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An Alternate Proof for the Top-Heavy Conjecture on Partition Lattices Using Shellability

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AN ALTERNATE PROOF FOR THE TOP-HEAVY CONJECTURE ON PARTITION LATTICES USING SHELLABILITY

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Abstract

A partially ordered set, or poset, is governed by an ordering that may or may not relate any pair of objects in the set. Both the bonds of a graph and the partitions of a set are partially ordered, and their poset structure can be depicted visually in a Hasse diagram. The partitions of $\{1, 2, ..., n\}$ form a particularly important poset known as the partition lattice Π_n. It is isomorphic to the bond lattice of the complete graph K_n, making it a special case of the family of bond lattices.

Dowling and Wilson's 1975 Top-Heavy Conjecture states that every bond lattice has at least as many elements in its upper half as in its lower half. The existing proof of this conjecture by Huh et al. in 2017 relies heavily on algebraic geometry. In this paper, we provide an alternate combinatorial proof for the Top-Heavy Conjecture on partition lattices. To do this, we define a specific class of forests on n vertices and construct an abstract simplicial complex Δ_n out of the edge sets of these graphs. Then, we show that Δ_n is a shellable complex for all n, and we use this result to prove that Π_n is a top-heavy lattice.

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Contents

CHAPTER 1

Introduction

1.1. Background

A **graph** is a mathematical object consisting of a series of vertices, or nodes, connected by edges. Graphs can be used to model any network where objects or people are connected, such as social media, flight paths, and machine learning decision trees.

A graph G is defined by its vertex set $V(G)$ and edge set $E(G)$. For instance, if vertices 1 and 2 have an edge between them in G, then $1, 2 \in V(G)$ and $12 \in E(G)$. We say that vertices 1 and 2 are "adjacent." A graph H is a **subgraph** of G if and only if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In an **induced subgraph** H of G, any vertices in H that are adjacent in G must also be adjacent in H. A graph is **connected** if it is possible to find a path between any two of its vertices. If a graph is disconnected, its largest connected subgraphs are known as **connected components**. A **tree** T is a connected graph with n vertices and $n-1$ edges, the minimum number of edges so that T remains connected. Equivalently, a connected graph T is a tree if and only if it contains no cycles, or paths from a vertex to itself that do not overlap.

A **bond** is a specific type of subgraph, defined as follows:

Definition 1.1. A subgraph H of graph G is a bond of G if and only if the following hold:
1. $V(H) = V(G)$
2. Each connected component of H is an induced subgraph of G.

In Figure 1, subgraphs H_1 and H_3 are bonds of G, while H_2 is not. Note that any two vertices in G are adjacent, making it the **complete graph** on 4 vertices. The complete graph on n vertices is denoted K_n.

One of the primary motivations for examining the bonds of a graph is that, all together, they form an important combinatorial structure known as a **partially ordered set**. Otherwise known as a "poset," it is defined as follows:

Definition 1.2. A poset $(P, \leq)$, where P is a finite set and $\leq$ is a binary relation on P, satisfies the following properties $\forall\ x, y, z \in P$:
1. Reflexivity: $x \leq x$
2. Antisymmetry: If $x \leq y$ and $y \leq x$, then $x = y$.
3. Transitivity: If $x \leq y$ and $y \leq z$, then $x \leq z$.

In totally ordered sets like the integers and real numbers, any two elements can be compared; one is always larger than the other. However, in a partially ordered set, two elements may or may not be related.

If we let the collection of bonds of a graph be our set P and take the subgraph relation as our ordering $\leq$, the resulting structure satisfies all the above properties. It is called the **bond lattice** of the graph. For a pair of bonds H_1 and H_2 in the poset, if H_1 is a subgraph of

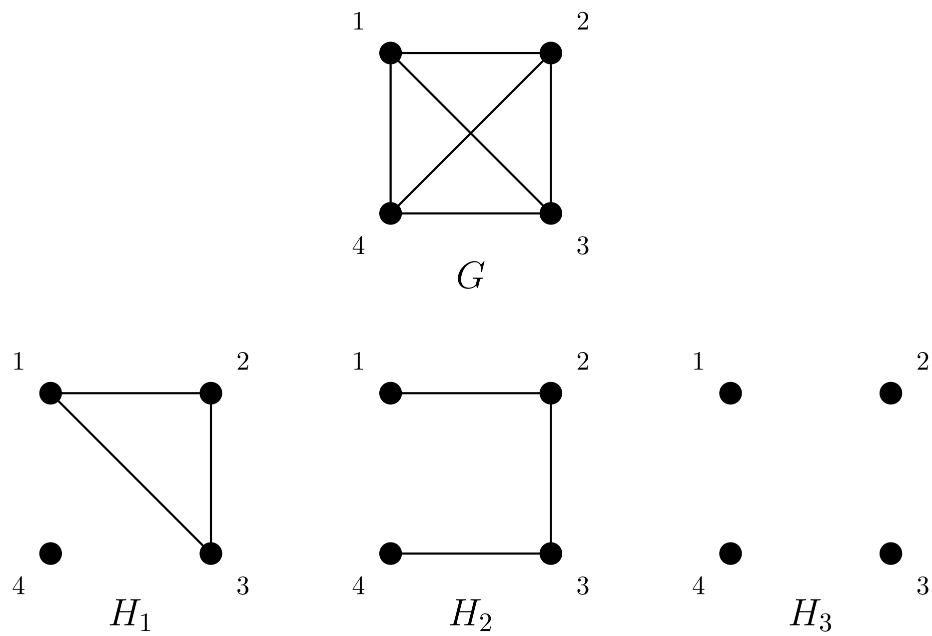

FIGURE 1. A graph and subgraphs

H_2, then H_2 is "above" H_1 in the ordering (or vice versa). But if neither bond is a subgraph of the other, they are unrelated in the ordering.

The Hasse diagram of a bond lattice is a graph-like structure that visualizes a bond lattice, with the nodes in the lattice representing individual bonds and the edges representing the subgraph relations between them. Each bond H is located on a certain level of the lattice, known as its **rank**. The rank of a bond is determined by its number of vertices minus its number of connected components. The edges on the lattice form paths between H and its subgraphs on the lower levels, as well as paths between H and its supergraphs on the higher levels. Figure 2 shows the bond lattice for K_4, graph G in the previous figure.

If G is a connected graph on n vertices, the lowest level of the bond lattice of G (rank 0) always contains one bond: the graph with n vertices and no edges. The highest level of the bond lattice (rank $n-1$) also contains a single bond: G itself.

There are many other instances of partial orders in mathematics. A **partition** is a way of dividing a finite set into separate parts, or "blocks." Partitions are written with slashes or vertical lines dividing the blocks; for example, 1/2/3, 12/3, and 123 are partitions of the set $\{1, 2, 3\}$ (also denoted $[3]$). The set of partitions of $[n]$ ordered by combining blocks is another common partially ordered set. For instance, in the partition poset of [4], 12/34 ranks above 1/2/34 because the former is formed by merging two blocks in the latter. However, 12/34 and 1/23/4 are unrelated.

