\ &= d^{(ab)^{i+k}} \end{aligned}$$

□

Thus, every element in the d-component can be written as $d^{(ab)^i}$ or $d^{(ab)^i a}$ for some $i \in \mathbb{N}$. Moreover, $d^{(ab)^{n_1 k}} = d$ so there can only be $2n_1k$ such possible elements. Furthermore, it is easy to verify that all secondary relations are satisfied. Therefore, the d-component has $2kn_1$ elements.

To verify that the c-component has $2kn_2$ elements, Figure 7 demonstrates that we can rotate G such that the c edge assumes the position of the d edge in a graph G' that is isotopic to G. Therefore, using the same procedure as above on G', we can show that the c component must have $2kn_2$ elements.

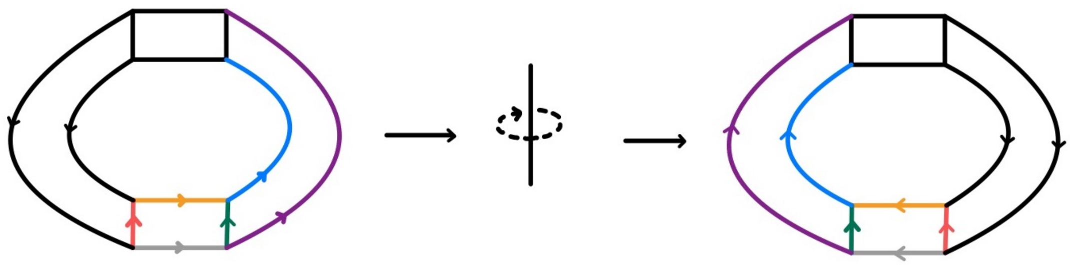


FIGURE 7. By rotating G 180 degrees along the y-axis, we obtain an isotopic graph.

2.5. The e and f Components

In this section we will show that the size of the e-component in the quandles has size kn_1. Figure 8 presents a Cayley graph of the e-component, which has a structure very similar to the a-component, but with different behavior on the top and bottom torus.

Theorem 2.11 and Theorem 2.12 explain the behavior of the N-quandle at the bottom-most torus containing the e element.

THEOREM 2.11. *For all $i, j \geq 0$,*

$$e^{c^i d^j a} = e^{\bar{c}^i \bar{d}^{j+1}}$$

PROOF. Lemma 2.6 gives us that $e^{c^i d^j a} = e^{a \bar{c}^i \bar{d}^j}$. Therefore,

$$\begin{aligned} e^{c^i d^j a} &= e^{a \bar{c}^i \bar{d}^j} \\ &= e^{e \bar{d} \bar{c}^i \bar{d}^j} \quad \text{(Lemma 2.2)} \\ &= e^{e \bar{c}^i \bar{d}\, \bar{d}^j} \quad \text{(Lemma 2.5)} \\ &= e^{\bar{c}^i \bar{d}^{j+1}} \end{aligned}$$

□

THEOREM 2.12. *For all $i, j \geq 0$,*

$$e^{c^i d^j e} = e^{\bar{c}^i \bar{d}^j}$$

This theorem follows immediately from Lemma 2.6.

Theorem 2.13, Theorem 2.14, Theorem 2.15 and Theorem 2.16 explain the behavior of the N-quandle at the top-most torus.

THEOREM 2.13. *When k is even,*

$$e^{(fe)^{\frac{k}{2}-1}fc^id^ja} = e^{(fe)^{\frac{k}{2}-1}f\overline{c}^{i-1}\overline{d}^j}$$

PROOF. By Lemma 2.6,

$$e^{(fe)^{\frac{k}{2}-1}fc^id^ja} = e^{(fe)^{\frac{k}{2}-1}fa\overline{c}^i\overline{d}^j} = e^{(fe)^{\frac{k}{2}-1}f(e\overline{d})\overline{c}^i\overline{d}^j} = e^{(fe)^{\frac{k}{2}}\overline{c}^i\overline{d}^{j+1}}$$

