# The Evolution of Gaussian-Type Elimination

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Abstract

Utilizing a matrix simplifies problems involving systems of linear equations. Gaussian elimination, a series of correctly executed row operations performed on matrices, allows us to systematically find the solution to problems. Although named after mathematician Johann Carl Friedrich Gauss, who was born in the late 18th century, elimination originated sometime between 200 BCE and 100 BCE in the Chinese text, *Nine Chapters on the Mathematical Arts*. The *Nine Chapters* offers a method different from Gaussian elimination and recommends using counting boards and counting rods to solve the problems it poses. Since then, the power of computational capabilities has tremendously grown. As a result, the method and computation of elimination have evolved to solve the increasingly larger and larger systems of equations. However, how exactly has elimination changed? How do the elimination methods used today compare with those used two millenniums ago? What do the changes in the computational tools of elimination reveal about its future? I propose a two-part research method consisting of textual analyses and interviews with experts. Through these, I would be able to understand how elimination has changed and how it will change.

**Introduction**

Real life problems involving variables of degree one can be modeled not only by systems of linear equations but also by matrix equations. If necessary, see the appendix for a short introduction to basic matrix arithmetic and operations. The method of Gaussian elimination allows us to produce an upper triangular matrix from the original matrix. Then, through back substitution, this upper triangular matrix reveals the solution to our problem (Strang 46-51). Thus, the algorithmic nature of Gaussian elimination allows us to systematically solve systems of linear equations by performing row operations on matrices.

Interestingly, although Gaussian elimination is named after Johann Carl Friedrich Gauss, a famous mathematician who lived in the late 18th century and early 19th century (Kleiner 79-80), the first method of elimination originated in China about 2,000 years ago (O'Connor & Robertson). Clearly, since then, technology has advanced tremendously. The passage of two millenniums has led me to wonder how the method and computation of elimination have changed and how future technologies will further change this powerful method.

**Background**

Currently, I am a student in a Linear Algebra course. One of my favorite topics is elimination. My interest in this method has led me to inquire about its history and declare it as my research topic. As previously stated, the first method of elimination using matrix models was developed by Chinese mathematicians between 200 BCE and 100 BCE (O'Connor & Robertson). In fact, the seventh chapter of the *Nine Chapters on the Mathematical Arts*, a Chinese mathematical text from around this time period, provided problems with two equations in two unknowns (Hart 30, 57). Scholar Roger Hart suggests that Chinese mathematicians took a different approach to elimination. The Chinese number system encouraged a method of elimination, the fangcheng procedure, whereby computation of fractions could be avoided (Hart

78). Moreover, the Chinese numeral system consisted of positioning rods in a certain way to represent a number. Calculations were performed on a flat surface, known as a counting board. For the problems of Chapter 7 of the *Nine Chapters on the Mathematical Arts*, the counting board functions the same way as the modern-day matrix. This means that the Chinese rod-numeral system, the counting square, and the fangcheng procedure were the first computational tools available for solving systems of linear equations through elimination.

Although solving small systems of equations by hand is a relatively easy task, the Chinese mathematicians wanted to avoid fractions by executing the fangcheng procedure (Hart 78). Indeed, simplicity is of interest especially when dealing with larger systems of equations. A system of equations with two variables and two equations can be solved in under a minute by employing Gaussian elimination by hand. Now imagine performing Gaussian elimination on a system of 100 equations with 100 variables in each. Finding the solution to this system would be a long, tedious procedure prone to error. For this reason, Gaussian elimination has been developed as a computer algorithm so that computers can carry out the process for us. In the 1980s, early computers that were able to perform Gaussian elimination utilized programming languages such as FORTRAN and BASIC (Magid 66; Rothenberg 364). Similarly, handheld programmable calculators such as the TI 58C Programmable Computer possessed the ability to perform Gaussian elimination (Rothenberg 369).

Since then, the power of our computational capabilities has exploded. Better software for matrix computation, such as MATLAB, have enabled users to work with matrices with ease. Steven J. Leon, author of the linear algebra textbook, *Linear Algebra with Applications*, states, “MATLAB is a powerful tool for matrix computations that is also user friendly” (Leon 467). The execution of elimination has come a long way in terms of speed and simplicity. However, even

today, enormous systems prove costly for computers to solve. A system of 1,000 equations in 1,000 variables takes about 1 second to solve on a regular PC according to Dr. Gilbert Strang, a math professor at MIT. Yet a matrix containing 100,000 equations in 100,000 variables would take one million seconds, about 11.6 days, to solve (Strang 102). In a world with increasingly enormous data sets, faster methods of finding solutions will need to be developed.