Much like a bond lattice, a **partition lattice** is a partial order on a collection of set partitions. Like bonds, every partition has a rank in the lattice, given by the number of elements in the set minus the number of blocks. Note that these figures can be drawn for all posets and are known more generally as **Hasse diagrams**. Below is the partition lattice for [4], often denoted Π_4:

This lattice should look familiar; it is identical in structure, or isomorphic, to the bond lattice of K_4 shown in Figure 2. More generally, the bond lattice of K_n is isomorphic to the

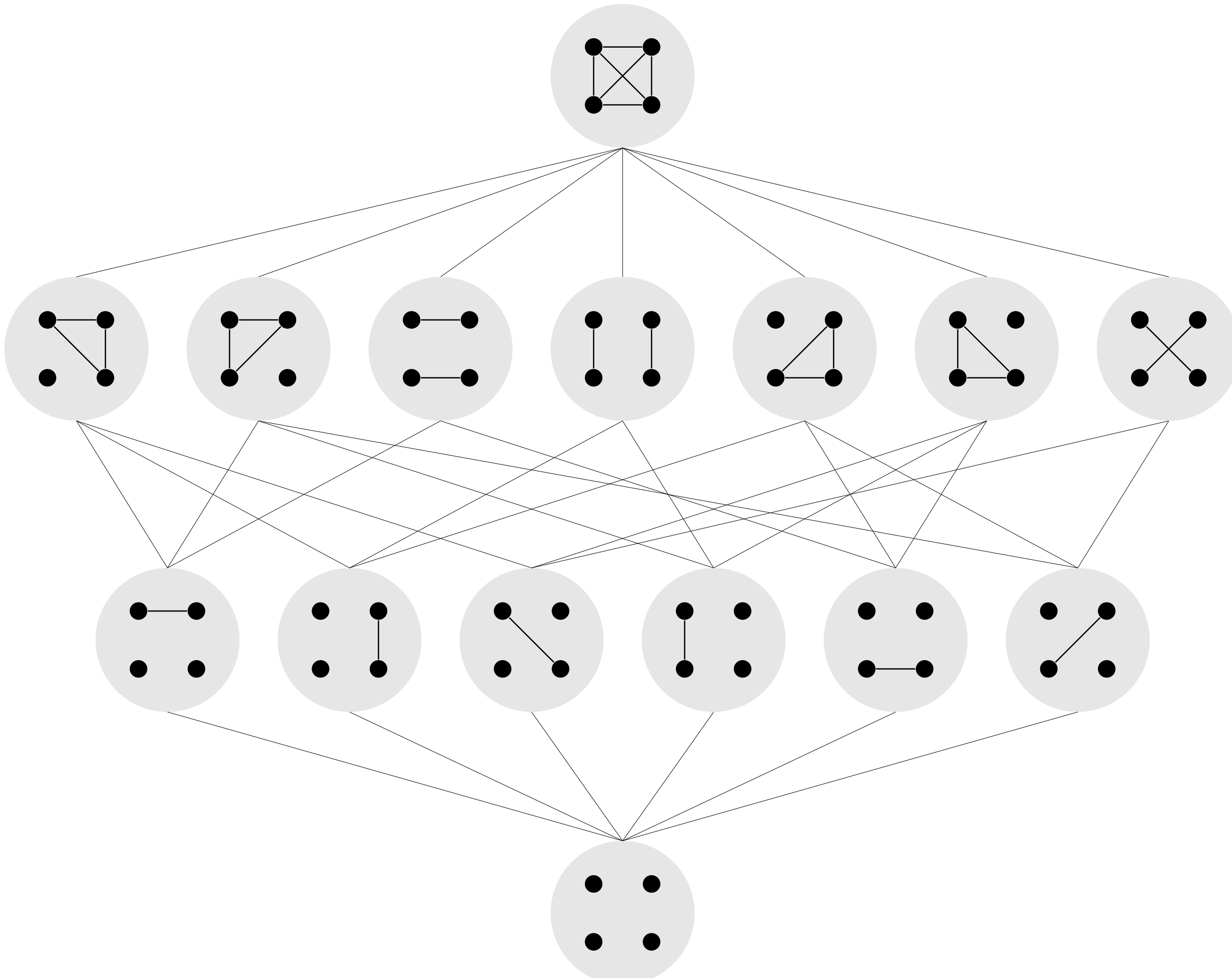

FIGURE 2. The bond lattice of the graph G in Figure 1

partition lattice Π_n for all $n \in \mathbb{N}$. This occurs because there is a one-to-one correspondence, or bijection, between bonds and partitions. If the numbered vertices in each connected component of a bond match the numbers in each block of a partition, then the bond and the partition are equivalent under the bijection. Moreover, this bijection preserves the poset structure; steps up or down in rank under the subgraph relation and the block ordering are equivalent. Thus, the bond lattice of K_n and the partition poset of $[n]$ are, for our purposes, identical. In this sense, partition lattices are special types of bond lattices that encode properties of complete graphs.

1.2. Motivation

In Figures 2 and 3, the upper half of the lattice contains more bonds than the lower half. As it turns out, this holds for all bond lattices. In fact, it is even true for a more abstract type of poset known as the lattice of flats of a **matroid**. Matroids provide a combinatorial generalization of the ideas of linear independence and span from linear algebra, and they are defined as follows:

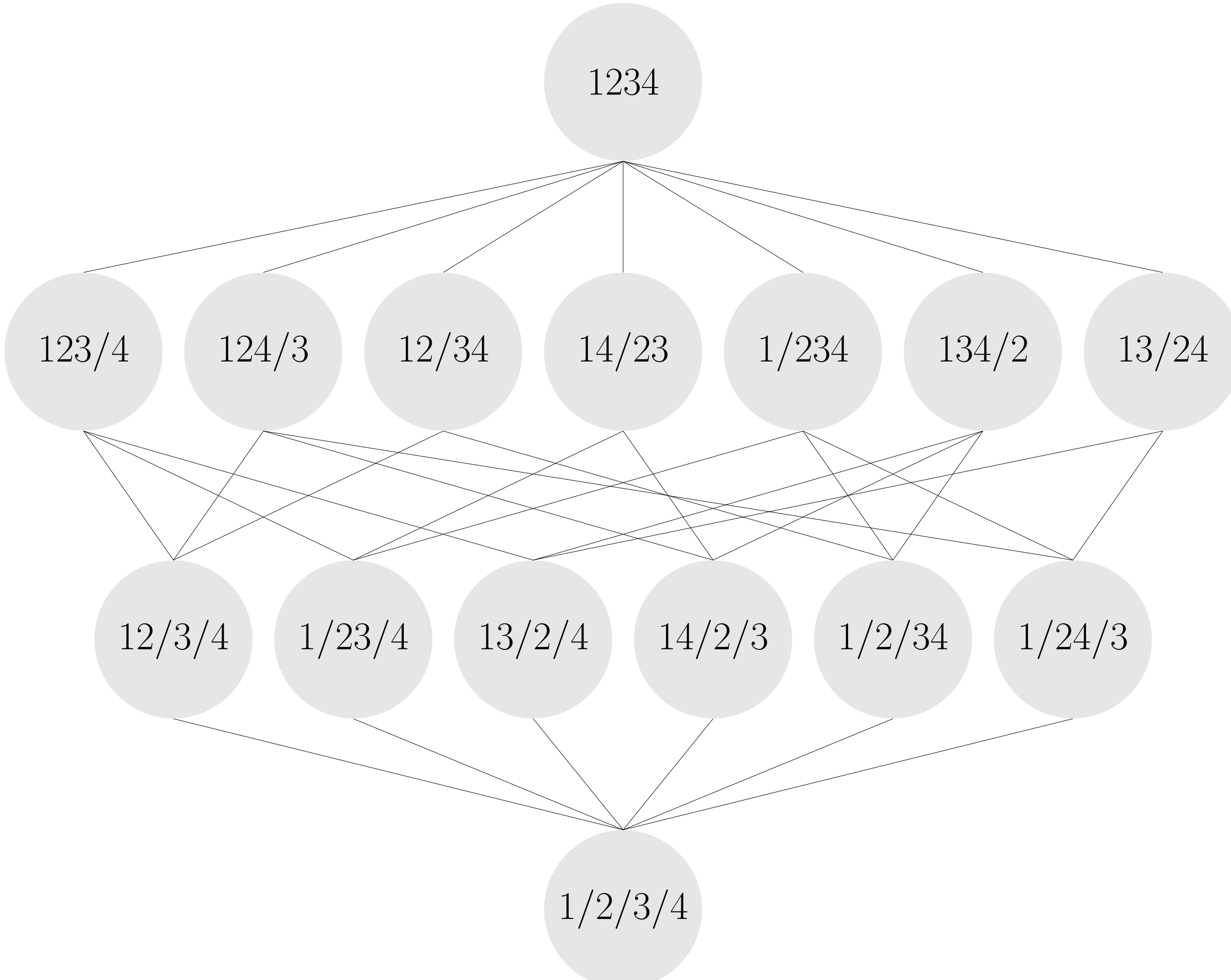

Figure 3. The Partition Lattice Π_4

Definition 1.3. A matroid (E, f), where E is a finite set and $f : E \to \mathbb{Z}$ is a function, satisfies the following properties $\forall\ S \subseteq E, a, b \in E$:

1. $f(\emptyset) = 0$.
2. $f(S) \leq f(S \cup \{a\}) \leq f(S) + 1$.
3. If $f(S \cup \{a\}) = f(S \cup \{b\}) = f(S)$, then $f(S \cup \{a, b\}) = f(S)$.

For any $S \subseteq E$, we call $f(S)$ the rank of S. The **flats** of E are the subsets with maximal rank; adding another element to a flat will cause its rank to increase by one. The flats of a matroid form a poset in the same manner as the bonds of a graph and the partitions of a set, with the same lattice structure and the same concept of rank. In fact, bond lattices are a special case of the more general matroids.

In 1975, Dowling and Wilson proposed the **Top-Heavy Conjecture** for all matroids [**3**].

Theorem 1.1 (Top-Heavy Conjecture). *Let $d \leq k/2$, $d, k \in \mathbb{N}$. For any matroid whose highest rank is d, the number of flats of rank k is less than or equal to the number of flats of rank $d - k$.*

The Top-Heavy Conjecture implies that the upper half of a matroid's lattice contains more flats than the lower half. Huh and Wang proved the conjecture in 2017 for a subclass of matroids known as realizable matroids, which includes bond lattices [**4**]. In 2023, Huh, Wang, Braden, Matherne, and Proudfoot verified the conjecture for all matroids [**2**]. However, these proofs make heavy use of elaborate algebraic geometry. In this paper, we provide a more combinatorics-centered proof of the Top-Heavy Conjecture for partition lattices - one that we believe can be generalized to other types of bond lattices and matroids.

CHAPTER 2

Tools for the Proof

2.1. Simplicial Complexes

The main tool in our proof of the Top-Heavy Conjecture for complete graphs is a combinatorial object known as an **abstract simplicial complex**. It is defined as follows:

DEFINITION 2.1. An abstract simplicial complex Δ is a finite collection of finite sets with the following properties:

1. $\emptyset \in \Delta$.
2. If $T \in \Delta$ and $S \subseteq T$, then $S \in \Delta$.

The elements of a simplicial complex Δ are called **faces**. The maximal faces, the ones which are not proper subsets of any other faces in Δ, are called **facets**. If we combine the second property above with the fact that facets are maximal, we see that Δ consists entirely of the facets and their subsets. Thus, a simplicial complex is uniquely determined by its facets.

For any face F in a simplicial complex Δ, the **dimension** of F is defined by $\dim(F) = |F| - 1$. The dimension of Δ is the dimension of its largest facet. A complex is called **pure** if all its facets have the same dimension. A complex can be drawn as a graph-like structure made of triangles of various dimensions, which correspond to the dimensions of the faces. For instance, faces of dimension 0 (1 element) are drawn as points, faces of dimension 1 (2 elements) are drawn as line segments, faces of dimension 2 (3 elements) are drawn as filled-in triangles, and faces of dimension 3 (4 elements) are drawn as filled-in tetrahedrons. An n-dimensional triangle is called an **n-simplex**.

In the simplicial complex in Figure 1, the facets are $\{a, b, c\}$, $\{a, c, d\}$, and $\{d, e\}$. The faces are all the subsets of those facets: $\{a, b, c\}$, $\{a, c, d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{c, d\}$, $\{d, e\}$ $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{e\}$, and $\emptyset$. The complex is not pure, since two of its facets have

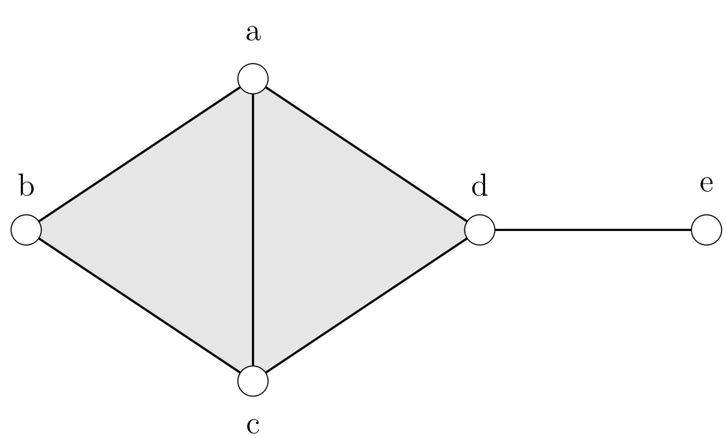

FIGURE 1. A Simplicial Complex

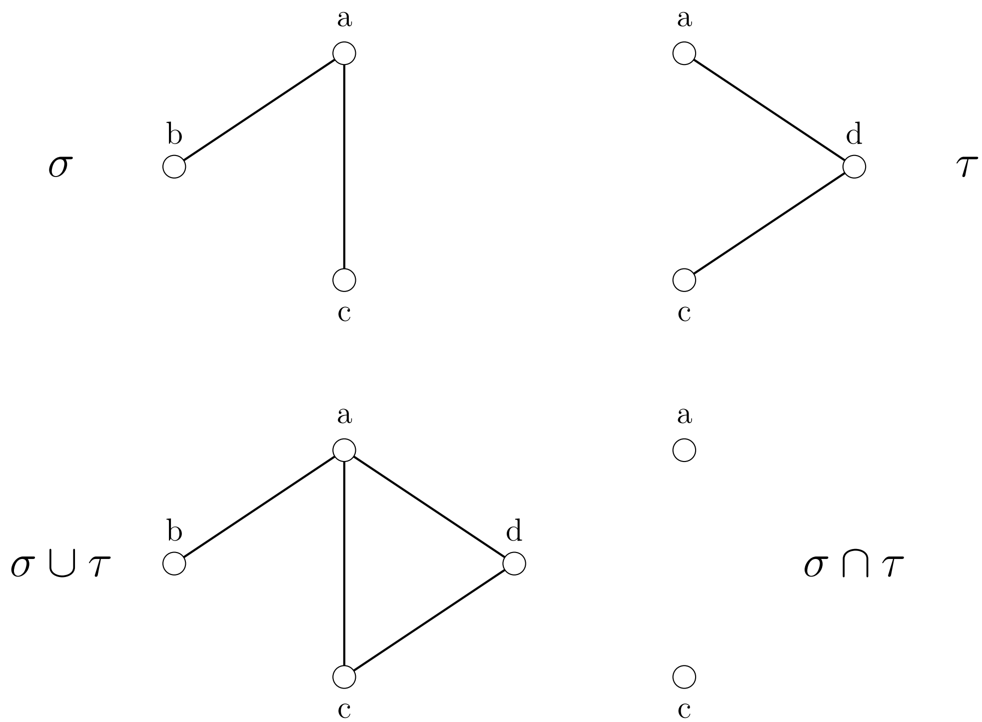

FIGURE 2. Several Simplicial Complexes

dimension 2 and the other facet has dimension 1. For future reference, it is common to write faces without set notation; for instance, $\{a, b, c\}$ could be abbreviated as abc or a, b, c.