Therefore,

$$\begin{aligned}
e^{(fe)^{\frac{k}{2}-1}fc^id^ja} &= e^{(fe)^{\frac{k}{2}}\overline{c}^i\overline{d}^{j+1}} \\
&= e^{(fe)^{\frac{k}{2}}(\overline{c}\overline{d})\overline{c}^{i-1}\overline{d}^j} \\
&= e^{(fe)^{\frac{k}{2}}(ef)^k\overline{c}^{i-1}\overline{d}^j} \\
&= e^{(ef)^{\frac{k}{2}}\overline{c}^{i-1}\overline{d}^j} \\
&= e^{(fe)^{\frac{k}{2}-1}f\overline{c}^{i-1}\overline{d}^j}
\end{aligned}$$

□

THEOREM 2.14. *When k is even,*

$$e^{(fe)^{\frac{k}{2}-1}fc^id^je} = e^{(fe)^{\frac{k}{2}-1}f\overline{c}^{i-1}\overline{d}^{j-1}}$$

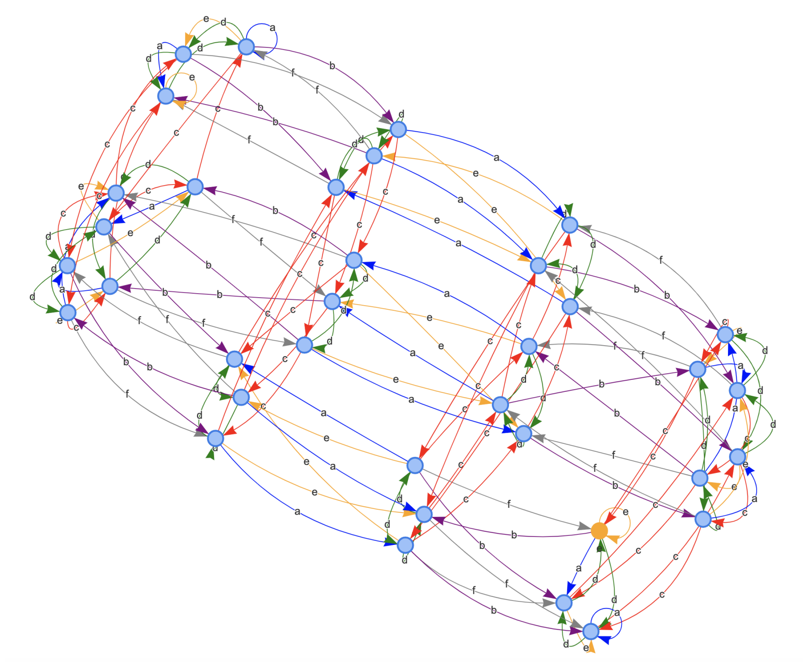


FIGURE 8. Cayley graph of the e-component when $n_1 = n_2 = 3$ and $k = 4$.

PROOF. By Lemma 2.6,

$$e^{(fe)^{\frac{k}{2}-1}fc^id^je} = e^{(fe)^{\frac{k}{2}-1}fe\overline{c}^i\overline{d}^j} = e^{(fe)^{\frac{k}{2}}\overline{c}^i\overline{d}^j}$$

Therefore,

$$\begin{aligned}
e^{(fe)^{\frac{k}{2}-1}fc^id^je} &= e^{(fe)^{\frac{k}{2}}\overline{c}^i\overline{d}^j} \\
&= e^{(fe)^{\frac{k}{2}}(\overline{c}\,\overline{d})\overline{c}^{i-1}\overline{d}^{j-1}} \quad \text{(Lemma 2.5)} \\
&= e^{(fe)^{\frac{k}{2}}(ef)^k\overline{c}^{i-1}\overline{d}^{j-1}} \quad \text{(Lemma 2.4)} \\
&= e^{(ef)^{\frac{k}{2}}\overline{c}^{i-1}\overline{d}^{j-1}} \\
&= e^{(fe)^{\frac{k}{2}-1}f\overline{c}^{i-1}\overline{d}^{j-1}}
\end{aligned}$$