Quantum computers provide exponential speedups of elimination (Harrow et al. 1). Currently, researchers are proposing and developing algorithms for performing elimination on a quantum computer (Barz et al. 1; Harrow et al. 1; Srinivasan et al. 2). In order to improve elimination, it is important to understand how the process of elimination has evolved. In other words, how has the execution of Gaussian-type elimination for nonsingular matrices changed since its origin in ancient China and how will future technologies such as quantum computers change the method? Joseph F. Grcar's journal article, "Mathematicians of Gaussian Elimination," provides details on each major contribution made to the development of Gaussian elimination. While he does list the computational technologies available up until the late 20$^{th}$ century, he does not focus on comparing them. An analysis of this aspect of Gaussian elimination is missing.

**Methods**

Two questions compose the research question that has been posed. First, how has Gaussian elimination changed throughout history? To address this question, I have two steps. First and foremost, when considering how Gaussian elimination has evolved, we must compare the actual processes performed when executing the algorithms. Looking at instructional texts on elimination suffices to answer how the method has changed. The majority of the texts that I will analyze for the first step will be mathematical books and textbooks intended for students of the

mathematical discipline. Of utmost importance is to analyze the *Nine Chapters on the Mathematical Arts*, where Gaussian-type elimination saw its origin. Unfortunately, the *Nine Chapters* exists only in reconstructed form since the original editions did not survive (Hart 30). This means that I will resort to secondary sources, such as Roger Hart's *The Chinese Roots of Linear Algebra*, in order to study the Chinese text. In addition, I will analyze more recent texts in order to see the methods that they present. Among the modern texts that I will analyze are Gilbert Strang's *Introduction to Linear Algebra*, Andy R. Magid's *Applied Matrix Models*, and Steven J. Leon's *Linear Algebra with Applications*. Step one will take one week.

The second step involves analyzing the changes in the computational aspect of Gaussian elimination. The same texts from the previous part will be used. In addition to these textual materials, I will obtain and learn how to use the computational technology mentioned in the texts. For instance, if I desire to examine how Gaussian elimination was executed on a programmable handheld computer in the 1980s, I will obtain that model of the calculator. Simply emulating the calculator on a modern computer is invalid, because the processing powers and user experience will differ. Only by utilizing the computational technologies as they were used during their time may I be able to understand how technology has changed. This step will be researched simultaneously with the first, but will likely take a week longer than the first. These two steps comprise the historical analysis portion that will take two weeks.

The second question involves predicting how future technologies will change the method of elimination. This question depends on having acquired sufficient knowledge about the first. Once I have a better understanding of how Gaussian elimination has changed, I will be able to make a reasonable prediction for how it will change in the near future. I recognize that I lack knowledge in quantum computers so I will have to rely on experts' opinions to support my

prediction. Around the end of the second week, I will contact researchers who have proposed or who are attempting to develop an elimination algorithm for use with quantum computers and ask them about elimination. Through email, I will ask the following questions among others:

- How similar is your elimination algorithm to Gaussian elimination?
- What have you changed?
- How will your algorithm change the computation of solutions?
- Will this improved method have new applications?

I expect to receive replies to my questions within a week of sending them. Therefore, the process of collecting data should take around three weeks. Writing the research paper will take an additional week. This means that I will complete my entire research project in one month.

**Expected Results**

The expected result of this research project is a research paper explaining the changes in Gaussian-type elimination, much like Joseph F. Grcar's journal article, "Mathematicians of Gaussian Elimination." Specifically, the research paper would describe changes in the method of elimination used throughout history. A comparative analysis of the computational tools would also be included. At the end of the paper, I will offer my own prediction of the future of elimination and I will incorporate the researchers' answers to my questions.

**Conclusion**

For problems that can be expressed as systems of linear equations, using matrices and elimination allows for a systematic method of finding solutions. However, elimination by hand is inefficient for large systems. Fortunately, the algorithmic nature of elimination enables computers to perform the process for us. But even then, enormous systems still take time to solve. The development of quantum computers will undoubtedly improve the efficiency of

elimination. It is therefore of importance to understand how the method and computation of elimination have changed since its origin in China two thousand years ago. With this knowledge, we will be able to understand how elimination could further change. My proposed research method will address these questions through textual analyses, comparisons of computational tools, and online interviews with current researchers.