It is important to note that the faces of a simplicial complex form a poset when ordered by inclusion, known as the **face poset**. The facets are the maximal elements in the face poset and occupy the highest positions in its Hasse diagram. The rank of each face is its size, or its dimension plus one. This poset structure inside every complex will become important later, when we set up our proof of the Top-Heavy Conjecture for partition lattices.

Like graphs, simplicial complexes can be broken up into smaller pieces. A **subcomplex** $\sigma \subseteq \Delta$ is a collection of sets in Δ that is also a complex. For example, in the complex pictured in Figure 2, $\sigma = \{ab, ac, a, b, c, \emptyset\}$ is a subcomplex. ab and ac are faces in the original complex, but they are the facets of σ. Because facets uniquely determine a complex, it is much easier to identify a subcomplex by its facets. We say that ab and ac "generate" σ, or $\sigma =< ab, ac >$. Basic set theory operations are appropriate to use on subcomplexes. For instance, if $\tau =< ad, cd >$, then $\sigma \cup \tau =< ab, ac, ad, cd >$ and $\sigma \cap \tau =< a, c >$. Note that $acd \notin \sigma \cup \tau$; we can tell this from the drawing in Figure 2, as the triangle is not filled in. Also note that all four complexes in Figure 2 are pure.

2.2. Shellability

It is not surprising that every simplicial complex is the union of its facets. Facets are the building blocks of a complex; looking at Figures 1 and 2, we can imagine constructing the complexes one facet at a time, "gluing" the pieces together at each step. A **shelling** is a specific way of building a complex out of its facets, an order in which they are glued together. It has two equivalent definitions:

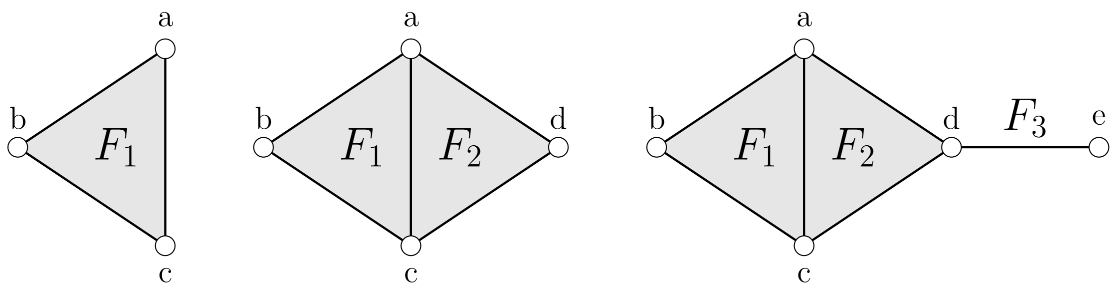

FIGURE 3. Example of Shelling

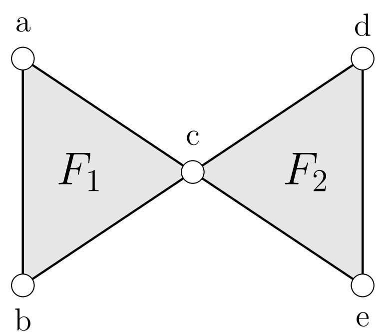

FIGURE 4. The Bowtie Complex

DEFINITION 2.2 (Definition 1 of Shelling). A shelling of a complex Δ is an ordering of its facets $F_1, F_2, ..., F_t$ such that $\forall\ 2 \le k \le t$, $(\bigcup_{i=1}^{k-1} < F_i >) \cap < F_k >$ is pure and of dimension $\dim(F_k) - 1$.

DEFINITION 2.3 (Definition 2 of Shelling). A shelling of a complex Δ is an ordering of its facets $F_1, F_2, ..., F_t$ such that $\forall\ 2 \le k \le t$, $\exists$ a unique minimal face $F_k' \subseteq F_k$ that is not contained in $F_i\ \forall\ i < k$.

A complex Δ is **shellable** if and only if there exists a shelling of Δ.

Figure 3 shows a step-by-step depiction of the shelling F_1, F_2, F_3 of the complex in Figure 1, where $F_1 = abc$, $F_2 = acd$, and $F_3 = de$. We can verify that this is a shelling by the first definition, observing that $< F_1 > \cap < F_2 >=< ac >$ and $(< F_1 > \cup < F_2 >) \cap < F_3 >=< d >$. $< ac >$ is pure with dimension 1, one less than $\dim(F_2) = 2$, and $< d >$ is pure with dimension 0, one less than $\dim(F_3) = 1$. At each step, the subcomplex by which the new facet is "glued" to the others is pure, and its dimension is one less than that of the new facet. Thus, F_1, F_2, F_3 is a shelling.

Alternatively, we can utilize the second definition. The face $d \subseteq F_2$ is not a subset of F_1, so we say it is new. It is also minimal, as it is the smallest subset of F_2 with this property. Finally, d is unique, as it is the only new minimal face in $< F_2 >$. Similarly, the face $e \subseteq F_3$ is not contained in F_1 or F_2, is the smallest new face in $< F_3 >$, and is the only face of its kind. Thus, F_2 and F_3 contain new unique minimal faces, so F_1, F_2, F_3 is a shelling. For each facet in the shelling, we call the new unique minimal face the **restriction set**.

Now, consider the ordering F_1, F_2 of the facets in the Bowtie Complex shown in Figure 4, where $F_1 = abc$ and $F_2 = cde$. We can see that this is not a shelling by both definitions. The first definition fails because $< F_1 > \cap < F_2 >=< c >$, which has dimension 0 while F_2 has dimension 2. The second definition fails because d and e are both new minimal faces in $< F_2 >$, so F_2 contains no unique new minimal face. The other possible ordering of the facets is F_2, F_1, but this is not a shelling for similar reasons. Therefore, the complex is not shellable.

We only need one shelling to prove that a simplicial complex Δ is shellable, but we must check every possible ordering of the facets before we can be sure Δ is not shellable. Because of this, shellability can be difficult to determine for larger complexes. This will become apparent later, as we will determine the shellability of a specific class of complexes up to the n^{th} dimension in order to prove partition lattices are top-heavy.

CHAPTER 3

Methods

3.1. The Increasing Galaxy Complex

Now, we formulate our combinatorial proof for the Top-Heavy Conjecture on partition lattices. To do this, we must define a simplicial complex arising from a particular class of graphs. For these definitions, assume that the vertices of the graphs are numbered by positive integers. Let an **increasing star** be a tree with one central vertex that is adjacent to and smaller than the other vertices (in numbering). Let an **increasing galaxy** be a forest on n vertices consisting of increasing stars. Here are the formal definitions:

Definition 3.1. An increasing star is a tree G with $V(G) \subseteq [n]$, such that $\exists!\ v \in V(G)$ with $uv \in E(G)$ and $v < u\ \forall\ u \in V(G)\ (u \neq v)$.

Definition 3.2. An increasing galaxy is a forest F with $V(F) = [n]$, such that each connected component of F is an increasing star.

The forest in Figure 1 contains two trees. In one tree, vertex 1 is adjacent to and less than 2, 3, 4, and 5. In the other tree, vertex 6 is adjacent to and less than 7. Thus, the trees are increasing stars, and the forest is an increasing galaxy on seven vertices. The edge set of this galaxy is $\{12, 13, 14, 15, 67\}$.