□

THEOREM 2.15. *When k is odd,*

$$e^{(fe)^{\frac{k-1}{2}}c^id^jb} = e^{(fe)^{\frac{k-1}{2}}\overline{c}^{i+1}\overline{d}^{j+2}}$$

PROOF. By Lemma 2.6,

$$e^{(fe)^{\frac{k-1}{2}}c^id^jb} = e^{(fe)^{\frac{k-1}{2}}b\overline{c}^i\overline{d}^j}$$

Therefore,

$$\begin{aligned}
e^{(fe)^{\frac{k-1}{2}}c^id^jb} &= e^{(fe)^{\frac{k-1}{2}}b\overline{c}^i\overline{d}^j} \\
&= e^{(fe)^{\frac{k-1}{2}}f\overline{d}\,\overline{c}^i\overline{d}^j} \\
&= e^{(fe)^{\frac{k-1}{2}}f\overline{c}^i\overline{d}^{j+1}} \\
&= e^{(ef)^{\frac{k+1}{2}}\overline{c}^i\overline{d}^{j+1}} \\
&= e^{(ef)^{\frac{k+1}{2}}(dc)\overline{c}^{i+1}\overline{d}^{j+2}} \\
&= e^{(ef)^{\frac{k+1}{2}}(fe)^k\overline{c}^{i+1}\overline{d}^{j+2}} \\
&= e^{(fe)^{k-\frac{k+1}{2}}\overline{c}^{i+1}\overline{d}^{j+2}} \\
&= e^{(fe)^{\frac{k-1}{2}}\overline{c}^{i+1}\overline{d}^{j+2}}
\end{aligned}$$

□

THEOREM 2.16. *When k is odd,*

$$e^{(fe)^{\frac{k-1}{2}}c^id^jf} = e^{(fe)^{\frac{k-1}{2}}\overline{c}^{i+1}\overline{d}^{j+1}}$$

PROOF. By Lemma 2.6,

$$e^{(fe)^{\frac{k-1}{2}}c^i d^j f} = e^{(fe)^{\frac{k-1}{2}}f\overline{c}^{\,i}\overline{d}^{\,j}}$$

Therefore,

$$\begin{aligned}
e^{(fe)^{\frac{k-1}{2}}c^i d^j f} &= e^{(fe)^{\frac{k-1}{2}}f\overline{c}^{\,i}\overline{d}^{\,j}} \\
&= e^{(fe)^{\frac{k+1}{2}}\overline{c}^{\,i}\overline{d}^{\,j}} \\
&= e^{(fe)^{\frac{k+1}{2}}(dc)\overline{c}^{\,i+1}\overline{d}^{\,j+1}} \\
&= e^{(ef)^{\frac{k+1}{2}}(fe)^k\overline{c}^{\,i+1}\overline{d}^{\,j+1}} \\
&= e^{(fe)^{k-\frac{k+1}{2}}\overline{c}^{\,i+1}\overline{d}^{\,j+1}} \\
&= e^{(fe)^{\frac{k-1}{2}}\overline{c}^{\,i+1}\overline{d}^{\,j+1}}
\end{aligned}$$

□

Therefore, every element in the e-component belongs to an $n_1 \times n_2$ sized torus and there are k such tori. Furthermore, it is easy to verify that all secondary relations are satisfied. Therefore, the e-component has kn_1n_2 elements.

To verify that the f-component also has kn_1n_2 elements, we can again use the automorphism from Figure 5.

2.6. Conclusion

We conclude that $Q_N(G)$ has $2kn_1 + 2kn_2 + 4kn_1n_2$ elements and is therefore finite.

Bibliography

[1] S. Winker, *Quandles, knot invariants, and the n-fold branched cover*, Ph.D. thesis, University of Illinois, Chicago (1984).
[2] B. Mellor and R. Smith, *N-quandles of links.* Topology and its Applications 294, 107662.