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Budget

| Item | Amount | Reason |
|---|---|---|
| On-Campus Housing (One Month) | ~$1,200 | Living on campus would allow me to access the university's resources such as the library's books |
| Stipend (food, supplies, etc.) | $400 | This will cover the materials and supplies that I will need during my monthlong stay |
| TI 58C Programmable Computer | ~$100<br>($84.95 + shipping on eBay) | This programmable calculator would allow me to experience performing elimination on an older calculator |
| | Total: ~$1700 | |

# Appendix

## Basic Matrix Arithmetic and Operations

In order to understand the usefulness of matrices, we will first consider a simple algebraic problem. Suppose that you bought 2 apples and 3 oranges from a store for a total of \$7 and that your friend bought 4 apples and 7 oranges from the same store for a total of \$15. You want to find the price of each fruit. To do so, set up a system of linear equations where *x* and *y* represent the price of apples and oranges, respectively. The two equations will be linear since the unknown variables are all linear. We obtain the following system:

$$\begin{cases} 2x + 3y = 7 \\ 4x + 7y = 15 \end{cases} \tag{1}$$

We will solve the system by using the method of elimination. The goal is to eliminate one variable in the second equation. We will eliminate the 4*x* in the second equation by subtracting 2 times equation 1 from equation 2 as follows:

$$\begin{array}{rl} 2 * (2x + 3y = 7) \Rightarrow 4x + 6y = 14 & \text{this is 2 times equation 1} \\ (4x + 7y = 15) & \text{equation 2} \\ \underline{-(4x + 6y = 14)} & \text{minus 2 times equation 1} \\ 0x + 1y = 1 & \text{new equation 2} \end{array}$$

Our new system becomes:

$$\begin{cases} 2x + 3y = 7 \\ 0x + 1y = 1 \end{cases}$$

Now it is easy to see that $y = 1$, and plugging this value into the first equation reveals that $x = 2$. Thus, apples cost \$2 and oranges cost \$1.

We can simplify (1) using matrices. A matrix is a rectangular array of numbers, obtained from a system of equations, usually enclosed by brackets (Cayley 17). We would have the following matrix equation for our apples and oranges problem.

$$\begin{bmatrix} 2x & 3y \\ 4x & 7y \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \end{bmatrix}$$

On the left side, we have a 2 by 2 matrix. It has two rows and two columns. On the right side, we have a 2 by 1 matrix. It has 2 rows and 1 column. We can further decompose the left matrix by depicting it as a product of two matrices:

$$\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \end{bmatrix} \tag{2}$$

Matrix multiplication for an *m* by *n* matrix and a *p* by *q* matrix is defined if *n* = *p*, and the resulting matrix will be *m* by *q*. We have two matrices, A and X, and their product AX. Notice that AX is a 2 by 1 matrix, since A has two rows and X has one column.

$$\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} = A$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = X$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + 3y \\ 4x + 7y \end{bmatrix} = AX$$

$2x + 3y$ is in the position $AX_{11}$. The subscript "11" indicates that $2x + 3y$ is in the row 1, column 1 of AX. To find $AX_{11}$, take the dot product of the first row of A with the first column of X. In other words, multiply the first number of the first row of A with the first number of the first column of X, multiply the second number of the first row of A with the second number of the first column of X, and find their sum (Cayley 20). This sum goes in the position $AX_{11}$. To find $AX_{21}$ (the number in the second row, first column of AX), you would compute the dot product of the second row of A with the first column of X. This is 4 * *x* + 7 * *y*.

We can further simplify (2) as an augmented matrix:

$$\begin{bmatrix} 2 & 3 & 7 \\ 4 & 7 & 15 \end{bmatrix} = M$$

*M* is a 2 by 3 matrix. It is simply matrix *A* with the 2 by 1 matrix containing the total costs attached at the end as the third column. Matrix *M* is much simpler than (1). The first column contains the coefficients of the variable *x*, the second column contains the coefficients of the variable *y*, and the last column contains each cost of the *x* and *y* combinations given by that row. Now we will perform row operations in order to change *M* so that it reveals the solutions. We want to obtain an upper triangular matrix (a matrix whose second row begins with a 0, third row begins with two 0s, fourth row begins with three 0s, etc.). These are the three row operations that we can perform:

(i) exchanging two rows

(ii) multiplying a row by a nonzero number

(iii) adding a multiple of a row to another row

We change *M* but not the solution (Mirsky 168). We will first divide row 1 of *M* by 2. We have

$$\begin{bmatrix} 1 & 3/2 & 7/2 \\ 4 & 7 & 15 \end{bmatrix} = M'$$

*M* has become a new matrix *M'*. Next, we will add –4 of row 1 to row 2 to obtain:

$$\begin{bmatrix} 1 & 3/2 & 7/2 \\ 0 & 1 & 1 \end{bmatrix} = M''$$

The second row tells us that $0x + 1y = 1$. Thus, $y = 1$. Substituting this value into the first row, $1x + \frac{3}{2}y = \frac{7}{2}$ reveals that $x = 2$. Gaussian elimination is the process of using rows to eliminate the variables in other rows in such a way that creates a triangular matrix that reveals the solutions after back substitution (Strang 50-51).