Together, the edge sets of all possible increasing galaxies on n vertices form a simplicial complex Δ_n. We call this the **increasing galaxy complex**. The edge set of each galaxy is a face in Δ_n, and the edge sets of the maximal galaxies - to which adding another edge cannot produce another galaxy - are the facets of Δ_n. For example, the increasing galaxy in Figure 1 is maximal because adding another edge would either produce a cycle, produce a tree that is not a star, or disrupt the inside-out ordering of the vertices, all of which would not result in another increasing galaxy. Thus, $\{12, 13, 14, 15, 67\}$ is a facet in Δ_7. In the future, we will say that an increasing galaxy G is "equivalent" to a facet F if $F = E(G)$.

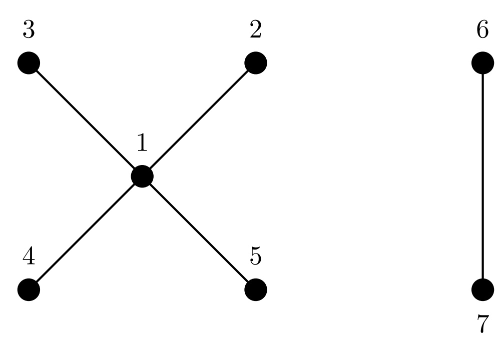

Figure 1. An Increasing Galaxy

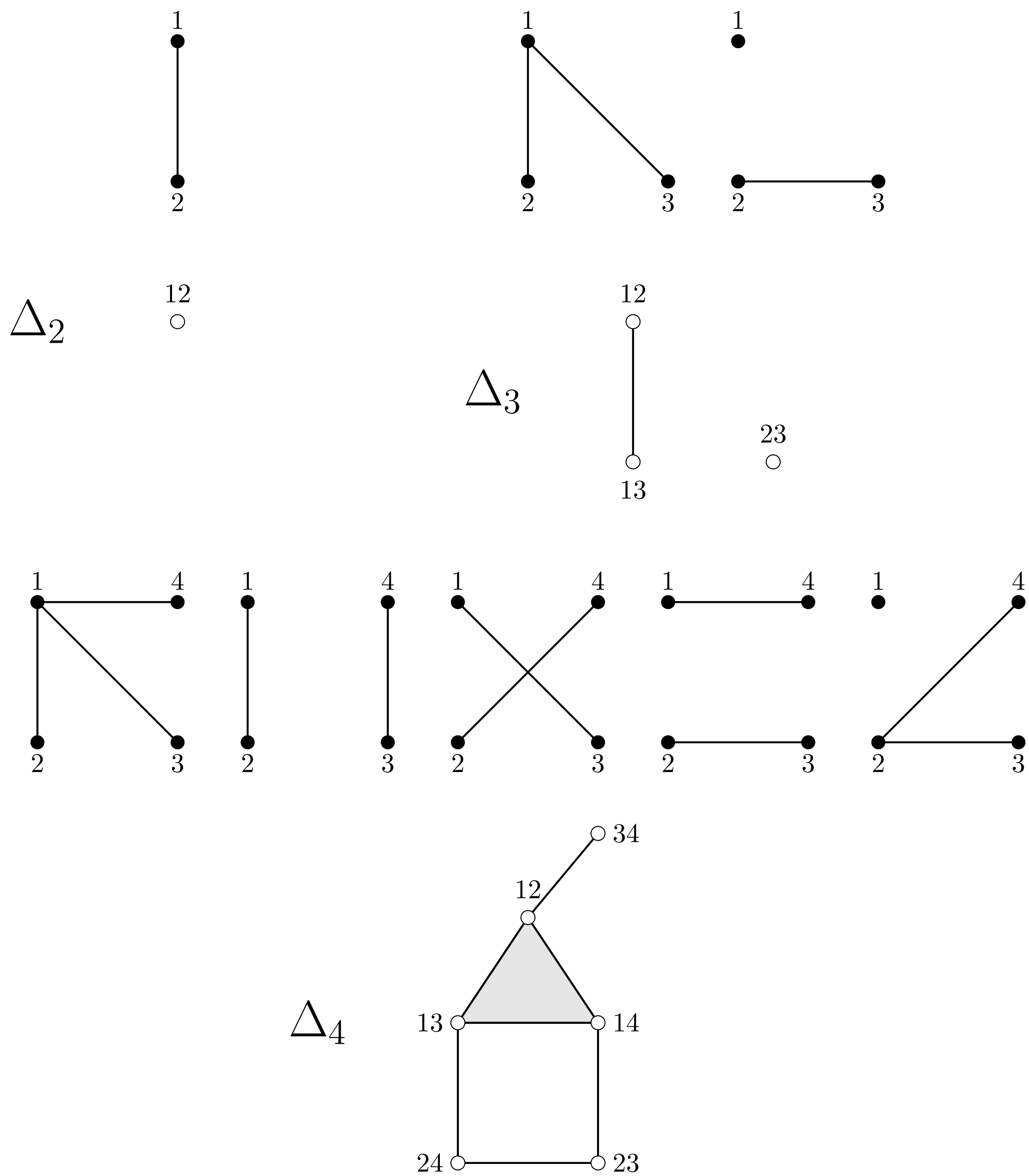

FIGURE 2. The Increasing Galaxy Complexes for $n = 2, 3, 4$

The only increasing galaxy on one vertex is the graph with one vertex and no edges. So Δ_1 is the trivial empty complex. Pictured in Figure 2 are the maximal increasing galaxies on two, three, and four vertices, and the resulting galaxy complexes generated by their edge sets. Note that the edges of the graphs become nodes in the drawing of the complex. As n increases, the complex quickly becomes larger and more difficult to draw (Δ_5 contains a filled-in tetrahedron and many filled-in triangles).

Now, notice that Δ_1 and Δ_2 are trivially shellable, $\{12, 13\}, \{23\}$ is a shelling of Δ_3, and $\{12, 13, 14\}, \{12, 34\}, \{13, 24\}, \{14, 23\}, \{23, 24\}$ is a shelling of Δ_4. If this pattern continues

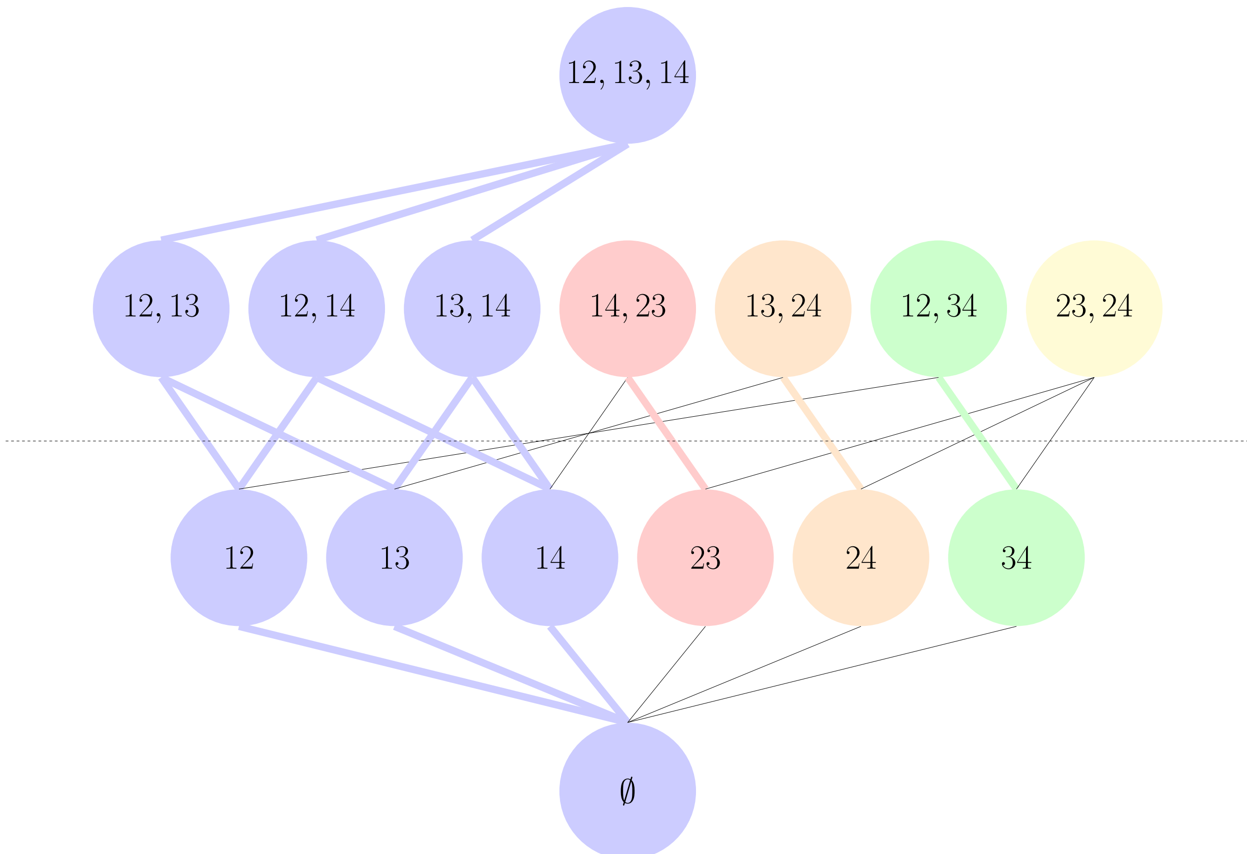

FIGURE 3. Boolean Decomposition of the Face Poset of Δ_4

and Δ_n is shellable for all $n \in \mathbb{N}$, we have a direct path to proving that Π_n is top-heavy. We outline this strategy below.

3.2. Outline of the Proof

In 1996, Björner and Wachs proved that if a complex Δ is shellable, its face poset $F(\Delta)$ can be split into a series of simple posets called **Boolean algebras** [**1**]. The Boolean algebra B_n is isomorphic to the poset given by the subsets of $[n]$, ordered by inclusion. For the theorem below, note that the poset interval $[a, b]$ is the section of poset P with lowest element a and highest element b; $[a, b]$ is a smaller poset within P.

THEOREM 3.1 (Björner-Wachs, 1996). *Let $F_1, F_2, \ldots, F_k$ be a shelling of simplicial complex Δ. Then $F(\Delta) = \bigsqcup_{i=1}^{n} [R(F_i), F_i]$, where $R(F_i)$ is the restriction set of facet F_i in the shelling, and $[R(F_i), F_i]$ is a Boolean algebra.*

As an example, the decomposition of $F(\Delta_4)$ into Boolean algebras is shown in Figure 3. The sections of each color in the Hasse diagram represent a Boolean interval from a restriction set to a facet. Working from our previous shelling of Δ_4, $[R(F_1), F_1] \cong B_3$ (blue), $[R(F_2), F_2] \cong B_1$ (green), $[R(F_3), F_3] \cong B_1$ (orange), $[R(F_4), F_4] \cong B_1$ (red), and $[R(F_5), F_5] \cong B_0$ (yellow). All together, their disjoint union is the entire poset, $F(\Delta_4)$.

This decomposition is useful here because Boolean algebras are always top-heavy posets; in fact, they are symmetric, with equally many elements of higher and lower rank. Let the **halfway point** of a poset be its highest rank divided by two; we can visualize it as a horizontal line drawn through the Hasse diagram, separating elements of high and low rank. The halfway point of $F(\Delta_4)$ is $\frac{\dim(\Delta_4)+1}{2} = \frac{3}{2}$. Notice how the Boolean intervals are positioned in the Hasse diagram in Figure 3: their halfway points are greater than or equal to the halfway point of the face poset. The halfway point of $[R(F_i), F_i]$ is $\frac{3}{2}$ for all $1 \leq i \leq 4$, while $[R(F_5), F_5]$ has halfway point 2. If a poset is made up of top-heavy intervals arranged in this manner, then it must be top-heavy.

Thus, if we can show Δ_n is shellable, and that the Boolean intervals corresponding to its facets are arranged at or above the halfway point of $F(\Delta_n)$, then we have that $F(\Delta_n)$ is top-heavy. We prove this formally below:

THEOREM 3.2. *Let Δ be a simplicial complex with k facets. Then $F(\Delta)$ is top-heavy if the following hold:*
(a) $F_1, F_2, ..., F_k$ is a shelling of Δ.
(b) $\forall\ i \in [k]$, $|F_i| + |R(F_i)| \geq \dim(\Delta) + 1$.

PROOF. Let $F(\Delta)$ denote the face poset of Δ. Since $F_1, F_2, ..., F_k$ is a shelling order, by Theorem 3.1, we have that

$$F(\Delta) = \bigsqcup_{i=1}^{n} [R(F_i), F_i]$$

where $[R(F_i), F_i]$ is a Boolean algebra P_i $\forall\ i \in [k]$. Since the rank of each face in $F(\Delta)$ is given by its size, the interval $[R(F_i), F_i]$ has length $|F_i| - |R(F_i)|$. Because the interval begins at rank $|R(F_i)|$, its halfway point in the face poset is given by

$$|R(F_i)| + \frac{|F_i| - |R(F_i)|}{2} = \frac{|F_i| + |R(F_i)|}{2}.$$

By our second assumption,

$$|F_i| + |R(F_i)| \geq \dim(\Delta) + 1 \Rightarrow \frac{|F_i| + |R(F_i)|}{2} \geq \frac{\dim(\Delta) + 1}{2}.$$

We know $\frac{|F_i|+|R(F_i)|}{2}$ is the halfway point of P_i within $F(\Delta)$, and $\frac{\dim(\Delta)+1}{2}$ is the halfway point of $F(\Delta)$ since $\dim(\Delta) + 1$ is its highest rank. Thus, each Boolean algebra comprising $F(\Delta)$ has a halfway point greater than or equal to the halfway point of $F(\Delta)$. Since $F(\Delta) = \bigsqcup_{i=1}^{n} P_i$, every face in Δ is contained in exactly one Boolean algebra P_i. Combining these facts, we get that for all $0 \leq i \leq r/2$, the number of elements of rank i in $F(\Delta)$ is less than or equal to the number of elements of rank $r - i$ where r is the maximum rank of $F(\Delta)$. Therefore, $F(\Delta)$ is top-heavy. ◻

Finally, returning to the Δ_4 example from before, observe that $F(\Delta_4)$ has the same number of elements at each rank as the partition lattice Π_4: one element of rank 0, six elements of rank 1, seven of rank 2, and one of rank 3. We call the vector $[1, 6, 7, 1]$ the **f-vector** of the complex, the number of faces of each dimension. Comparing the Hasse diagrams, we could draw Π_4 by rearranging the edges in Δ_4. Clearly, if one is top-heavy, then so is the other. We will soon prove that this is true for all n. Therefore, if we can show $F(\Delta_n)$ is top-heavy, we have our desired result that Π_n is top-heavy!

In summary, here is the outline of our final proof:

1.) Show that the increasing galaxy complex Δ_n is shellable.
2.) Show that $|F_i| + |R(F_i)| \geq \dim(\Delta_n) + 1$ for every facet F_i in the shelling.
3.) Show that $F(\Delta_n)$ and Π_n have the same number of elements at each rank.

To show these things, we will need to prove two fascinating correspondences: the bijection between faces of Δ_n and partitions of $[n]$, and the bijection between facets of Δ_n and partitions of $[n-1]$.

3.3. The Faces-Partitions Bijection

Theorem 3.3. *Let Δ_n be the increasing galaxy complex on n vertices. Then Δ_n has B_n faces, where B_n is the nth Bell number (the number of partitions of $[n]$).*

Proof. Define the mapping $\phi : \Delta_n \to \Pi_n$ as follows: let F be an increasing galaxy equivalent to a face in Δ_n, and let $u, v \in V(F)$. If u and v are in the same connected component of F, then let u and v occupy the same block in partition $\phi(F) \in \Pi_n$. This map is well-defined, as F has n vertices, so partitioning its vertices by component gives a partition $P \in \Pi_n$. We prove ϕ is a bijection.

ϕ is injective: Let F and G be increasing galaxies on n vertices, with $F \neq G$. Suppose, toward contradiction, that $\phi(F) = \phi(G)$. Then F and G have components $F_1, F_2, ..., F_k$ and $G_1, G_2, ..., G_k$, respectively, where F_i and G_i have the same vertices ($k \in \mathbb{N}, 1 \leq i \leq k$). Let $V(F_i) = V(G_i) = \{v_1, v_2, ..., v_m\}$, where v_1 is the smallest vertex in the set. Since F_i and G_i are components of increasing galaxies, they are increasing stars, so $E(F_i) = E(G_i) = \{v_1v_2, v_1v_3, ..., v_1v_m\}$ with v_1 as the central vertex. But then F_i and G_i have the same vertex set and edge set, so
$F_i = G_i \Rightarrow F = G$, a contradiction. Therefore, $\phi(F) \neq \phi(G)$, and ϕ is injective.

ϕ is surjective: Let $P = P_1/P_2/.../P_k$ be a partition in Π_n ($k \in \mathbb{N}$). Take each block $P_i = \{v_1, v_2, ..., v_m\}$, where v_1 is the smallest element in P_i, and draw a graph S_i with $V(S_i) = \{v_1, v_2, ..., v_m\}$ and $E(S_i) = \{v_1v_2, v_1v_3, ..., v_1v_m\}$. This graph is an increasing star on m vertices, with central vertex v_1. Therefore, $F = \bigcup_{i=1}^{k} S_i$ is an increasing galaxy on n vertices. Since the vertices in the components of F match the vertices in the blocks of P, we have that $\phi(F) = P$. Therefore, ϕ is surjective.

Since $\phi : \Delta_n \to \Pi_n$ is a bijection, Δ_n has B_n faces. □

3.4. The Facets-Partitions Bijection

Theorem 3.4. *Let Δ_n be the increasing galaxy complex on n vertices. Then Δ_n has B_{n-1} facets, where B_{n-1} is the $(n-1)$th Bell number (the number of partitions of $[n-1]$).*

PROOF. Let A_n be the set of facets of Δ_n, and let $\Pi_{2,...,n}$ denote the partition poset of $\{2, ..., n\}$. Note that $\Pi_{n-1} \cong \Pi_{2,...,n}$. Define the mapping $\theta : A_n \to \Pi_{2,...,n}$ as follows: let F be a maximal increasing galaxy equivalent to a facet in Δ_n, and let $u, v \in V(F)$ with $u \neq 1, v \neq 1$. If u and v are in the same connected component of F that does not contain 1, then let u and v occupy the same block in partition $\theta(F) \in \Pi_{2,...,n}$. If u is in the same component as 1, then let u be a solitary block, or singleton, in $\theta(F)$. This map is well-defined, as F has $n-1$ vertices other than 1, so partitioning these vertices by component gives a partition $P \in \Pi_{2,...,n}$. We prove θ is a bijection.

θ is injective: Let F and G be maximal increasing galaxies on n vertices, with $F \neq G$. Suppose, toward contradiction, that $\theta(F) = \theta(G) = B_1/.../B_k/\{c_1\}/.../\{c_m\}$, where $|B_i| > 1$ $\forall\ i \in [k]$ and $\{c_j\}$ is a singleton $\forall\ j \in [m]$. Then F and G have components $F_1, F_2, ..., F_k, F'$ and $G_1, G_2, ..., G_k, G'$, respectively, where $V(F_i) = V(G_i) = B_i$
$\forall\ i \in [k]$ and $V(F') = V(G') = \{1, c_1, ..., c_m\}$.

Since F' and G' are components of increasing galaxies, they are increasing stars, so $E(F') = E(G') = \{1c_1, 1c_2, ..., 1c_m\}$ with 1 as the central vertex. Similarly, each pair of components F_i and G_i has some smallest vertex $v_i \in B_i$, so the components have the same edge set with v_i as the central vertex. But then F_i and G_i have the same vertex set and edge set, as do F' and G', so $F_i = G_i, F' = G' \Rightarrow F = G$, a contradiction. Therefore, $\theta(F) \neq \theta(G)$, and θ is injective.

θ is surjective: Let $P = B_1/.../B_k/\{c_1\}/.../\{c_m\}$ be a partition in $\Pi_{2,...,n}$, where
$|B_i| \geq 2\ \forall\ i \in [k]$ and c_j is a singleton $\forall\ j \in [m]$. For each block $B_i = \{v_1, v_2, ..., v_r\}$, where v_1 is the smallest element in B_i, draw a graph S_i with $V(S_i) = B_i$ and $E(S_i) = \{v_1v_2, v_1v_3, ..., v_1v_r\}$. This graph is an increasing star with central vertex v_1.

Next, draw another graph S' with $V(S') = \{1, c_1, ...c_m\}$ and $E(S') = \{1c_1, 1c_2, ..., 1c_m\}$. This is also an increasing star, and its central vertex is 1. Therefore, $F = \bigcup_{i=1}^{k} S_i \cup S'$ is an increasing galaxy on n vertices. Since the vertices in the components S_i match the vertices in the non-singleton blocks of P, and the vertices in S' (besides 1) match the vertices in the singleton blocks of P, we have that $\phi(F) = P$.

Finally, we must show F is a maximal increasing star, so that it corresponds to a facet of Δ_n. To do this, we show it is impossible to add an edge to F and produce another increasing galaxy. Let u and v be two distinct, non-adjacent vertices in F. There are six cases:

1. $u, v \in V(S_i)$. Adding the edge uv would produce the cycle u, v_1, v, which would not result in an increasing galaxy.

2. $u, v \in V(S')$. Adding the edge uv would produce the cycle $u, 1, v$, which would not result in an increasing galaxy.

3. $u \in V(S_i)$, $v \in V(S_j)$, $i \neq j$. Since $|B_i| \geq 2$ and $|B_j| \geq 2$, there exists some vertex $a \in V(S_i)$ adjacent to u and some vertex $b \in V(S_j)$ adjacent to v. Adding the edge uv would produce the path a, u, v, b. This rules out an increasing galaxy because the longest possible path in a star contains three vertices.

4. $u \in V(S_i)$, $v \in V(S')$, $v \neq 1$. Then v is adjacent to 1, and since $|B_i| \geq 2$, there exists some vertex $a \in V(S_i)$ adjacent to u. Therefore, adding the edge uv would produce the path $a, u, v, 1$, ruling out an increasing galaxy.

5. $u \in V(S_i)$, $u \neq v_1$, $v \in V(S')$, $v = 1$. Adding the edge uv would produce the path $1, u, v_1$. For this path to exist in a star, u must be the central element. But $u > 1$, so this would not be an increasing star.

6. $u \in V(S_i)$, $u = v_1$, $v \in V(S')$, $v = 1$. Since $|B_i| \geq 2$, there exists some vertex $a \in V(S_i)$ adjacent to u. Therefore, adding the edge uv would produce the path $1, v_1, a$. But this cannot produce an increasing star since $v_1 > 1$.

Therefore, F is a maximal increasing galaxy, and it corresponds to facet of Δ_n. So θ is surjective.

Since $\theta : A_n \to \Pi_{2,\ldots,n}$ is a bijection, Δ_n has B_{n-1} facets. □

CHAPTER 4

Results

4.1. The Proof

With the above theorems in our arsenal, we are ready for the main proof! One final helpful definition: a **linear extension** of a poset $(P, \leq)$ is a total ordering $\lhd$ on P such that if $x < y$ in the poset, then $x \lhd y$. A linear extension is a total order on the elements of a poset that obeys the original partial order.

THEOREM 4.1 (Top-Heavy Conjecture for Partition Lattices). *Let* $d \leq \frac{n-1}{2}$, $d, n \in \mathbb{N}$. *In the partition lattice* Π_n*, the number of elements of rank* d *is less than or equal to the number of elements of rank* $n-1-d$.

PROOF. As outlined above, there are three parts to this proof:
1.) Show that the increasing galaxy complex Δ_n is shellable.
2.) Show that $|F_i| + |R(F_i)| \geq \dim(\Delta_n) + 1$ for every facet F_i in the shelling.
3.) Show that $F(\Delta_n)$ and Π_n have the same number of elements at each rank.

1.) First, we prove Δ_n is shellable. Let θ be the bijection between the facets of Δ_n and the partitions of $\{2, 3, ..., n\}$ defined in Theorem 3.4. For each facet F_i, let $\Pi_i = \theta(F_i)$. We prove that if $(\Pi_1, \Pi_2, ..., \Pi_k)$ is a linear extension of the partition lattice $\Pi_{2,..,n}$, then $(F_1, F_2, ..., F_k)$ is a shelling of Δ_n.

Let $\Pi_i = B_1/B_2/.../B_k/\{c_1\}/\{c_2\}/.../\{c_m\}$, where $|B_j| \geq 2 \;\forall\; j \in [k]$ and $c_1, ..., c_m$ are singletons. Define the edge set $E_i = \{ab \,|\, \forall\, j \in [k],\, a = \min(B_j),\, b \in B_j,\, a \neq b\}$. We prove $(F_1, F_2, ..., F_k)$ satisfies the second definition of a shelling (Definition 2.3) by showing that E_i is a new unique minimal face in F_i; that is, $R(F_i) = E_i$.

First, we show that E_i is new. Suppose, toward contradiction, that $E_i \subseteq F_h$, where $h < i$. Since $E_i \subseteq F_h$, Π_h must contain blocks $\beta_1, \beta_2, ..., \beta_k$ where $B_j \subseteq \beta_j \,\forall\, j \in [k]$. Since $\Pi_h \neq \Pi_i$, for at least one singleton c_q in Π_i ($q \in [m]$), c_q is in some block β_j in Π_h. But then Π_h is formed by merging blocks in Π_i, so $\Pi_i < \Pi_h$ in the partition poset.
$\therefore \Pi_i \lhd \Pi_h$ in the linear extension.
$\therefore F_i$ comes before F_h in the facet ordering, a contradiction. Thus, E_i is a new face.

Next, we show that E_i is a minimal new face in F_i. Consider any face $E \subseteq F_i$ such that $|E| < |E_i|$. Then $\exists$ some edge ab such that $ab \in E_i$ but $ab \notin E$. Assume $a, b \in B_j$ ($j \in [k]$) and $a < b$. Then $E \subseteq F_h$, where

$$\Pi_h = B_1/B_2/.../(B_j - \{b\})/.../B_k/\{c_1\}/\{c_2\}/.../\{c_m\}/\{b\}.$$

But then Π_i is formed by merging blocks in Π_h, so $\Pi_h < \Pi_i$ in the partition poset.
$\therefore \Pi_h \lhd \Pi_i$ in the linear extension.
$\therefore F_h$ comes before F_i in the facet ordering. Since $E \subseteq F_h$, E is not a new face. Thus, E_i is a minimal new face.

Finally, we show that E_i is a unique minimal new face in F_i. Let $G \subseteq F_i$ be a new face that is distinct from E_i. Then $E_i \subset G$; otherwise, there is some edge in E_i that is not in G, and the above result follows that G is not new. Therefore, $|E_i| < |G|$ and G is not minimal. So E_i is the only minimal new face in F_i.

Since every facet F_i contains a unique minimal new face $R(F_i)$, then $(F_1, F_2, ..., F_k)$ is a shelling of Δ_n.

2.) For the next part of our proof, we show that $|F_i|+|R(F_i)| \geq \dim(\Delta_n)+1$ for each facet F_i. For the increasing galaxy equivalent to F_i, $B_1, B_2, ..., B_k$, and $\{1, c_1, c_2, ..., c_m\}$ are the vertex sets of the connected components. Note that for any forest F, $|E(F)| = |V(F)| - C$, where C is the number of components of F. Since the galaxy with edge set F_i has $k+1$ components, we have that

$$|F_i| = n - k - 1.$$

Similarly, the galaxy equivalent to $R(F_i) = E_i$ has k components with vertex sets $B_1, B_2, ..., B_k$. The only vertices in $[n]$ missing from this galaxy are $1, c_1, c_2, ..., c_m$, so it has $n - m - 1$ vertices. Therefore,

$$|R(F_i)| = n - m - k - 1.$$

Combining these results, we have that

$$|F_i| + |R(F_i)| = 2(n-1) - 2k - m.$$

Since $\dim(\Delta_n) + 1 = n - 1$, we must show that

$$2(n-1) - 2k - m \geq n - 1$$
$$\Leftrightarrow 2k + m \leq n - 1.$$

Since every vertex in $[n]$ occurs in a non-singleton B_j $(j \in [k])$, is a singleton c_q $(q \in [m])$, or is equal to 1, we have that

$$n = \sum_{j=1}^{k} |B_j| + m + 1$$

$$\therefore\ n - 1 = \sum_{j=1}^{k} |B_j| + m.$$

Finally, since $|B_j| \geq 2$, we see that

$$2k \leq \sum_{j=1}^{k} |B_j|$$

$$\therefore\ 2k + m \leq \sum_{j=1}^{k} |B_j| + m$$

$$\therefore\ 2k + m \leq n - 1.$$

Therefore, $|F_i| + |R(F_i)| \geq \dim(\Delta_n) + 1$ as desired.

3.) Since Δ_n is shellable, and $|F_i| + |R(F_i)| \geq \dim(\Delta_n) + 1$ for each facet F_i in the shelling, by Theorem 3.2, the face poset $F(\Delta_n)$ is top-heavy. We conclude by showing that Π_n is also top-heavy.

By Theorem 3.3, there exists a bijection ϕ between the faces of Δ_n and the partitions of $[n]$, so $F(\Delta_n)$ and Π_n have an equal number of elements. Let F be a face in Δ_n of size r. Then F has rank r in $F(\Delta_n)$. Since the galaxy with edge set F is a forest on n vertices, it has $n-r$ components. Thus, as defined in Theorem 3.3, the partition $\phi(F)$ has $n-r$ blocks, giving it rank $n-(n-r)=r$ in Π_n. Therefore, ϕ preserves rank, so $F(\Delta_n)$ and Π_n have the same number of elements of each rank. Since $F(\Delta_n)$ is top-heavy, Π_n is also top-heavy, and we are done.

□

4.2. Future Research

We now have a combinatorial proof that partition lattices are top-heavy. Partition lattices are useful objects, encoding properties of partitions and complete graphs; however, they are a special case of the bond lattice, which is merely one type of realizable matroid. Future research could explore the use of shelling arguments to prove that other classes of bond lattices and realizable matroids are top-heavy.

Additionally, this paper makes no mention of the topological consequences of shellability. For instance, Björner and Wachs showed that any shellable complex is equivalent under homotopy (deformation) to a wedge of spheres, a series of spheres of various dimensions joined together in a chain-like structure [**1**]. Future work could explore the homotopies of the increasing galaxy complex or other shellable complexes.

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